

On consistency of the MLE under finite mixtures of location-scale distributions with a structural parameter

Guanfu Liu^a, Pengfei Li^b, Yukun Liu^{c,*}, Xiaolong Pu^c

^a*School of Statistics and Information, Shanghai University of International Business and Economics, Shanghai 201620, China*

^b*Department of Statistics and Actuarial Sciences, University of Waterloo, Waterloo, ON, Canada, N2L 3G1*

^c*School of Statistics, East China Normal University, Shanghai 200241, China*

Abstract

We provide a general and rigorous proof for the strong consistency of maximum likelihood estimators of the cumulative distribution function of the mixing distribution and structural parameter under finite mixtures of location-scale distributions with a structural parameter. The consistency results do not require the parameter space of location and scale to be compact. We illustrate the results by applying them to finite mixtures of location-scale distributions with the component density function being one of the commonly used density functions: normal, logistic, extreme-value, or t . An extension of the strong consistency results to finite mixtures of multivariate elliptical distributions is also discussed.

Keywords: Consistency, Finite mixture of location-scale distributions, Maximum likelihood estimator, Structural parameter.

1. Introduction

Suppose we have an independent and identically distributed (*i.i.d.*) sample X_1, \dots, X_n from the following finite mixture model:

$$g(x; \Psi, \sigma) = \sum_{j=1}^m \alpha_j f(x; \mu_j, \sigma) = \int_{\mathbb{R}} f(x; \mu, \sigma) d\Psi(\mu). \quad (1)$$

*Corresponding author

Email address: ykliu@sfs.ecnu.edu.cn (Yukun Liu)

Here $f(x; \mu, \sigma)$, the component density function, is assumed to come from a location-scale distribution family, namely, $f(x; \mu, \sigma) = \sigma^{-1} f((x - \mu)/\sigma; 0, 1)$ with $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$ being the location and scale parameters, respectively.

5 The positive integer m is called the order of the mixture model, $(\alpha_1, \dots, \alpha_m)$ with $\alpha_j \geq 0$ and $\sum_{j=1}^m \alpha_j = 1$ are called the mixing proportions, and $\Psi(\mu) = \sum_{j=1}^m \alpha_j I(\mu_j \leq \mu)$ is called the cumulative distribution function of the mixing distribution. The parameter σ , appearing in all m component density functions, is called a structural parameter, and Model (1) is called a finite mixture

10 of location-scale distributions with a structural parameter. Note that $\Psi(\cdot)$ includes unknown μ_j and α_j parameters. Hence, (Ψ, σ) covers all the unknown parameters in (1). In this paper, we investigate the strong consistency of the maximum likelihood estimator (MLE) of (Ψ, σ) under Model (1).

Finite mixtures of location-scale distributions with a structural parameter

15 have many applications. They play an important role in medical studies and genetics. For example, Roeder (1994) applied the finite normal mixture model with a structural parameter to analyze sodium–lithium countertransport activity in red blood cells. Finite mixtures of logistic distributions and of extreme value distributions with a structural parameter are widely used to analyze failure-time

20 data. For instance, a finite mixture of logistic distributions with a structural parameter was used by Naya et al. (2006) to study the thermogravimetric analysis trace. A finite mixture of extreme value distributions with a structural parameter was found to provide an adequate fit to the logarithm of the number of cycles to failure for a group of 60 electrical appliances (Lawless, 2003; Example 4.4.2). More applications can be found in McLachlan & Peel (2000) and

25 Lawless (2003).

The maximum likelihood method has been widely used to estimate the unknown parameters in finite mixture models (McLachlan & Peel, 2000; Chen, 2017). The consistency of the MLE under finite mixture models has been

30 studied by Kiefer & Wolfowitz (1956), Redner (1981), and Chen (2017). As pointed out by Chen (2017), the results in Kiefer & Wolfowitz (1956) require that $g(x; \Psi, \sigma)$ can be continuously extended to a compact space of (Ψ, σ) . This

turns out to be impossible because $f(x; \mu, \sigma)$ is not well defined at $\sigma = 0$. To make the results in Kiefer & Wolfowitz (1956) applicable to our current setup, we must constrain the parameter σ to be in a compact subset of \mathbb{R}^+ . The consistency results in Redner (1981) require even more restrictive conditions: the parameter space for (μ, σ) must be a compact subset of $\mathbb{R} \times \mathbb{R}^+$; see Chen (2017) for more discussion. It is worth mentioning that Bryant (1991) established the strong consistency of the estimators obtained by the linear-optimization-based method. His result can be viewed as a generalization of the classical consistency result for MLE. However, it requires that the parameter space be closed and that m be equal to the true order of the mixture model. By utilizing the properties of the normal distribution, Chen (2017) proved the strong consistency of the MLE under finite normal mixture models with a structural parameter without imposing the compactness assumption on the parameter space. To the best of our knowledge, general consistency results for the MLE of (Ψ, σ) under Model (1) are not available in the literature except for the normal mixture model.

Because of the importance of finite mixtures of location-scale distributions with a structural parameter, it is necessary to study the consistency of the MLE of the underlying parameters, (Ψ, σ) , under Model (1). The goal of this paper is to provide a general and rigorous proof of this consistency. In Section 2, we present the main consistency results. We emphasize that we do not require the parameter space of (μ, σ) to be compact. The detailed proofs are given in Section 3. Section 4 illustrates the consistency results by applying them to Model (1) with $f(x; \mu, \sigma)$ being one of the commonly used component density functions: normal, logistic, extreme-value, or t . An extension of the consistency results to finite mixtures of multivariate elliptical distributions is discussed in Section 5.

2. Main results

With the *i.i.d.* sample X_1, \dots, X_n from (1), the log-likelihood function of (Ψ, σ) is given by

$$\ell_n(\Psi, \sigma) = \sum_{i=1}^n \log\{g(X_i; \Psi, \sigma)\}.$$

The MLE of (Ψ, σ) is defined as

$$(\hat{\Psi}, \hat{\sigma}) = \arg \max_{\Psi \in \Psi_m, \sigma > 0} \ell_n(\Psi, \sigma),$$

where

$$\Psi_m = \left\{ \sum_{j=1}^m \alpha_j I(\mu_j \leq \mu) : \alpha_j \geq 0, \sum_{j=1}^m \alpha_j = 1, \mu_j \in \mathbb{R} \right\}.$$

In this section, we establish the consistency property of $(\hat{\Psi}, \hat{\sigma})$ without imposing compactness on the parameter space of (μ, σ) . To discuss the consistency of $\hat{\Psi}$, we define

$$D(\Psi_1, \Psi_2) = \int_{\mathbb{R}} |\Psi_1(\mu) - \Psi_2(\mu)| \exp(-|\mu|) d\mu. \quad (2)$$

60 We show that $D(\Psi_1, \Psi_2)$ is a distance on Ψ_m in the Appendix. Suppose $\Psi_0 \in \Psi_m$ is the true cumulative distribution function of the mixing distribution. We say that $\hat{\Psi}$ is strongly consistent if $D(\hat{\Psi}, \Psi_0) \rightarrow 0$ almost surely as $n \rightarrow \infty$.

The strong consistency of $(\hat{\Psi}, \hat{\sigma})$ depends on the following regularity conditions.

C1. The finite mixture model in (1) is identifiable. That is, if (Ψ_1, σ_1) and (Ψ_2, σ_2) with $\Psi_1 \in \Psi_m$, $\Psi_2 \in \Psi_m$, $\sigma_1 > 0$, and $\sigma_2 > 0$ satisfy

$$\int_{\mathbb{R}} f(x; \mu, \sigma_1) d\Psi_1(\mu) = \int_{\mathbb{R}} f(x; \mu, \sigma_2) d\Psi_2(\mu)$$

65 for all x , then $\Psi_1 = \Psi_2$ and $\sigma_1 = \sigma_2$.

C2. $\int_{\mathbb{R}} |\log\{g(x; \Psi_0, \sigma_0)\}| g(x; \Psi_0, \sigma_0) dx < \infty$, where (Ψ_0, σ_0) is the true value of (Ψ, σ) .

C3. There exist positive constants v_0, v_1 , and β with $\beta > 1$ such that for all x

$$f(x; 0, 1) \leq \min\{v_0, v_1|x|^{-\beta}\}.$$

C4. The function $f(x; 0, 1)$ is continuous with respect to x .

Theorem 1. *Assume Conditions C1–C4 and that the true density is $g(x; \Psi_0, \sigma_0)$.*

70 *If an estimator $(\bar{\Psi}, \bar{\sigma})$ of (Ψ, σ) satisfies*

$$\ell_n(\bar{\Psi}, \bar{\sigma}) - \ell_n(\Psi_0, \sigma_0) \geq c > -\infty \quad (3)$$

for some constant c , then $\bar{\Psi} \rightarrow \Psi_0$ with respect to the metric $D(\cdot, \cdot)$ in (2) and $\bar{\sigma} \rightarrow \sigma_0$ almost surely as $n \rightarrow \infty$.

We comment that Conditions C1 and C2 are the standard regularity conditions for the consistency of the MLE. Condition C2 implies Wald’s integrability
 75 condition (Wald, 1949). Conditions C3 and C4 ensure that Lemma 1 in Section 3 is correct and the mixture density $g(x; \Psi, \sigma)$ can be continuously extended to a compact space for (Ψ, σ) . Conditions C2–C4 together guarantee that $\bar{\sigma}$ is away from 0 and bounded above almost surely as $n \rightarrow \infty$. Then the MLE consistency results in Kiefer & Wolfowitz (1956) can be applied; more discussion is
 80 given in Section 3. In Section 4, we show that Model (1) with $f(x; \mu, \sigma)$ being one of the four commonly used component density functions (normal, logistic, extreme-value, or t) satisfies these conditions.

It can be seen that the MLE $(\hat{\Psi}, \hat{\sigma})$ satisfies (3) with $c = 0$. Therefore, by Theorem 1, both $\hat{\Psi}$ and $\hat{\sigma}$ are strongly consistent under Conditions C1–C4.

85 **Corollary 1.** *Assume Conditions C1–C4 and that the true density is $g(x; \Psi_0, \sigma_0)$. Then $\hat{\Psi} \rightarrow \Psi_0$ with respect to the metric $D(\cdot, \cdot)$ in (2) and $\hat{\sigma} \rightarrow \sigma_0$ almost surely as $n \rightarrow \infty$.*

3. Proofs

3.1. Some useful lemmas

90 As discussed in Chen (2017), except for Conditions C1 and C2, a key regularity condition for the MLE consistency results in Kiefer & Wolfowitz (1956) is that the definition of $g(x; \Psi, \sigma)$ can be continuously extended to a compact space of (Ψ, σ) . This extension could fail under our current setup because $f(x; \mu, \sigma)$ is

not well defined at $\sigma = 0$. To make the consistency results in Kiefer & Wolfowitz
 95 (1956) applicable, a key step is to show that there exist positive constants ϵ and
 Δ such that as $n \rightarrow \infty$, the event sequence $\{\epsilon \leq \bar{\sigma} \leq \Delta\}$ occurs almost surely.

We first present a technical lemma that gives a uniform upper bound for the
 number of observations in the σ^{1-a} neighborhood of μ . Here $a = (1 + \beta)/(2\beta)$,
 where β is given in Condition C3. With the condition that $\beta > 1$ in Condition
 100 C3, we have $0 < a < 1$.

For convenience of presentation, we let $b = 2(\beta + 1)/(\beta - 1)$ and $\epsilon_0 =$
 $(3mbv_0/\sigma_0)^{-1/(1-a)}$. Further, let

$$G_n(x) = n^{-1} \sum_{i=1}^n I(X_i \leq x)$$

be the empirical cumulative distribution function, and let $G(x)$ be the cumula-
 tive distribution function of the X_i 's. Define

$$E = \liminf_{n \rightarrow \infty} \left\{ \sup_{\mu \in \mathbb{R}} \sum_{i=1}^n I(|X_i - \mu| < \epsilon_0^{1-a}) \leq n/(mb) \right\}.$$

Lemma 1. *Suppose $\{X_1, \dots, X_n\}$ is an i.i.d. sample from Model (1). Further,
 assume Conditions C3 and C4 are satisfied. Then we have*

$$P(E) = 1,$$

and therefore almost surely there exists an n_0 such that when $n \geq n_0$, we have

$$\sup_{\mu \in \mathbb{R}} \sum_{i=1}^n I(|X_i - \mu| < \epsilon_0^{1-a}) \leq n/(mb).$$

Proof. Since

$$\sup_{\mu \in \mathbb{R}} \sum_{i=1}^n I(|X_i - \mu| < \epsilon_0^{1-a}) \leq \sup_{\mu \in \mathbb{R}} [n\{G_n(\mu + \epsilon_0^{1-a}) - G_n(\mu - \epsilon_0^{1-a})\}],$$

it suffices to show that

$$P \left(\liminf_{n \rightarrow \infty} \left\{ \sup_{\mu \in \mathbb{R}} \{G_n(\mu + \epsilon_0^{1-a}) - G_n(\mu - \epsilon_0^{1-a})\} \leq 1/(mb) \right\} \right) = 1. \quad (4)$$

Note that

$$\begin{aligned} \sup_{\mu \in \mathbb{R}} \{G_n(\mu + \epsilon_0^{1-a}) - G_n(\mu - \epsilon_0^{1-a})\} &\leq \sup_{\mu \in \mathbb{R}} |G_n(\mu + \epsilon_0^{1-a}) - G(\mu + \epsilon_0^{1-a})| \\ &\quad + \sup_{\mu \in \mathbb{R}} |G(\mu + \epsilon_0^{1-a}) - G(\mu - \epsilon_0^{1-a})| \\ &\quad + \sup_{\mu \in \mathbb{R}} |G_n(\mu - \epsilon_0^{1-a}) - G(\mu - \epsilon_0^{1-a})|. \end{aligned}$$

Under Conditions C3 and C4,

$$\sup_{\mu \in \mathbb{R}} |G(\mu + \epsilon_0^{1-a}) - G(\mu - \epsilon_0^{1-a})| \leq 2v_0\epsilon_0^{1-a}/\sigma_0.$$

Let

$$A_1 = \left\{ \lim_{n \rightarrow \infty} \sup_{\mu \in \mathbb{R}} |G_n(\mu + \epsilon_0^{1-a}) - G(\mu + \epsilon_0^{1-a})| = 0 \right\}$$

and

$$A_2 = \left\{ \lim_{n \rightarrow \infty} \sup_{\mu \in \mathbb{R}} |G_n(\mu - \epsilon_0^{1-a}) - G(\mu - \epsilon_0^{1-a})| = 0 \right\}.$$

By the Glivenko–Cantelli theorem,

$$P(A_1) = P(A_2) = 1.$$

Hence, $P(A_1 \cap A_2) = 1$. Then almost surely there exists an n_0 such that when $n \geq n_0$,

$$\sup_{\mu \in \mathbb{R}} \{G_n(\mu + \epsilon_0^{1-a}) - G_n(\mu - \epsilon_0^{1-a})\} \leq 3v_0\epsilon_0^{1-a}/\sigma_0 = 1/(mb),$$

which implies (4). This completes the proof. \square

105 The next lemma helps us to show that asymptotically we can restrict σ to be in a bounded interval away from 0 and ∞ . To formally present the result, we define some notation. Let $K_0 = \int_{\mathbb{R}} \log\{g(x; \Psi_0, \sigma_0)\}g(x; \Psi_0, \sigma_0)dx$. Condition C2 ensures that $|K_0| < \infty$. Further, define $\Delta = v_0/\exp(K_0 - 1)$. We choose a positive number ϵ that satisfies the following conditions:

110 D1. $\epsilon \leq \epsilon_0 = (3mbv_0/\sigma_0)^{-1/(1-a)}$;

D2. $\epsilon \leq (v_1/v_0)^{-2/(\beta+1)}$;

D3. $b^{-1} \log v_0 + (1 - b^{-1}) \log v_1 + \frac{1}{4}(\beta - 1) \log \epsilon \leq K_0 - 1$.

Clearly, an ϵ satisfying the above conditions exists, since $\beta > 1$ as assumed in Condition C3.

Lemma 2. *Suppose $\{X_1, \dots, X_n\}$ is an i.i.d. sample from Model (1). Further, assume Conditions C1–C4 are satisfied. Then we have*

$$P \left(\lim_{n \rightarrow \infty} \left\{ \sup_{\Psi \in \Psi_m, \sigma \in [\Delta, \infty)} \ell_n(\Psi, \sigma) - \ell_n(\Psi_0, \sigma_0) \right\} = -\infty \right) = 1 \quad (5)$$

and

$$P \left(\lim_{n \rightarrow \infty} \left\{ \sup_{\Psi \in \Psi_m, \sigma \in (0, \epsilon]} \ell_n(\Psi, \sigma) - \ell_n(\Psi_0, \sigma_0) \right\} = -\infty \right) = 1. \quad (6)$$

115 *Proof.* We start with (5). Recall that $f(x; \mu, \sigma) = \sigma^{-1} f((x - \mu)/\sigma; 0, 1)$. By Condition C3, we have

$$\begin{aligned} \ell_n(\Psi, \sigma) &= \sum_{i=1}^n \log \left\{ \sum_{j=1}^m \frac{\alpha_j}{\sigma} f \left(\frac{X_i - \mu_j}{\sigma}; 0, 1 \right) \right\} \\ &\leq \sum_{i=1}^n \log \left\{ \sum_{j=1}^m \frac{\alpha_j}{\sigma} v_0 \right\} \\ &\leq n (\log v_0 - \log \sigma), \end{aligned} \quad (7)$$

where v_0 is given in Condition C3.

Hence, with $\Delta = v_0 / \exp(K_0 - 1)$, we have

$$\sup_{\Psi \in \Psi_m, \sigma \geq \Delta} \ell_n(\Psi, \sigma) - \ell_n(\Psi_0, \sigma_0) \leq n(K_0 - 1) - \ell_n(\Psi_0, \sigma_0).$$

By the strong law of large numbers and the definition of K_0 , we have (5).

We next consider (6). Let $A = \{i : \min_{1 \leq j \leq m} |X_i - \mu_j| \leq \sigma^{1-a}\}$ and $n(A)$ be the number of indices in set A . For an index set S , define $\ell_n(\Psi, \sigma; S) = \sum_{i \in S} \log g(X_i; \Psi, \sigma)$. Similarly to (7), it can be shown that

$$\ell_n(\Psi, \sigma; A) \leq n(A) (\log v_0 - \log \sigma). \quad (8)$$

By Condition C3, we have

$$\begin{aligned} \ell_n(\Psi, \sigma; A^c) &\leq \sum_{i \in A^c} \log \left\{ \sum_{j=1}^m \frac{\alpha_j}{\sigma} v_1 \left(\frac{|X_i - \mu_j|}{\sigma} \right)^{-\beta} \right\} \\ &= \sum_{i \in A^c} \left[\log v_1 + (\beta - 1) \log \sigma + \log \left\{ \sum_{j=1}^m \alpha_j |X_i - \mu_j|^{-\beta} \right\} \right]. \end{aligned}$$

Since $\min_{1 \leq j \leq m} |X_i - \mu_j| \geq \sigma^{1-a}$ for all $i \in A^c$, it follows that

$$\begin{aligned} \ell_n(\Psi, \sigma; A^c) &\leq \sum_{i \in A^c} \{\log v_1 + (\beta a - 1) \log \sigma\} \\ &= n(A^c) \left\{ \log v_1 + \frac{1}{2}(\beta - 1) \log \sigma \right\}, \end{aligned} \quad (9)$$

where the equality holds because $\beta a = (\beta + 1)/2 > 1$.

Combining (8) and (9) gives

$$\begin{aligned} \ell_n(\Psi, \sigma) &= \ell_n(\Psi, \sigma; A) + \ell_n(\Psi, \sigma; A^c) \\ &\leq n(\log v_0 - \log \sigma) + n(A^c) \left\{ \log v_1 - \log v_0 + \frac{1}{2}(\beta + 1) \log \sigma \right\} \\ &= n(\log v_0 - \log \sigma) + n(1 - b^{-1}) \left\{ \log v_1 - \log v_0 + \frac{1}{2}(\beta + 1) \log \sigma \right\} \\ &\quad + \{n(A^c) - n(1 - b^{-1})\} \left\{ \log v_1 - \log v_0 + \frac{1}{2}(\beta + 1) \log \sigma \right\} \\ &= n \left\{ b^{-1} \log v_0 + (1 - b^{-1}) \log v_1 + \frac{1}{4}(\beta - 1) \log \sigma \right\} \\ &\quad + \{n/b - n(A)\} \left\{ \log v_1 - \log v_0 + \frac{1}{2}(\beta + 1) \log \sigma \right\}, \end{aligned}$$

where in the last step we have used the fact that $b = 2(\beta + 1)/(\beta - 1)$. Then, for $\sigma \in (0, \epsilon]$, since $\beta > 1$ and ϵ satisfies D3, we have

$$\begin{aligned} &\ell_n(\Psi, \sigma) - \ell_n(\Psi_0, \sigma_0) \\ &\leq n \left\{ b^{-1} \log v_0 + (1 - b^{-1}) \log v_1 + \frac{1}{4}(\beta - 1) \log \epsilon \right\} - \ell_n(\Psi_0, \sigma_0) \\ &\quad + \{n/b - n(A)\} \left\{ \log v_1 - \log v_0 + \frac{1}{2}(\beta + 1) \log \sigma \right\} \\ &\leq n(K_0 - 1) - \ell_n(\Psi_0, \sigma_0) \end{aligned} \quad (10)$$

$$+ \{n/b - n(A)\} \left\{ \log v_1 - \log v_0 + \frac{1}{2}(\beta + 1) \log \sigma \right\}. \quad (11)$$

By the strong law of large numbers and the definition of K_0 , we have

$$P \left(\lim_{n \rightarrow \infty} \{n(K_0 - 1) - \ell_n(\Psi_0, \sigma_0)\} = -\infty \right) = 1. \quad (12)$$

Again for $\sigma \in (0, \epsilon]$, since ϵ satisfies D2, it follows that

$$\log v_1 - \log v_0 + \frac{1}{2}(\beta + 1) \log \sigma \leq \log v_1 - \log v_0 + \frac{1}{2}(\beta + 1) \log \epsilon \leq 0. \quad (13)$$

Since for each $i \in \{1, 2, \dots, n\}$,

$$I\left(\min_{1 \leq j \leq m} |X_i - \mu_j| \leq \sigma^{1-a}\right) \leq \sum_{j=1}^m I(|X_i - \mu_j| \leq \sigma^{1-a}),$$

we have

$$\begin{aligned} n(A) &= \sum_{i=1}^n I\left(\min_{1 \leq j \leq m} |X_i - \mu_j| \leq \sigma^{1-a}\right) \\ &\leq \sum_{i=1}^n \sum_{j=1}^m I(|X_i - \mu_j| \leq \sigma^{1-a}) \\ &\leq m \cdot \sup_{\mu \in \mathbb{R}} \sum_{i=1}^n I(|X_i - \mu| \leq \sigma^{1-a}). \end{aligned}$$

Because $\sigma \in (0, \epsilon]$ and ϵ satisfies D1, it follows from Lemma 1 that

$$n(A) \leq m \cdot \sup_{\mu \in \mathbb{R}} \sum_{i=1}^n I(|X_i - \mu| \leq \epsilon_0^{1-a}) \leq n/b.$$

This together with (13) implies that for large enough n ,

$$\{n/b - n(A)\} \left\{ \log v_1 - \log v_0 + \frac{1}{2}(\beta + 1) \log \sigma \right\} \leq 0,$$

125 almost surely. Combining (10)–(13) leads to (6). This completes the proof. \square

3.2. Proof of Theorem 1

The results in (5) and (6) imply that

$$P\left(\liminf_{n \rightarrow \infty} \{\epsilon \leq \bar{\sigma} \leq \Delta\}\right) = 1.$$

Hence, we can confine $\bar{\sigma}$ to $[\epsilon, \Delta]$ asymptotically, and the results of Kiefer & Wolfowitz (1956) apply. For completeness, we outline the key steps of the proof of Theorem 1 by following Kiefer & Wolfowitz (1956).

In the first step, we compactify the parameter space Ψ_m . Let

$$\bar{\Psi}_m = \{\gamma + \rho\Psi : \Psi \in \Psi_m, \gamma \geq 0, \rho \geq 0, 0 \leq \gamma + \rho \leq 1\}.$$

We extend the definition of $D(\cdot, \cdot)$ to $\bar{\Psi}_m$ without modifications. Then $\bar{\Psi}_m$ is a compact metric space with respect to $D(\cdot, \cdot)$. See the Appendix for a proof of its compactness. Let $\theta = (\Psi, \sigma)$ and define

$$\bar{D}(\theta_1, \theta_2) = D(\Psi_1, \Psi_2) + |\sigma_1 - \sigma_2|.$$

Since $D(\cdot, \cdot)$ is a distance and

$$D(\Psi_1, \Psi_2) \leq 2 \int_{\mathbb{R}} \exp(-|\mu|) d\mu = 4,$$

130 we can verify that $\bar{D}(\cdot, \cdot)$ is a bounded distance on $\bar{\Psi}_m \times [\epsilon, \Delta]$. Hence, $\bar{\Psi}_m \times [\epsilon, \Delta]$ is compact with respect to $\bar{D}(\cdot, \cdot)$.

In the second step, we argue that $g(x; \Psi, \sigma)$ is continuous for all x on $\bar{\Psi}_m \times [\epsilon, \Delta]$ under the distance $\bar{D}(\cdot, \cdot)$. That is, for any $\theta = (\Psi, \sigma) \in \bar{\Psi}_m \times [\epsilon, \Delta]$, if $\bar{D}(\tilde{\theta}, \theta) \rightarrow 0$, then we have

$$g(x; \tilde{\Psi}, \tilde{\sigma}) \rightarrow g(x; \Psi, \sigma).$$

For the given (Ψ, σ) , we define

$$H(\mu, \sigma^*) = \Psi(\mu) I(\sigma \leq \sigma^*)$$

and define \tilde{H} to be H with $(\tilde{\Psi}, \tilde{\sigma})$ in place of (Ψ, σ) . Then the mixture density can be written as

$$g(x; \Psi, \sigma) = \int_{\mathbb{R} \times \mathbb{R}^+} f(x; \mu, \sigma^*) dH(\mu, \sigma^*).$$

We further define

$$\tilde{D}(H_1, H_2) = \int_{\mathbb{R} \times \mathbb{R}^+} |H_1(\mu, \sigma^*) - H_2(\mu, \sigma^*)| \exp\{-|\mu| - |\sigma^*|\} d\mu d\sigma^*.$$

Lemma 2.4 of Chen (2017) implies that if $f(x; \mu, \sigma)$ is continuous for (μ, σ) , $\lim_{|\mu|+|\sigma| \rightarrow \infty} f(x; \mu, \sigma) = 0$, and $\tilde{D}(\tilde{H}, H) \rightarrow 0$, then we have $g(x; \tilde{\Psi}, \tilde{\sigma}) \rightarrow g(x; \Psi, \sigma)$.

135 Condition C4 ensures that $f(x; \mu, \sigma)$ is continuous for (μ, σ) ; Condition C3 ensures that $\lim_{|\mu|+|\sigma| \rightarrow \infty} f(x; \mu, \sigma) = 0$. Hence, to argue that $g(x; \Psi, \sigma)$ is continuous for all x , we need to show only that if $\bar{D}(\tilde{\theta}, \theta) \rightarrow 0$, then we must

have $\tilde{D}(\tilde{H}, H) \rightarrow 0$. Note that

$$\begin{aligned}
\tilde{D}(\tilde{H}, H) &= \int_{\mathbb{R} \times \mathbb{R}^+} |\tilde{\Psi}(\mu)I(\tilde{\sigma} \leq \sigma^*) - \Psi(\mu)I(\sigma \leq \sigma^*)| \exp\{-|\mu| - |\sigma^*|\} d\mu d\sigma^* \\
&\leq \int_{\mathbb{R} \times \mathbb{R}^+} |\tilde{\Psi}(\mu) - \Psi(\mu)| \exp\{-|\mu| - |\sigma^*|\} d\mu d\sigma^* \\
&\quad + \int_{\mathbb{R} \times \mathbb{R}^+} |I(\tilde{\sigma} \leq \sigma^*) - I(\sigma \leq \sigma^*)| \exp\{-|\mu| - |\sigma^*|\} d\mu d\sigma^* \\
&\leq \int_{\mathbb{R}} |\tilde{\Psi}(\mu) - \Psi(\mu)| \exp\{-|\mu|\} d\mu \\
&\quad + 2 \int_{\mathbb{R}^+} |I(\tilde{\sigma} \leq \sigma^*) - I(\sigma \leq \sigma^*)| \exp\{-|\sigma^*|\} d\sigma^* \\
&\leq D(\tilde{\Psi}, \Psi) + 2|\tilde{\sigma} - \sigma| \exp\{-\min(\tilde{\sigma}, \sigma)\}. \tag{14}
\end{aligned}$$

Further, when $\bar{D}(\tilde{\theta}, \theta) \rightarrow 0$, we must have $D(\tilde{\Psi}, \Psi) \rightarrow 0$ and $\tilde{\sigma} \rightarrow \sigma$, which, together with (14), implies that $\tilde{D}(\tilde{H}, H) \rightarrow 0$. Hence, $g(x; \Psi, \sigma)$ is continuous for all x on $\bar{\Psi}_m \times [\epsilon, \Delta]$ under the distance $\bar{D}(\cdot, \cdot)$.

In the third step, we show that for any $\theta = (\Psi, \sigma) \in \bar{\Psi}_m \times [\epsilon, \Delta]$ such that $(\Psi, \sigma) \neq (\Psi_0, \sigma_0)$, we can find a δ such that

$$E \{ \log g(X; B_\delta(\theta)) \} - E \{ \log g(X; \Psi_0, \sigma_0) \} < 0, \tag{15}$$

where

$$B_\delta(\theta) = \left\{ \theta^* = (\Psi^*, \sigma^*) : \bar{D}(\theta^*, \theta) < \delta, \theta^* \in \bar{\Psi}_m \times [\epsilon, \Delta] \right\}$$

and

$$g(x; B_\delta(\theta)) = \sup_{\theta^* \in B_\delta(\theta)} g(x; \Psi^*, \sigma^*).$$

By Kiefer & Wolfowitz (1956) and Chen (2017) and because of Conditions C1, C2, and C4, we can obtain (15) by arguing that

$$\lim_{\delta \rightarrow 0^+} E \left[\left\{ \log \frac{g(X; B_\delta(\theta))}{g(X; \Psi_0, \sigma_0)} \right\}^+ \right] < \infty. \tag{16}$$

Here $x^+ = \max(x, 0)$. By Condition C3,

$$\log \frac{g(X; B_\delta(\theta))}{g(X; \Psi_0, \sigma_0)} \leq \log(v_0/\sigma_0) - \log g(X; \Psi_0, \sigma_0).$$

Hence,

$$\left\{ \log \frac{g(X; B_\delta(\boldsymbol{\theta}))}{g(X; \Psi_0, \sigma_0)} \right\}^+ \leq |\log(v_0/\sigma_0)| + |\log g(X; \Psi_0, \sigma_0)|,$$

which, together with Condition C2, implies (16). Hence, (15) is proved.

In the last step, we combine the above results to complete the proof of Theorem 1. For any $\delta > 0$, let

$$B^{(0)} = B_\delta(\boldsymbol{\theta}_0).$$

The result in (15) leads to a finite open cover of the compact set $\{\bar{\Psi}_m \times [\epsilon, \Delta]\} \setminus B^{(0)}$. Then by the finite covering theorem, we can find $\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_K$ from $\bar{\Psi}_m \times [\epsilon, \Delta]$ and positive $\delta_1, \dots, \delta_K$ for some positive integer K such that

$$\{\bar{\Psi}_m \times [\epsilon, \Delta]\} \setminus B^{(0)} \subset \bigcup_{k=1}^K B_{\delta_k}(\boldsymbol{\theta}_k)$$

and

$$E \{ \log g(X; B_{\delta_k}(\boldsymbol{\theta}_k)) \} - E \{ \log g(X; \Psi_0, \sigma_0) \} < 0. \quad (17)$$

Define

$$B^{(k)} = B_{\delta_k}(\boldsymbol{\theta}_k), \quad k = 1, \dots, K, \quad B^{(K+1)} = \{\bar{\Psi}_m \times (0, \epsilon]\} \cup \{\bar{\Psi}_m \times [\Delta, \infty)\}.$$

Then the whole parameter space of $\boldsymbol{\theta} = (\Psi, \sigma)$ can be written as

$$\bar{\Psi}_m \times (0, \infty) = \bigcup_{k=0}^{K+1} B^{(k)}.$$

Note that it can be easily verified that

$$\sup_{\Psi \in \bar{\Psi}_m, \sigma \in (0, \epsilon]} \ell_n(\Psi, \sigma) = \sup_{\Psi \in \bar{\Psi}_m, \sigma \in (0, \epsilon]} \ell_n(\Psi, \sigma)$$

and

$$\sup_{\Psi \in \bar{\Psi}_m, \sigma \in [\Delta, \infty)} \ell_n(\Psi, \sigma) = \sup_{\Psi \in \bar{\Psi}_m, \sigma \in [\Delta, \infty)} \ell_n(\Psi, \sigma).$$

Hence, by the strong law of large numbers, (5)–(6), and (17), we have for $k = 1, \dots, K + 1$

$$P \left(\lim_{n \rightarrow \infty} \left\{ \sup_{\boldsymbol{\theta} \in B^{(k)}} \ell_n(\Psi, \sigma) - \ell_n(\Psi_0, \sigma_0) \right\} = -\infty \right) = 1.$$

Therefore, for any estimator $(\bar{\Psi}, \bar{\sigma})$ of (Ψ, σ) such that

$$\ell_n(\bar{\Psi}, \bar{\sigma}) - \ell_n(\Psi_0, \sigma_0) \geq c > -\infty$$

for some constant c , we must have for any $\delta > 0$

$$P\left(\liminf_{n \rightarrow \infty} \{\bar{\theta} \in B^{(0)}\}\right) = P\left(\liminf_{n \rightarrow \infty} \{\bar{D}(\bar{\theta}, \theta_0) < \delta\}\right) = 1.$$

This implies that $\bar{D}(\bar{\theta}, \theta_0) \rightarrow 0$ almost surely. Equivalently, $D(\bar{\Psi}, \Psi_0) \rightarrow 0$ and
 145 $\bar{\sigma} \rightarrow \sigma_0$ almost surely. This completes the proof of Theorem 1.

4. Examples

In this section, we illustrate the consistency results of Section 2 by showing that the four commonly used component density functions satisfy Conditions C1–C4. Consequently, by Corollary 1, the MLE of (Ψ, σ) is strongly consistent
 150 if $f(x; \mu, \sigma)$ is one of these four functions. As preparation, we present a sufficient condition for verifying Condition C2.

Proposition 1. *If $\int_{\mathbb{R}} \{\log f(x; \mu, 1)\} f(x; 0, 1) dx > -\infty$ for any given μ and Condition C3 is satisfied, then Condition C2 is satisfied.*

Proof. Let

$$\Psi_0(\mu) = \sum_{j=1}^{m_0} \alpha_{0j} I(\mu_{0j} \leq \mu).$$

Without loss of generality, we assume $\sigma_0 = 1$. Otherwise, we can take the
 155 transformation $t = x/\sigma_0$.

Note that

$$\begin{aligned}
& \int_{\mathbb{R}} |\log\{g(x; \Psi_0, \sigma_0)\}| g(x; \Psi_0, \sigma_0) dx \\
&= \int_{\mathbb{R}} \left| \log \left\{ \sum_{j=1}^{m_0} \alpha_{0j} f(x; \mu_{0j}, 1) \right\} \right| \sum_{j=1}^{m_0} \alpha_{0j} f(x; \mu_{0j}, 1) dx \\
&= \int_{\mathbb{R}} \left| \log \left\{ \sum_{j=1}^{m_0} \alpha_{0j} \frac{f(x; \mu_{0j}, 1)}{v_0} \right\} + \log v_0 \right| \sum_{j=1}^{m_0} \alpha_{0j} f(x; \mu_{0j}, 1) dx \\
&\leq |\log v_0| + \int_{\mathbb{R}} \left| \log \left\{ \sum_{j=1}^{m_0} \alpha_{0j} \frac{f(x; \mu_{0j}, 1)}{v_0} \right\} \right| \sum_{j=1}^{m_0} \alpha_{0j} f(x; \mu_{0j}, 1) dx \\
&= |\log v_0| - \int_{\mathbb{R}} \log \left\{ \sum_{j=1}^{m_0} \alpha_{0j} \frac{f(x; \mu_{0j}, 1)}{v_0} \right\} \sum_{j=1}^{m_0} \alpha_{0j} f(x; \mu_{0j}, 1) dx,
\end{aligned}$$

where in the last step we have used the fact that $f(x; 0, 1) \leq v_0$ in Condition C3. Hence, to verify Condition C2, it suffices to show that

$$\int_{\mathbb{R}} \log \left\{ \sum_{j=1}^{m_0} \alpha_{0j} f(x; \mu_{0j}, 1) \right\} \sum_{j=1}^{m_0} \alpha_{0j} f(x; \mu_{0j}, 1) dx > -\infty. \quad (18)$$

By Jensen's inequality,

$$\begin{aligned}
& \int_{\mathbb{R}} \log \left\{ \sum_{j=1}^{m_0} \alpha_{0j} f(x; \mu_{0j}, 1) \right\} \sum_{j=1}^{m_0} \alpha_{0j} f(x; \mu_{0j}, 1) dx \\
&\geq \int_{\mathbb{R}} \left\{ \sum_{j=1}^{m_0} \alpha_{0j} \log f(x; \mu_{0j}, 1) \right\} \sum_{j=1}^{m_0} \alpha_{0j} f(x; \mu_{0j}, 1) dx \\
&= \sum_{j=1}^{m_0} \sum_{h=1}^{m_0} \alpha_{0j} \alpha_{0h} \int_{\mathbb{R}} \{\log f(x; \mu_{0j}, 1)\} f(x; \mu_{0h}, 1) dx \\
&= \sum_{j=1}^{m_0} \sum_{h=1}^{m_0} \alpha_{0j} \alpha_{0h} \int_{\mathbb{R}} \{\log f(t; \mu_{0j} - \mu_{0h}, 1)\} f(t; 0, 1) dt.
\end{aligned}$$

By the condition $\int_{\mathbb{R}} \{\log f(x; \mu, 1)\} f(x; 0, 1) dx > -\infty$ for any given μ , we have (18). This completes the proof. \square

160 Note that it is easy to verify that the four commonly used component density functions all satisfy Condition C4. We now verify that they all satisfy Conditions C1–C3.

Example 1 (Normal distribution). *Let $f(x; 0, 1) = (2\pi)^{-1/2} \exp(-x^2/2)$, the probability density function of the standard normal distribution.*

165 (a) The identifiability of Model (1) with the normal component density function follows from Teicher (1963) and Yakowitz & Spragins (1968). Hence, Condition C1 is satisfied.

(b) Condition C3 is satisfied if we choose $v_0 = v_1 = (2\pi)^{-1/2}$ and $\beta = 2$.

(c) For any given μ ,

$$\int_{\mathbb{R}} \{\log f(x; \mu, 1)\} f(x; 0, 1) dx = -0.5 \log(2\pi) - 0.5(\mu^2 + 1) > -\infty.$$

Thus, Condition C2 is also satisfied by Proposition 1.

170 **Example 2** (Logistic distribution). *Let $f(x; 0, 1) = e^x/(1+e^x)^2$, the probability density function of the standard logistic distribution.*

(a) Following Theorem 2.1 of Holzmann et al. (2004), Model (1) with a logistic component density function is identifiable. Hence, Condition C1 is satisfied.

175 (b) It can be verified that Condition C3 is satisfied with $v_0 = v_1 = 1$ and $\beta = 2$.

(c) Since $\log(1 + e^x) \leq \log 2 + |x| \leq \log 2 + 0.5(1 + x^2)$ for any x , it follows that

$$\begin{aligned} \int_{\mathbb{R}} \{\log f(x; \mu, 1)\} f(x; 0, 1) dx &= -\mu - 2 \int_{\mathbb{R}} \{\log(1 + e^{x-\mu})\} f(x; 0, 1) dx \\ &\geq -\mu - 2 \log 2 - \int_{\mathbb{R}} \{1 + (x - \mu)^2\} f(x; 0, 1) dx \\ &= -\mu - 2 \log 2 - (1 + \mu^2 + \pi^2/3) > -\infty. \end{aligned}$$

Thus, Condition C2 is also satisfied by Proposition 1.

180 **Example 3** (Extreme-value distribution). *Let $f(x; 0, 1) = \exp\{-x - \exp(-x)\}$, the probability density function of the standard extreme value type I distribution or the Gumbel distribution.*

- (a) The identifiability of Model (1) with the component density function being the probability density function of the Gumbel distribution follows from
185 Ahmad et al. (2010). Hence, Condition C1 is satisfied.
- (b) It can be verified that Condition C3 is satisfied with $v_0 = v_1 = 1$ and $\beta = 2$.
- (c) Note that

$$\begin{aligned} \int_{\mathbb{R}} \{\log f(x; \mu, 1)\} f(x; 0, 1) dx &= \int_{\mathbb{R}} \{-(x - \mu) - e^{-(x-\mu)}\} \exp(-x - e^{-x}) dx \\ &= \mu - \gamma - e^\mu > -\infty. \end{aligned}$$

Here γ is the Euler–Mascheroni constant. Hence, Condition C2 is satisfied
190 by Proposition 1.

Example 4 (Student’s t distribution). *Let $f(x; 0, 1) = (1 + x^2/\nu)^{-(\nu+1)/2} C_\nu$, the probability density function of Student’s t distribution with ν degrees of freedom. Here ν is a given positive integer, $C_\nu = \Gamma\{(\nu + 1)/2\} \{\sqrt{\nu\pi} \Gamma(\nu/2)\}^{-1}$, and $\Gamma(\cdot)$ is the Gamma function.*

- 195 (a) The identifiability of Model (1) with the component density function being the probability density function of the t -distribution follows from Holzmann et al. (2006). Hence, Condition C1 is satisfied.
- (b) It can be verified that Condition C3 is satisfied with $v_0 = 1, v_1 = \nu$, and $\beta = 2$.
- 200 (c) Note that

$$\begin{aligned} &\int_{\mathbb{R}} \{\log f(x; \mu, 1)\} f(x; 0, 1) dx \\ &= \log C_\nu - \frac{\nu + 1}{2} C_\nu \int_{\mathbb{R}} \log \left\{ 1 + \frac{(x - \mu)^2}{\nu} \right\} (1 + x^2/\nu)^{-(\nu+1)/2} dx. \end{aligned}$$

By the comparison test for improper integrals,

$$0 \leq \int_{\mathbb{R}} \log \left\{ 1 + \frac{(x - \mu)^2}{\nu} \right\} (1 + x^2/\nu)^{-(\nu+1)/2} dx < \infty$$

for any given μ . Hence, $\int_{\mathbb{R}} \log f(x; \mu, 1) f(x; 0, 1) dx > -\infty$ and Condition C2 is satisfied by Proposition 1.

5. Extension to multivariate case

In this section, we extend the results in Corollary 1 to finite mixtures of multivariate elliptical distributions, a special class of finite mixtures of multivariate location-scale distributions. The identifiability of this special class of models has been well studied in Holzmann et al. (2006).

Suppose we have *i.i.d.* p -dimensional random vectors $\mathbf{X}_1, \dots, \mathbf{X}_n$ from the following finite mixture model:

$$g(\mathbf{x}; \Psi, \Sigma) = \sum_{j=1}^m \alpha_j f(\mathbf{x}; \mu_j, \Sigma) = \int_{\mathbb{R}^p} f(\mathbf{x}; \mu, \Sigma) d\Psi(\mu). \quad (19)$$

Here $f(\mathbf{x}; \mu, \Sigma)$, the component density function, is assumed to take the form

$$f(\mathbf{x}; \mu, \Sigma) = |\Sigma|^{-1/2} f_0((\mathbf{x} - \mu)^\tau \Sigma^{-1} (\mathbf{x} - \mu)),$$

where $\mathbf{x}, \mu \in \mathbb{R}^p$, Σ is a $p \times p$ positive definite matrix, and $f_0(x)$ is a density generator, i.e., a non-negative function on $[0, \infty)$ such that $f_0(\mathbf{x}^\tau \mathbf{x})$ is a probability density function. The MLE $(\hat{\Psi}, \hat{\Sigma})$ of (Ψ, Σ) is defined as for the univariate case. Next we extend the distance $D(\cdot, \cdot)$ from the univariate to the multivariate case:

$$D^*(\Psi_1, \Psi_2) = \int_{\mathbb{R}} |\Psi_1(\mu) - \Psi_2(\mu)| \exp(-|\mu|) d\mu. \quad (20)$$

Here for $\mu = (\mu_1, \dots, \mu_p)^\tau$, $|\mu|$ is interpreted as $\sum_{l=1}^p |\mu_l|$. Similarly to the proof for $D(\cdot, \cdot)$ in the Appendix, we can verify that $D^*(\cdot, \cdot)$ is a distance.

The consistency of $(\hat{\Psi}, \hat{\Sigma})$ relies on the following regularity conditions:

C1*. The finite mixture model in (19) is identifiable.

C2*. $\int_{\mathbb{R}^p} |\log\{g(\mathbf{x}; \Psi_0, \Sigma_0)\}|g(\mathbf{x}; \Psi_0, \Sigma_0)d\mathbf{x} < \infty$, where (Ψ_0, Σ_0) is the true value of (Ψ, Σ) .

C3*. There exist positive constants v_0, v_1 , and β with $\beta > p$ such that for all $x \geq 0$

$$f_0(x) \leq \min \left\{ v_0, v_1 x^{-\beta/2} \right\}.$$

C4*. For $x \geq 0$, $f_0(x)$ is continuous in x .

215 Under the regularity conditions C1*–C4*, we have the strong consistency of $(\hat{\Psi}, \hat{\Sigma})$ in the following theorem.

Theorem 2. *Assume Conditions C1*–C4* and that the true density is $g(x; \Psi_0, \Sigma_0)$. Then $\hat{\Psi} \rightarrow \Psi_0$ with respect to the metric $D^*(\cdot, \cdot)$ in (20) and $\hat{\Sigma} \rightarrow \Sigma_0$ almost surely as $n \rightarrow \infty$.*

220 One of the key steps of the proof of Theorem 2 is to establish a similar result to Lemma 1 for the multivariate case. This result can be obtained by combining the proof of Corollary 3 in Alexandrovich (2014) with that of Lemma 1. See the Appendix for a proof of Theorem 2.

As an illustration, we consider finite mixtures of multivariate normal distributions with a common and unknown variance-covariance matrix Σ .

225 **Example 5.** *(Multivariate normal distribution). Let $f_0(x) = (2\pi)^{-1/2} \exp(-x/2)$, the density generator for the multivariate normal density function. Clearly, $f_0(x)$ satisfies Condition C4*.*

(a) *The identifiability of finite mixtures of multivariate normal distributions is covered by Holzmann et al. (2006). Hence, Condition C1* is satisfied.*

(b) *It can be verified that Condition C3* is satisfied with $v_0 = (2\pi)^{-p/2}$,*

$$v_1 = (2\pi)^{-p/2}(p+1)^{(p+1)/2},$$

and $\beta = p + 1$.

(c) Following the proof of Proposition 1, to verify Condition C2*, it suffices to show that

$$\int_{\mathbb{R}^p} \{\log f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}_0)\} f(\mathbf{x}; \mathbf{0}, \boldsymbol{\Sigma}_0) d\mathbf{x} > -\infty$$

for all $\boldsymbol{\mu}$. Note that

$$\begin{aligned} & \int_{\mathbb{R}^p} \{\log f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}_0)\} f(\mathbf{x}; \mathbf{0}, \boldsymbol{\Sigma}_0) d\mathbf{x} \\ &= -0.5p \log(2\pi) - 0.5 \log |\boldsymbol{\Sigma}_0| - 2p - \boldsymbol{\mu}^\top \boldsymbol{\Sigma}_0^{-1} \boldsymbol{\mu} \\ &> -\infty. \end{aligned}$$

Hence, Condition C2* is satisfied.

Since finite mixtures of multivariate normal distributions with a common and
 235 unknown variance-covariance matrix satisfy regularity conditions C1*-C4*, the
 MLE $(\hat{\Psi}, \hat{\boldsymbol{\Sigma}})$ under this model is strongly consistent.

6. Summary and discussion

In this paper, we establish the strong consistency of the cumulative distribu-
 tion function of the mixing distribution and the structural parameter in finite
 240 mixtures of location-scale distributions with a structural parameter in both the
 univariate and multivariate cases. We further demonstrate that some commonly
 used finite mixtures of location-scale distributions satisfy the regularity condi-
 tions.

For the model setups in (1) and (19), Ψ is assumed to have finite support,
 245 and the scale parameter σ or $\boldsymbol{\Sigma}$ is assumed to be the same in all the component
 density functions. Two considerations underlie these assumptions. First, if
 Ψ is fully nonparametric, then the mixture model may not be identifiable. For
 example, the normal mixture model is not identifiable if Ψ is fully nonparametric
 (Chen, 2017). Assuming that Ψ has finite support ensures that finite mixtures
 250 of some commonly used location-scale distributions are identifiable. Second, if
 the σ or $\boldsymbol{\Sigma}$ can vary in different component density functions, then the log-
 likelihood is unbounded (Chen et al., 2008; Chen & Tan, 2009). Hence, the

usual MLEs of the unknown parameters are not well defined. We may need to consider other estimation methods such as the penalized MLE considered in
255 Chen et al. (2008), Chen & Tan (2009), and Alexandrovich (2014). We leave the consistency properties of such estimators to future research.

We next discuss the applicability of Corollary 1 and Theorem 2. The results in Corollary 1 and Theorem 2 are applicable only to the MLE, i.e., the global maximum point of the log-likelihood function. Commonly used algorithms such
260 as the EM-algorithm may lead to a local maximum point of the log-likelihood, which is not guaranteed by our result to be consistent. In practice, we suggest trying multiple initial values to increase the chance of locating the global maximum point.

A. Appendix

265 A.1. Proof that $D(\cdot, \cdot)$ is a distance

To show that $D(\cdot, \cdot)$ is a distance., it suffices to show

- (a) $D(\Psi_1, \Psi_2) \geq 0$;
- (b) $D(\Psi_1, \Psi_3) \leq D(\Psi_1, \Psi_2) + D(\Psi_2, \Psi_3)$ for any $\Psi_1, \Psi_2, \Psi_3 \in \Psi_m$;
- (c) $D(\Psi_1, \Psi_2) = D(\Psi_2, \Psi_1)$;
- 270 (d) $D(\Psi_1, \Psi_2) = 0$ if and only if $\Psi_1(\mu) = \Psi_2(\mu)$ for all $\mu \in \mathbb{R}$.

Based on the definition of $D(\cdot, \cdot)$ in (2), it is easy to verify that (a)–(c) are satisfied. Next we discuss (d). If $\Psi_1(\mu) = \Psi_2(\mu)$ for all $\mu \in \mathbb{R}$, then obviously

$$D(\Psi_1, \Psi_2) = \int_{\mathbb{R}} |\Psi_1(\mu) - \Psi_2(\mu)| \exp(-|\mu|) d\mu = 0.$$

We now argue that if $D(\Psi_1, \Psi_2) = 0$, we must have $\Psi_1(\mu) = \Psi_2(\mu)$ for all $\mu \in \mathbb{R}$. We denote the distinct values of the supports of $\Psi_1(\mu)$ and $\Psi_2(\mu)$ as $\{t_1 < t_2 < \dots < t_{m^*}\}$ and define $t_{m^*+1} = +\infty$. We write $\Psi_1(\mu)$ and $\Psi_2(\mu)$ as

$$\Psi_1(\mu) = \sum_{j=1}^{m^*} \alpha_{j1} I(t_j \leq \mu) \text{ and } \Psi_2(\mu) = \sum_{j=1}^{m^*} \alpha_{j2} I(t_j \leq \mu).$$

If $\Psi_1(\mu) \neq \Psi_2(\mu)$ for some μ , then we can find a j_0 such that

$$\alpha_{j_1} = \alpha_{j_2} \text{ for } j \leq j_0 - 1 \quad \text{and} \quad \alpha_{j_0 1} \neq \alpha_{j_0 2}.$$

This implies that

$$\begin{aligned} D(\Psi_1, \Psi_2) &\geq \int_{\mu \in [t_{j_0}, t_{j_0+1})} |\Psi_1(\mu) - \Psi_2(\mu)| \exp(-|\mu|) d\mu \\ &\geq |\alpha_{j_0 1} - \alpha_{j_0 2}| \int_{t_{j_0}}^{t_{j_0+1}} \exp(-|\mu|) d\mu > 0. \end{aligned}$$

Hence, if $D(\Psi_1, \Psi_2) = 0$, we must have $\Psi_1(\mu) = \Psi_2(\mu)$ for all $\mu \in \mathbb{R}$. This completes the proof.

A.2. Proof of the compactness of $\bar{\Psi}_m$

275 We prove that $\bar{\Psi}_m$ is compact with respect to the distance $D(\cdot, \cdot)$ according to the following equivalent definition of compactness of a metric space. A metric space is compact if and only if every sequence in this space has a convergent subsequence whose limit is also in this space.

In the proof of the compactness of $\bar{\Psi}_m$, we need the following results from
280 real analysis.

Result (i). If $\{c_n, n = 1, 2, \dots\}$ is a bounded real sequence, then $\{c_n, n = 1, 2, \dots\}$ has a convergent subsequence $\{c_{n_k}, k = 1, 2, \dots\}$, and its limit is finite.

285 *Rresult (ii).* If $\{c_n, n = 1, 2, \dots\}$ is an unbounded real sequence, then $\{c_n, n = 1, 2, \dots\}$ has a subsequence $\{c_{n_k}, k = 1, 2, \dots\}$ that diverges to ∞ or $-\infty$.

Result (i) is just the classic Bolzano–Weierstrass theorem from real analysis. Hence, we give only the proof for Result (ii). Suppose $\{c_n, n = 1, 2, \dots\}$ has no upper bound. Then for any given positive integer k , there exists n_k such that
290 $c_{n_k} > k$. Then $\lim_{k \rightarrow \infty} c_{n_k} = \infty$. Hence, Result (ii) holds.

Next we return to the proof of the compactness of $\bar{\Psi}_m$. Let $\{\Psi_L, L = 1, 2, \dots\}$ be a sequence in $\bar{\Psi}_m$, where

$$\Psi_L(\mu) = \gamma_L + \rho_L \sum_{j=1}^m \alpha_{jL} I(\mu_{jL} \leq \mu)$$

with $\gamma_L \geq 0$, $\rho_L \geq 0$, $0 \leq \gamma_L + \rho_L \leq 1$, $\alpha_{jL} \geq 0$, $\sum_{j=1}^m \alpha_{jL} = 1$, and $-\infty < \mu_{1L} \leq \mu_{2L} \leq \dots \leq \mu_{mL} < \infty$.

Using Results (i)–(ii) and G. Cantor’s “diagonal method,” we can find a subsequence $\{L_k, k = 1, 2, \dots\}$ such that

$$\gamma_0 = \lim_{k \rightarrow \infty} \gamma_{L_k}, \quad \rho_0 = \lim_{k \rightarrow \infty} \rho_{L_k}, \quad \alpha_{j0} = \lim_{k \rightarrow \infty} \alpha_{jL_k}, \quad \mu_{j0} = \lim_{k \rightarrow \infty} \mu_{jL_k},$$

where $\gamma_0 \geq 0$, $\rho_0 \geq 0$, $0 \leq \gamma_0 + \rho_0 \leq 1$, $\alpha_{j0} \geq 0$, $\sum_{j=1}^m \alpha_{j0} = 1$, and $-\infty \leq \mu_{10} \leq \mu_{20} \leq \dots \leq \mu_{m0} \leq \infty$. We define

$$\Psi_0(\mu) = \gamma_0 + \rho_0 \sum_{j=1}^m \alpha_{j0} I(\mu_{j0} \leq \mu).$$

Further, we define two index sets S_1 and S_2 as $S_1 = \{j : \mu_{j0} = -\infty\}$ and $S_2 = \{j : \mu_{j0} = \infty\}$, respectively. Let $S = S_1 \cup S_2$. Then $\Psi_0(\mu)$ can be rewritten as

$$\Psi_0(\mu) = \gamma'_0 + \rho'_0 \sum_{j \notin S} \alpha'_{j0} I(\mu_{j0} \leq \mu),$$

where $\gamma'_0 = \gamma_0 + \rho_0 \sum_{j \in S_1} \alpha_{j0}$, $\rho'_0 = \rho_0 \sum_{j \notin S} \alpha_{j0}$, and $\alpha'_{j0} = \alpha_{j0} / \sum_{l \notin S} \alpha_{l0}$ for $j \notin S$. This implies that $\Psi_0 \in \bar{\Psi}_m$.

Let $\Psi_{L_k} = \gamma_{L_k} + \rho_{L_k} \sum_{j=1}^m \alpha_{jL_k} I(\mu_{jL_k} \leq \mu)$. To finish the proof, we need to argue that as $k \rightarrow \infty$,

$$D(\Psi_{L_k}, \Psi_0) \rightarrow 0.$$

Note that

$$\begin{aligned} D(\Psi_{L_k}, \Psi_0) &= \int_{\mathbb{R}} |\Psi_{L_k}(\mu) - \Psi_0(\mu)| \exp(-|\mu|) d\mu \\ &= \int_{\mathbb{R}} \left| \gamma_{L_k} - \gamma_0 + \sum_{j=1}^m \{ \rho_{L_k} \alpha_{jL_k} I(\mu_{jL_k} \leq \mu) - \rho_0 \alpha_{j0} I(\mu_{j0} \leq \mu) \} \right| \exp(-|\mu|) d\mu \\ &\leq \int_{\mathbb{R}} |\gamma_{L_k} - \gamma_0| \exp(-|\mu|) d\mu \end{aligned} \tag{21}$$

$$+ \sum_{j=1}^m \int_{\mathbb{R}} |\rho_{L_k} \alpha_{jL_k} I(\mu_{jL_k} \leq \mu) - \rho_0 \alpha_{j0} I(\mu_{j0} \leq \mu)| \exp(-|\mu|) d\mu. \tag{22}$$

Since $\gamma_0 = \lim_{k \rightarrow \infty} \gamma_{L_k}$, we have that as $k \rightarrow \infty$,

$$\int_{\mathbb{R}} |\gamma_{L_k} - \gamma_0| \exp(-|\mu|) d\mu = 2 |\gamma_{L_k} - \gamma_0| \rightarrow 0. \tag{23}$$

By the triangular inequality and the facts that $0 \leq \rho_0 \leq 1$ and $0 \leq \alpha_{j0} \leq 1$, we have

$$\begin{aligned} & \int_{\mathbb{R}} |\rho_{L_k} \alpha_{jL_k} I(\mu_{jL_k} \leq \mu) - \rho_0 \alpha_{j0} I(\mu_{j0} \leq \mu)| \exp(-|\mu|) d\mu \\ & \leq \int_{\mathbb{R}} |\rho_{L_k} \alpha_{jL_k} - \rho_0 \alpha_{j0}| \exp(-|\mu|) d\mu \end{aligned} \quad (24)$$

$$+ \int_{\mathbb{R}} |I(\mu_{jL_k} \leq \mu) - I(\mu_{j0} \leq \mu)| \exp(-|\mu|) d\mu \quad (25)$$

Similarly to (23), as $k \rightarrow \infty$ we have

$$\int_{\mathbb{R}} |\rho_{L_k} \alpha_{jL_k} - \rho_0 \alpha_{j0}| \exp(-|\mu|) d\mu \rightarrow 0. \quad (26)$$

For (25), we have that if μ_{j0} is finite,

$$\int_{\mathbb{R}} |I(\mu_{jL_k} \leq \mu) - I(\mu_{j0} \leq \mu)| \exp(-|\mu|) d\mu \leq |\mu_{jL_k} - \mu_{j0}| \rightarrow 0 \quad (27)$$

as $k \rightarrow \infty$, and if $\mu_{j0} = \infty$ or $-\infty$,

$$\int_{\mathbb{R}} |I(\mu_{jL_k} \leq \mu) - I(\mu_{j0} \leq \mu)| \exp(-|\mu|) d\mu = \exp(-|\mu_{jL_k}|) \rightarrow 0 \quad (28)$$

as $k \rightarrow \infty$.

Combining (24)–(28) leads to

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}} |\rho_{L_k} \alpha_{jL_k} I(\mu_{jL_k} \leq \mu) - \rho_0 \alpha_{j0} I(\mu_{j0} \leq \mu)| \exp(-|\mu|) d\mu = 0,$$

which, together with (21)–(23), implies that

$$\lim_{k \rightarrow \infty} D(\Psi_{L_k}, \Psi_0) = 0.$$

This completes the proof.

A.3. Proof of Theorem 2

Our proof of Theorem 2 is similar to that of Theorem 1. Hence, we simply outline the key steps.

In the first step, we establish a result similar to Lemma 1. Let $a^* = (\beta + p)/(2\beta)$ and $b^* = 2(\beta - p + 2)/(\beta - p)$, where $\beta > p$ is given in Condition C3*. Hence, $0 < a^* < 1$ and $b^* > 1$. We choose ϵ_0^* such that

$$\frac{v_0 \pi^{p/2}}{|\Sigma_0|^{1/2} \Gamma(p/2 + 1)} (\epsilon_0^*)^{(1-a^*)/2} = 1/(2mb^*).$$

For the matrix Σ , “ $\Sigma > 0$ ” means that Σ is a positive definite matrix.

Lemma 3. Suppose $\{\mathbf{X}_1, \dots, \mathbf{X}_n\}$ is an *i.i.d.* sample from $g(\mathbf{x}; \Psi_0, \Sigma_0)$. Further, assume Condition C3*. Let

$$E^* = \liminf_{n \rightarrow \infty} \left\{ \sup_{\{\Sigma > 0, |\Sigma| \leq \epsilon_0^*\}} \sup_{\mu \in \mathbb{R}^p} \sum_{i=1}^n I \left\{ (\mathbf{x}_i - \mu)^\tau \Sigma^{-1} (\mathbf{x}_i - \mu) \leq |\Sigma|^{-a^*/p} \right\} \leq n/(mb^*) \right\}.$$

Then $P(E^*) = 1$.

Proof. Following the proof of Corollary 3 in Alexandrovich (2014), we can show that

$$P \left(\liminf_{n \rightarrow \infty} \left\{ \sup_{\{\Sigma > 0, |\Sigma| \leq \epsilon_0^*\}} \sup_{\mu \in \mathbb{R}^p} \sum_{i=1}^n I \left\{ (\mathbf{x}_i - \mu)^\tau \Sigma^{-1} (\mathbf{x}_i - \mu) \leq |\Sigma|^{-a^*/p} \right\} \leq a_n \right\} \right) = 1,$$

where

$$a_n = \frac{3}{4} \sqrt{n \log \log n} + \frac{nv_0 \pi^{p/2}}{|\Sigma_0|^{1/2} \Gamma(p/2 + 1)} (\epsilon_0^*)^{(1-a^*)/2} = \frac{3}{4} \sqrt{n \log \log n} + \frac{n}{2mb^*}.$$

When n is large enough, $a_n \leq n/(mb^*)$. Hence, $P(E^*) = 1$. This completes the proof. \square

In the second step, we establish a result similar to Lemma 2. We first define some notation. Based on *i.i.d.* p -dimensional random vectors $\mathbf{X}_1, \dots, \mathbf{X}_n$ from $g(\mathbf{x}; \Psi, \Sigma)$, the log-likelihood of (Ψ, Σ) is

$$\ell_n(\Psi, \sigma) = \sum_{i=1}^n \log\{g(X_i; \Psi, \Sigma)\}.$$

Let $K_0^* = \int_{\mathbb{R}^p} \log\{g(\mathbf{x}; \Psi_0, \Sigma_0)\} g(\mathbf{x}; \Psi_0, \Sigma_0) d\mathbf{x}$. Condition C2* ensures that $|K_0^*| < \infty$. Further, define $\Delta^* = \{v_0 / \exp(K_0^* - 1)\}^2$. We choose a positive number ϵ^* that satisfies the following conditions:

D1*. $\epsilon^* \leq \epsilon_0^*$;

D2*. $\epsilon^* \leq (v_1/v_0)^{-4/(\beta-p+2)}$;

D3*. $(1/b^*) \log v_0 + (1 - 1/b^*) \log v_1 + \frac{1}{8}(\beta - p) \log \epsilon^* \leq K_0^* - 1$.

Clearly, an ϵ^* satisfying the above conditions exists, since $\beta > p$ as assumed in Condition C3*.

Lemma 4. Suppose $\{X_1, \dots, X_n\}$ is an i.i.d. sample from $g(\mathbf{x}; \Psi_0, \Sigma_0)$. Further, assume Conditions C1*-C4* are satisfied. Then we have

$$P \left(\lim_{n \rightarrow \infty} \left\{ \sup_{\Psi \in \Psi_m, \Sigma > 0, |\Sigma| \geq \Delta^*} \ell_n(\Psi, \Sigma) - \ell_n(\Psi_0, \Sigma_0) \right\} = -\infty \right) = 1 \quad (29)$$

and

$$P \left(\lim_{n \rightarrow \infty} \left\{ \sup_{\Psi \in \Psi_m, \Sigma > 0, |\Sigma| \leq \epsilon^*} \ell_n(\Psi, \Sigma) - \ell_n(\Psi_0, \Sigma_0) \right\} = -\infty \right) = 1. \quad (30)$$

Proof. We start with (29). By Condition C3*, we have

$$\begin{aligned} \ell_n(\Psi, \Sigma) &= \sum_{i=1}^n \log \left\{ \sum_{j=1}^m \frac{\alpha_j}{|\Sigma|^{1/2}} f_0((\mathbf{X}_i - \boldsymbol{\mu}_j)^\tau \Sigma^{-1} (\mathbf{X}_i - \boldsymbol{\mu}_j)) \right\} \\ &\leq n(\log v_0 - 0.5 \log |\Sigma|), \end{aligned} \quad (31)$$

where v_0 is given in Condition C3*.

Hence, with $\Delta^* = \{v_0 / \exp(K_0^* - 1)\}^2$, we have

$$\sup_{\Psi \in \Psi_m, \Sigma > 0, |\Sigma| \geq \Delta^*} \ell_n(\Psi, \Sigma) - \ell_n(\Psi_0, \Sigma_0) \leq n(K_0^* - 1) - \ell_n(\Psi_0, \Sigma_0).$$

By the strong law of large numbers and the definition of K_0^* , we have (29).

We next consider (30). Let

$$A^* = \left\{ i : \min_{1 \leq j \leq m} (\mathbf{X}_i - \boldsymbol{\mu}_j)^\tau \Sigma^{-1} (\mathbf{X}_i - \boldsymbol{\mu}_j) \leq |\Sigma|^{-a^*/p} \right\}.$$

315 Similarly to (31), it can be shown that

$$\ell_n(\Psi, \Sigma; A^*) \leq n(A^*)(\log v_0 - 0.5 \log |\Sigma|). \quad (32)$$

By Condition C3*, we have

$$\begin{aligned} \ell_n(\Psi, \Sigma; (A^*)^c) &\leq \sum_{i \in (A^*)^c} \log \left[\sum_{j=1}^m \frac{\alpha_j}{|\Sigma|^{1/2}} v_1 \{(\mathbf{X}_i - \boldsymbol{\mu}_j)^\tau \Sigma^{-1} (\mathbf{X}_i - \boldsymbol{\mu}_j)\}^{-\beta/2} \right] \\ &\leq n((A^*)^c) \{ \log v_1 + 0.5(a^*\beta/p - 1) \log |\Sigma| \} \\ &= n((A^*)^c) \{ \log v_1 + 0.25(\beta - p) \log |\Sigma| \}, \end{aligned} \quad (33)$$

where in the last step we have used the fact that $a^*\beta = (\beta + p)/2$.

Combining (32) and (33) gives

$$\begin{aligned}
\ell_n(\Psi, \Sigma) &= \ell_n(\Psi, \Sigma; A^*) + \ell_n(\Psi, \Sigma; (A^*)^c) \\
&\leq n(\log v_0 - 0.5 \log |\Sigma|) + n((A^*)^c) \{\log v_1 - \log v_0 + 0.25(\beta - p + 2) \log |\Sigma|\} \\
&= n(\log v_0 - 0.5 \log |\Sigma|) + n(1 - 1/b^*) \{\log v_1 - \log v_0 + 0.25(\beta - p + 2) \log |\Sigma|\} \\
&\quad + \{n((A^*)^c) - n(1 - 1/b^*)\} \{\log v_1 - \log v_0 + 0.25(\beta - p + 2) \log |\Sigma|\} \\
&= n \left\{ (1/b^*) \log v_0 + (1 - 1/b^*) \log v_1 + \frac{1}{8}(\beta - p) \log |\Sigma| \right\} \\
&\quad + \{n/b^* - n(A^*)\} \{\log v_1 - \log v_0 + 0.25(\beta - p + 2) \log |\Sigma|\},
\end{aligned}$$

where in the last step we have used the fact that $b^* = 2(\beta - p + 2)/(\beta - p)$.

Then, for Σ satisfying $|\Sigma| \leq \epsilon^*$, since ϵ^* satisfies D3*, we have

$$\begin{aligned}
&\ell_n(\Psi, \Sigma) - \ell_n(\Psi_0, \Sigma_0) \\
&\leq n(K_0^* - 1) - \ell_n(\Psi_0, \Sigma_0) \tag{34}
\end{aligned}$$

$$+ \{n/b^* - n(A^*)\} \{\log v_1 - \log v_0 + 0.25(\beta - p + 2) \log |\Sigma|\}. \tag{35}$$

By the strong law of large numbers and the definition of K_0^* , we have

$$P \left(\lim_{n \rightarrow \infty} \{n(K_0^* - 1) - \ell_n(\Psi_0, \Sigma_0)\} = -\infty \right) = 1. \tag{36}$$

Again for Σ satisfying $|\Sigma| \leq \epsilon^*$, since ϵ^* satisfies D2*, we have

$$\begin{aligned}
\log v_1 - \log v_0 + 0.25(\beta - p + 2) \log |\Sigma| &\leq \log v_1 - \log v_0 + 0.25(\beta - p + 2) \log \epsilon^* \leq 0, \\
\end{aligned} \tag{37}$$

which together with Lemma 3 implies that, for large enough n ,

$$\{n/b^* - n(A^*)\} \{\log v_1 - \log v_0 + 0.25(\beta - p + 2) \log |\Sigma|\} \leq 0,$$

almost surely. Combining (34)–(37) leads to (30). This completes the proof. \square

Lemma 4 implies that asymptotically we can confine $(\hat{\Psi}, \hat{\Sigma})$ to Θ , where
³²⁰ $\Theta = \Psi_m \times \{\Sigma : \Sigma > 0, |\Sigma| \in [\epsilon^*, \Delta^*]\}$. Note that Θ is completely regular (Chen & Tan, 2009), so the techniques in Wald (1949) and Kiefer & Wolfowitz (1956) can be applied to establish the strong consistency of $(\hat{\Psi}, \hat{\Sigma})$. This completes the proof of Theorem 2.

Acknowledgements

325 We are grateful to the editor, the associate editor, and three anonymous
referees for their insightful and constructive comments which led to an im-
proved presentation of this article. The work of Dr. Liu and Professor Pu is
supported by grants from the National Natural Science Foundation of China
(11771144, 11371142, 11771145, 11471119), the outstanding doctoral disserta-
330 tion cultivation plan of action (PY2015049), the Program of Shanghai Subject
Chief Scientist (14XD1401600), and the 111 Project (B14019). Dr. Li's research
is supported in part by NSERC Grant RGPIN-2015-06592.

Reference

- Ahmad, K. E., Jaheen, Z. F., & Modhesh, A. A. (2010). Estimation of a dis-
335 criminant function based on small sample size from a mixture of two Gumbel
distributions. *Communications in Statistics: Simulation and Computation*,
39, 713–725.
- Alexandrovich, G. (2014). A note on the article ‘Inference for multivariate
normal mixtures’ by J. Chen and X. Tan. *Journal of Multivariate Analysis*,
340 *129*, 245–248.
- Bryant, P. G. (1991). Large-sample results for optimization-based clustering
methods. *Journal of Classification*, *8*, 31–44.
- Chen, J. (2017). Consistency of the MLE under mixture models. *Statistical
Science*, *32*, 47–63.
- 345 Chen, J., & Tan, X. (2009). Inference for multivariate normal mixtures. *Journal
of Multivariate Analysis*, *100*, 1367–1383.
- Chen, J., Tan, X., & Zhang, R. (2008). Inference for normal mixtures in mean
and variance. *Statistica Sinica*, *18*, 443–465.
- Holzmann, H., Munk, A., & Gneiting, T. (2006). Identifiability of finite mixtures
350 of elliptical distributions. *Scandinavian Journal of Statistics*, *33*, 753–763.

- Holzmann, H., Munk, A., & Stratmann, B. (2004). Identifiability of finite mixtures – with applications to circular distributions. *Sankhya*, *66*, 440–449.
- Kiefer, J., & Wolfowitz, J. (1956). Consistency of the maximum likelihood estimator in the presence of infinitely many incidental parameters. *Annals of Mathematical Statistics*, *27*, 887–906.
- Lawless, J. F. (2003). *Statistical Models and Methods for Lifetime Data*. (2nd ed.). Hoboken: John Wiley and Sons.
- McLachlan, G. J., & Peel, D. (2000). *Finite Mixture Models*. New York: Wiley.
- Naya, S., Cao, R., López De Ullibarri, I., Artiaga, R., Barbadillo, F., & García, A. (2006). Logistic mixture model versus Arrhenius for kinetic study of material degradation by dynamic thermogravimetric analysis. *Journal of Chemometrics*, *20*, 158–163.
- Redner, R. (1981). Note on the consistency of the maximum likelihood estimate for nonidentifiable distributions. *Annals of Statistics*, *9*, 225–228.
- Roeder, K. (1994). A graphical technique for determining the number of components in a mixture of normals. *Journal of the American Statistical Association*, *89*, 487–495.
- Teicher, H. (1963). Identifiability of finite mixtures. *Annals of Mathematical Statistics*, *34*, 1265–1269.
- Wald, A. (1949). Note on the consistency of the maximum likelihood estimate. *Annals of Mathematical Statistics*, *20*, 595–601.
- Yakowitz, S. J., & Spragins, J. D. (1968). On the identifiability of finite mixtures. *Annals of Mathematical Statistics*, *39*, 209–214.