Notes on occupation time fluctuations of binary branching particle systems

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Abstract
In this paper, we study the functional limits of occupation time fluctuations of a kind of site-dependent binary branching particle systems. Our result extends the Theorem 2.2 in Bojdecki et al (Stochastic Process. Appl. 116 (2006), P.1-P.18) and gets some new information on the behaviors of occupation time processes.

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1. Introduction
Consider a kind of site-dependent binary branching particle systems which is described as follows. Particles in $\mathbb{R}^d$ start off at time $t = 0$ from a Poisson random field with Lebesgue intensity measure $\lambda$, and they evolve independently. The space motion consists of a stochastic process $\vec{\xi} = \{\vec{\xi}(t), t \geq 0\} = \{(\xi_1(t), \xi_2(t), \ldots, \xi_d(t)), t > 0\}$, where for every $0 < k \leq d$, $\xi_k = \{\xi_k(t), t \geq 0\}$ is a symmetric $\alpha_k$-stable Lévy process ($0 < \alpha_k \leq 2$) and $\xi_1, \ldots, \xi_d$ are independent each other. In addition, they split at a rate $\gamma$ and the branching law at location $x$ has the following generating function
$$g(s, x) = s + \frac{\sigma(x)}{2}(1 - s)^2 = (1 - \sigma(x))s + \sigma(x)\frac{1 + s^2}{2}, \quad 0 \leq s \leq 1,$$
where $\sigma(\cdot) \in [0, 1]$ is a measurable function on $\mathbb{R}^d$. Intuitively, in this model, the particles’ abilities of splitting into new particles are affected by their sites. In more detail, regarding $\sigma = \{\sigma(x), x \in \mathbb{R}^d\}$ as a static random medium, this branching mechanism can be explained as that, facing the chance of splitting, the particle at site $x$ either refuse it with $1 - \sigma(x)$ probability or accept with $\sigma(x)$ probability. Once the particle accepts the chance, it obeys the common binary branching. In addition, the setting of the process $\vec{\xi}$ indicates that the particles may have different motions in different directions. While this assumption can not technically change the arguments in this paper, we believe that it will be helpful in some situations such as investigation on the problem whether/when/how the anisotropy caused by the micro particles affects the

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large-scaled macro phenomena. By the notation in Li [9], we refer to such a model as a \((d, \tilde{\alpha}, \sigma(x))\)-branching particle system.

Let \(N(s)\) denote the random counting measure of a \((d, \tilde{\alpha}, \sigma(x))\)-branching particle system at time \(s\), i.e. \(N(s)(A)\) is the number of particles in the set \(A \subset \mathbb{R}^d\) at time \(s\). We call the measure-valued process

\[
L(t) = \int_0^t N(s) ds, \quad t \geq 0,
\]

the occupation time and call the process

\[
X(t) = \int_0^t (N(s) - \mathbb{E}(N_s)) ds, \quad t \geq 0,
\]

the occupation time fluctuation, where \(\mathbb{E}(N(s))\) is the expectation functional understood as \(\mathbb{E}(N(s)), \phi = \mathbb{E}((N(s), \phi))\) for any \(\phi \in \mathcal{S}(\mathbb{R}^d)\), the space of smooth rapidly decreasing functions. Here and sometimes in the sequel, we write \(\langle \mu, f \rangle = \int fd\mu\) where \(\mu\) is a measure and \(f\) a measurable function.

Under the assumption of \(\tilde{\alpha} := \sum_{k=1}^d 1/\alpha_k \in (1, 2)\), the author [9] pointed out without proof that

**Theorem 1.1** If \(\int_{\mathbb{R}^d} \sigma(x) dx < \infty\), then \(X(n\cdot)/\sqrt{n}\) converges in the integral sense to a centered Gaussian process \(X(\cdot)\) with covariance function

\[
\text{Cov}(\langle X(r), \phi_1 \rangle, \langle X(t), \phi_2 \rangle) = (r \wedge t) \int_{\mathbb{R}^d} \left[ 2\phi_1(x)G\phi_2(x) + \gamma \sigma(x)G\phi_1(x)G\phi_2(x) \right] \text{d}x,
\]

for any \(r, t > 0\) and \(\phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^d)\), where \(G\) is the Green operator of the process \(\xi\).

(see Remark 2.1 (2) in Li [9]). As the first step of this paper, we will prove this result with more details. Note that the same conclusions hold in the cases of \(\tilde{\alpha} \geq 2\) (see Theorem 2.2 in Li [9]).

On the other hand, if \(N(s)\) is the random counting measure of a classical \((d, \alpha, 1)\)-branching particle system satisfying \(d/\alpha \in (1, 2)\), it is proved in Bojdecki et al [2, Theorem 2.2] that \(X(n\cdot)/n^{(3-d/\alpha)/2}\) converges weakly in the Skorokhod space \(C([0, 1], \mathcal{S}'(\mathbb{R}^d))\) (here \(\mathcal{S}'(\mathbb{R}^d)\) is the space of tempered distributions) to a centered Gaussian process \(X(\cdot)\) with covariance function

\[
\text{Cov}(\langle X(r), \phi_1 \rangle, \langle X(t), \phi_2 \rangle) = K\langle \lambda, \phi_1 \rangle \langle \lambda, \phi_2 \rangle \left( r^h + t^h - \frac{1}{2} (r + t)^h + |r - t|^h \right)
\]

for any \(r, t > 0\) and \(\phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^d)\), where \(K\) is a constant and \(h = 3 - d/\alpha\). It is not surprising that the results with \(\tilde{\alpha}\) instead of \(d/\alpha\) holds for \((d, \tilde{\alpha}, \sigma(x))\)-branching particle systems with \(\sigma(x) \equiv 1\) and \(\tilde{\alpha} \in (1, 2)\). Obviously, there is a big gap between the functional limits of the occupation time fluctuations related to the case of \(\int_{\mathbb{R}^d} \sigma(x) dx < \infty\) and those related to the case of \(\sigma(x) \equiv 1\). A natural and interesting question is what will happen when we fill the gap. Unfortunately, it seems to be hard to answer this question with a complete conclusion. As a compromise, in this short paper, we true to the problem what extent a similar result of (1.3) holds in.

We propose the following assumptions:
**Assumption (A):** There is a function \( L(t) > 0, t > 0, \) such that
\[
\lim_{t \to \infty} \frac{L(t)}{t^\delta} = 0,
\]
for any \( \delta > 0 \) and
\[
\lim_{t \to \infty} \sigma(t^{H(\bar{\alpha})} x)L(t) = \rho(x), \quad \text{a.s.}
\]
with respect to Lebesgue measure, where \( \rho(x) > 0 \) with positive measure. Furthermore, there exists a constant \( M > 0 \) such that for all large \( t \), there is a set \( B_t \subset \mathbb{R}^d \) satisfying
\[
\sigma(t^{H(\bar{\alpha})} x)L(t) \leq M,
\]
for all \( x \in \mathbb{R}^d \setminus B_t \) and as \( t \to \infty \)
\[
L(t)t^{2(\bar{\alpha}-1)}\lambda(B_t) \to 0.
\]

For convenience of reference, we call \( L(t) \) the speed of \( \sigma(x) \) converging to 0 according the curve \( t^{H(\bar{\alpha})} \) (t varies) and \( \rho(x) \) the density of \( \sigma(x) \) in the speed \( L(t) \). Note that the speed and the density functions essentially reflect the branching behavior of particles nearby infinity.

As a result, we get that

**Theorem 1.2** Assume (A) and \( 1 < \bar{\alpha} < 2 \). Then \( X(n \cdot) L(n)/n^{(3-\bar{\alpha})/2} \) converges in the integral sense to a centered Gaussian process \( X(\cdot) \) with covariance function
\[
\text{Cov}(\langle X(r), \phi_1 \rangle, \langle X(t), \phi_2 \rangle) = C(r,t)\langle \lambda, \phi_1 \rangle\langle \lambda, \phi_2 \rangle,
\]
where
\[
C(r,t) = \gamma \int_0^r ds \int_{\mathbb{R}^d} \rho(x)dx \int_0^{r-s} p_u(x)du \int_0^{t-s} p_v(x)dv,
\]
for all \( r,t \geq 0 \). Moreover, if there exists \( \eta_0 \in (0,1) \) such that
\[
\limsup_{t \to \infty} L(t)t^{2(\bar{\alpha}-1)+\eta_0}\lambda(B_t) < \infty,
\]
then \( X_n \) converges weakly to \( X \) in \( C([0,1], \mathcal{S}'(\mathbb{R}^d)) \).

Obviously, if \( \lim_{x \to \infty} \sigma(x) = \sigma > 0 \), then the assumption (A) holds for \( L(t) \equiv 1 \) and \( B_t = \emptyset \). Therefore, Theorem 1.2 generalizes Theorem 2.2 in Bodjecki et al [2]. Moreover, this result shows that the speed of \( \sigma(x) \) converging to 0 changes the growth of the occupation time fluctuation. In addition, the covariance function of \( X \) in Theorem 1.2 explicitly points out that the branching mechanisms of particles in the neighbor of initial site have no contribution to the temporal structure of the limit process, because the density function \( \rho(x) \) essentially characterizes the slight varies of branching laws in the neighbor of infinity. Although there are much literature on the occupation time fluctuations of branching particle systems, see for example [2-7, 9-13] and the reference therein, since the study methods are mainly analytic and do not shed much light on the 'physical' meaning of the results (see Bodjecki et al [5]), one know little
of the mechanism how the particles’ behaviors affect the limit processes. Theorem 1.2
provides some new information in this field.

Without other statement, in this paper, we use $K$ to denote an unspecified positive
finite constant which may not necessarily be the same in each occurrence.

The remainder of this paper is organized as follows. In Section 2, we collect some
necessary results and formulas. In Section 3 we prove the main results.

2. Preliminary

Let $\vec{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_d)$ and denote the diagonal matrix $diag(\alpha_1^{-1}, \alpha_2^{-1}, \cdots, \alpha_d^{-1})$ by
$H(\vec{\alpha})$. By convention, $r^H = \sum_{k=0}^{\infty} \frac{(H \ln r)^k}{k!}$ for any $r > 0$ and $d \times d$ matrix $H$. Therefore
\[ r^{H(\vec{\alpha})} = diag(r^{\alpha_1^{-1}}, r^{\alpha_2^{-1}}, \cdots, r^{\alpha_d^{-1}}). \]

Suppose $N = \{N(t), t \geq 0\}$ is the random counting measure of a $(d, \vec{\alpha}, \sigma(x))$-branching
particle system. The corresponding spatial motion is denoted by $\vec{\xi}$. Then $i$-th component of $\{\vec{\xi}(t), t \geq 0\}$, $\xi_i(t)$, is a symmetric $\alpha_i$-stable Levy process. $\vec{\xi}$ is an operator-
self-similar process with independent increments (see [8, 15]). We have
\[ \{\vec{\xi}(rt)\}^d = r^{H(\vec{\alpha})}\{\vec{\xi}(t)\}, \tag{2.1} \]
for all $r \geq 0$. We denote its semigroup by $\{T_t\}_{t \geq 0}$ and the transition density by $p_t$, i.e.,
\[ T_s f(x) := \mathbb{E}(f(\vec{\xi}(t + s)) | \vec{\xi}(t) = x) = \int_{\mathbb{R}^d} p_s(y - x) f(y)dy, \]
for all $s, t \geq 0$, $x \in \mathbb{R}^d$ and bounded measurable functions $f$. To avoid misunderstanding, we sometimes write $T_s f(x)$ as $T_s(f(\cdot))(x)$. (2.1) means that
\[ p_{rt}(x) = r^{-\vec{\alpha}} p_t(r^{-H(\vec{\alpha})}x), \tag{2.2} \]
where $\vec{\alpha} = \sum_{k=1}^{d} \alpha_k^{-1}$. For every $\phi \in \mathcal{S}(\mathbb{R}^d)$, the Green potential operator of $\vec{\xi}$ satisfies that
\[ G\phi(x) = \int_0^\infty T_t \phi(x)dt. \]
It is well-known that $G\phi(x)$ is bounded when $\vec{\alpha} > 1$. Li [9] had shown that
\[ \mathbb{E}(\langle N(s), \phi \rangle) = \int_{\mathbb{R}^d} F_\phi(s, x)dx = \int_{\mathbb{R}^d} \phi(x)dx = \langle \lambda, \phi \rangle. \tag{2.3} \]
Let $F_n = n^{1/2}$ if $\sigma(x)$ is integral and $F_n = n^{(\beta-\vec{\alpha})/2}/\sqrt{L(n)}$ if Assumption (A) holds. For any $t \geq 0$, define
\[ X_n(t) := \frac{X(nt)}{F_n} = \frac{1}{F_n} \int_0^{nt} (N(s) - \lambda)ds \]
according to the formula (1.1). Furthermore, define a sequence of random variables $\tilde{X}_n$
in $\mathcal{S}'(\mathbb{R}^{d+1})$ as follows: For any $n \geq 0$ and $\psi \in \mathcal{S}(\mathbb{R}^{d+1})$, let
\[ \langle \tilde{X}_n, \psi \rangle = \int_0^1 \langle X_n(t), \psi(\cdot, t) \rangle dt. \tag{2.4} \]
Without other statement, in the sequel, \( \psi \in \mathcal{S}(\mathbb{R}^{d+1}) \) always has the form \( \psi(x, t) = \phi(x) h(t) \), where \( \phi \in \mathcal{S}(\mathbb{R}^d) \) and \( h \in \mathcal{S}(\mathbb{R}) \) are nonnegative functions. Let

\[
\hat{h}(s) = \int_s^1 h(t) \, dt \quad \text{and} \quad \psi_n(x, s) = \frac{1}{F_n} \phi(x) \hat{h}(\frac{s}{n}).
\]  

(2.5)

From Li [9], we have that\( \int_0^t \psi_n(x,s) \, ds \) converges in integral sense (see Bojdecki et al [5]), it suffices to prove that \{\langle \psi_n \rangle \} is tight. For calculation in the proofs, the tightness of \( \psi_n \) is used. Except some necessary modification to fit the complexity added by the inhomogeneity of the branching, the main schemes of proofs are same as those of Mitoma [14] is used. To save space, we will omit some similar and simple procedures of calculation in the proofs.

### 3. Proofs of the main results

We first prove Theorem 1.1.

**Proof.** Without loss generality, we verify the case \( t = 1 \), namely, proving that \( \langle \hat{X}_n, \psi \rangle \) converges in distribution to \( \langle \hat{X}, \psi \rangle := \int_0^1 \langle X(s), \psi(\cdot, s) \rangle \, ds \) for all \( \psi \in \mathcal{S}(\mathbb{R}^{d+1}) \). By the lemma in Bojdecki et al [2, P.9], it is sufficient to check that

\[
\lim_{n \to \infty} \mathbb{E}(e^{-\langle \hat{X}_n, \psi \rangle}) = \exp \left( \frac{1}{2} \int_0^1 \int_0^1 \text{Cov} \left( \langle X(s), \psi(\cdot, s) \rangle, \langle X(t), \psi(\cdot, t) \rangle \right) ds dt \right) \tag{3.1}
\]
for each non-negative $\psi \in S(\mathbb{R}^{d+1})$. Below, we discuss the case of $\psi(x,t) = \phi(x)h(t)$ with $\phi \geq 0$ and $h \geq 0$. For general non-negative $\psi$, the proof is the same with slightly more complicated notation and is omitted.

First of all, applying (2.10) and (2.11) into (2.7), we obtain that

$$I_1(n,\psi_n) \leq Kn \left[ \int_{\mathbb{R}^d} \sigma(x)dx + \int_{\mathbb{R}^d} \sigma(x)G\sigma(x)dx \right]. \quad (3.2)$$

Since $G\phi$ is bounded and $\sigma$ is bounded and integral, we further get that

$$I_1(n,\psi_n) \leq Kn \left[ \int_{\mathbb{R}^d} \sigma(x)dx + \int_{\mathbb{R}^d} \sigma(x)G\sigma(x)dx \right]. \quad (3.3)$$

Secondly, we deal with the limit of $I_2(n,\psi_n)$. (2.8) yields that

$$I_2(n,\psi_n) = \frac{\gamma}{2} \int_{\mathbb{R}^d} \sigma(x)dx \int_0^1 \left( \int_0^{n-s} T_u \psi_n(x,s+u)du \right)^2 ds = \int_0^1 h(t)dt \int_0^1 h(r)dr \int_0^{r\wedge t} \Psi_n(s)ds, \quad (3.4)$$

where

$$\Psi_n(s) = \frac{n\gamma}{2F_n^2} \int_{\mathbb{R}^d} \sigma(x)dx \int_0^{n(t-s)} T_u \phi(x)du \int_0^{n(r-s)} T_v \phi(x)dv.$$ 

Since $\int_{\mathbb{R}^d} \sigma(x)dx < \infty$ and $F_n^2 = n$, by the monotone convergence theorem we get that as $n \to \infty$,

$$\Psi_n(s) \to \frac{\gamma}{2} \int_{\mathbb{R}^d} \sigma(x)(G\phi(x))^2dx < \infty.$$ 

Therefore, as $n \to \infty$,

$$I_2(n,\psi_n) \to \frac{\gamma}{2} \int_{\mathbb{R}^d} \sigma(x)(G\phi(x))^2dx \int_0^1 h(t)dt \int_0^1 h(r)(r \wedge t)dr. \quad (3.5)$$

Thirdly, we point out that

$$\lim_{n \to \infty} I_3(n,\psi_n) = \int_0^1 h(t)dt \int_0^1 h(r)(t \wedge r)dr \int_{\mathbb{R}^d} \phi(x)G\phi(x)dx. \quad (3.6)$$

The proof is same as that in Li [9] and omitted.
At last, combining (2.6) with (3.3), (3.5) and (3.6), we arrive at (3.1). The proof of Theorem 1.1 is complete.

Now we give the proof of Theorem 1.2.

**Proof.** To prove the first statement, by the same arguments used in the proof of Theorem 1.1, it suffices to prove that
\[
\lim_{n \to \infty} I_1(n, \psi_n) = \lim_{n \to \infty} I_3(n, \psi_n) = 0, \tag{3.7}
\]
and
\[
\lim_{n \to \infty} I_2(n, \psi_n) = \frac{1}{2} \left( \int_{\mathbb{R}^d} \phi(z) \, dz \right)^2 \int_0^1 \int_0^1 h(r) h(t) C(r, t) \, dr \, dt, \tag{3.8}
\]
where \(C(r, t)\) is given by (1.8). Due to the fact \(\sigma(x) \in [0, 1]\) and the assumption (1.4), it is easy to see that by some obvious modification, the arguments used in Bojdecki et al [2] (to discuss the terms of \(I_2\) and \(I_3\) therein) are still valid on proving (3.7). The details are omitted.

We pass to (3.8). By using (2.5), (2.10) and \(F_n^2 = n^{3-H} / L(n)\), from (2.8) we get that
\[
I_2(n, \psi_n) = \frac{2}{2} \int_{\mathbb{R}^d} \sigma(x) \, dx \int_0^n \left( \int_0^{n-s} T_u \psi_n(x, s + u) \, du \right)^2 \, ds = \frac{1}{2} \int_0^1 \Theta_n(s) \, ds, \tag{3.9}
\]
where
\[
\Theta_n(s) = \int_{\mathbb{R}^d} n^\alpha \sigma(x) L(n) \left( \int_0^{1-s} T_{nu} \phi(x) \tilde{h}(s + u) \, du \right)^2 \, dx.
\]
Then using (2.2), we have that
\[
\Theta_n(s) = \int_{\mathbb{R}^d} n^\alpha \sigma(x) L(n) \left( \int_0^{1-s} \int_{\mathbb{R}^d} p_{nu}(y - x) \phi(y) \, dy \tilde{h}(s + u) \, du \right)^2 \, dx
= \int_{\mathbb{R}^d} n^\alpha \sigma(x) L(n) \left( \int_0^{1-s} \int_{\mathbb{R}^d} n^-H(\tilde{\alpha})(y - x) \phi(y) \, dy \tilde{h}(s + u) \, du \right)^2 \, dx.
\]
Furthermore, substituting \(x' = n^{-H(\tilde{\alpha})} x\), we arrive at
\[
\Theta_n(s) = \int_{\mathbb{R}^d} \sigma(n^{H(\tilde{\alpha})} x) L(n) \left( \int_0^{1-s} \int_{\mathbb{R}^d} p_u(x - n^{-H(\tilde{\alpha})} y) \phi(y) \, dy \tilde{h}(s + u) \, du \right)^2 \, dx.
\]
Let
\[
\Omega = \{(s, u, v, x, y, z) : s \in [0, 1], 0 \leq u, v \leq 1 - s, x, y, z \in \mathbb{R}^d\}.
\]
Define
\[
\Gamma_n(s, u, v, x, y, z) = \sigma(n^{H(\tilde{\alpha})} x) L(n) \Delta_n(s, u, v, x, y, z),
\]
on \(\Omega\), where
\[
\Delta_n(s, u, v, x, y, z) = \tilde{h}(s + u) \tilde{h}(s + v) p_u(x - n^{-H(\tilde{\alpha})} y) \phi(y) p_v(x - n^{-H(\tilde{\alpha})} z) \phi(z).
\]
Then, from (3.9)

$$I_2(n, \psi_n) = \frac{\gamma}{2} \int_\Omega \Gamma_n(s, u, v, x, y, z) ds du dv dx dy dz.$$  \hspace{1cm} (3.10)

From the assumption (1.5), it is easy to see that as $n \to \infty$

$$\Gamma_n(s, u, v, x, y, z) \to \Gamma(s, u, v, x, y, z) = \rho(x) \Delta(s, u, v, x, y, z),$$  \hspace{1cm} (3.11)

almost everywhere in $\Omega$, where

$$\Delta(s, u, v, x, y, z) = \tilde{h}(s+u)\tilde{h}(s+v)p_u(x)p_v(x)\phi(y)\phi(z).$$

By some direct calculations, one have that

$$\frac{\gamma}{2} \int_\Omega \rho(x) \Delta(s, u, v, x, y, z) ds du dv dx dy dz = r.h.s \text{ of (3.8).}$$

Therefore, to ensure (3.8) it suffice to prove that

$$\lim_{n \to \infty} \int_\Omega \Gamma_n(s, u, v, x, y, z) ds du dv dx dy dz = \int_\Omega \Gamma(s, u, v, x, y, z) ds du dv dx dy dz.$$  \hspace{1cm} (3.12)

Note that the assumptions of (1.6) and $\sigma(x) \leq 1$ imply that

$$\Gamma_n(s, u, v, x, y, z) \leq (M + L(n)I_{\mathcal{P}_n}(x)) \Delta_n(s, u, v, x, y, z) \to M\Delta(s, u, v, x, y, z),$$  \hspace{1cm} (3.13)

almost everywhere in $\Omega$ as $n \to \infty$. In addition,

$$\int_\Omega M\Delta_n(s, u, v, x, y, z) ds du dv dx dy dz$$

$$= M \int_0^1 ds \int_{\mathbb{R}^d} \tilde{h}(s+u) \int_0^{1-s} \tilde{h}(s+v) \left[ n^{2s} T_u \phi_n(x) T_v \phi_n(x) \right] du dv, \hspace{1cm} (3.14)$$

where we use the notation $\phi_n(x) := \phi(n^{H(\alpha)}x)$ and the fact that for all $t > 0$

$$\int_{\mathbb{R}^d} p_t(x - n^{-H(\alpha)}y) \phi(y) dy = n^\alpha \int_{\mathbb{R}^d} p_t(x - y) \phi(n^{H(\alpha)}y) dy = n^\alpha T_{n^\alpha} \phi_n(x).$$

For any $f \in L(\mathbb{R}^d)$, denote its Fourier transform by $\hat{f}$. It is well-known that $|\hat{\phi}|$ is bounded and integrable if $\phi \in \mathcal{S}(\mathbb{R}^d)$ and that for any $t > 0$,

$$n^{\alpha t} T_{n^{\alpha t}} \phi_n(z) = \hat{\phi}(n^{-H(\alpha)}z) e^{-t \sum_{i=1}^d |z_i|^{\alpha i}}.$$

The Plancherel formula and (3.14) yield that

$$\int_\Omega M\Delta_n(s, u, v, x, y, z) ds du dv dx dy dz$$

$$= \frac{M}{(2\pi)^d} \int_0^1 ds \int_{\mathbb{R}^d} \tilde{h}(s+u) du \int_0^{1-s} \tilde{h}(s+v) dv \times \left[ \hat{\phi}(n^{-H(\alpha)}z) \phi(n^{-H(\alpha)}z) e^{-(u+v) \sum_{i=1}^d |z_i|^{\alpha i}} \right] dz.$$  \hspace{1cm} (3.15)
Hence we can readily verify that
\[
\lim_{n \to \infty} \int_{\Omega} M \Delta_n(s, u, v, x, y, z) ds du vdxdydz = \int_{\mathbb{R}^d} L(n) 1_{B_n}(x) \Delta_n(s, u, v, x, y, z) ds du vdxdydz
\]
\[
= CM \int_{0}^{1} ds \int_{0}^{1-s} \tilde{h}(s+u)du \int_{0}^{1-s} \tilde{h}(s+v) \left| \hat{\phi}(0) \right|^2 dv
\]
\[
= \int_{1}^{0} ds \int_{0}^{1-s} du \int_{0}^{1-s} dv \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} M \Delta(s, u, v, x, y, z)dz,
\]
where \( C = \frac{1}{2} \prod_{k=1}^{d} \frac{\Gamma(1/\alpha_k)}{\alpha_k} \). At the same time, using (2.2) again, we have that
\[
\int_{\Omega} L(n) 1_{B_n}(x) \Delta_n(s, u, v, x, y, z) ds du vdxdydz
\]
\[
= \frac{L(n)}{n^\alpha} \int_{\mathbb{R}^d} 1_{B_n}(n^{-H(\vec{a})}x) dx \int_{0}^{1} ds \int_{0}^{1-s} \tilde{h}(s+u)du \int_{0}^{1-s} \tilde{h}(s+v) dv
\]
\[
\times \int_{\mathbb{R}^d} p_u(n^{-H(\vec{a})}(x-y)) \phi(y)dy \int_{\mathbb{R}^d} p_u(n^{-H(\vec{a})}(x-z)) \phi(z)dz
\]
\[
= \frac{L(n)}{n^\alpha} \int_{\mathbb{R}^d} 1_{B_n}(n^{-H(\vec{a})}x) dx \int_{0}^{1} ds \int_{0}^{1-s} \tilde{h}(s+u)du \int_{0}^{1-s} \tilde{h}(s+v) dv
\]
\[
\times \int_{\mathbb{R}^d} p_u(x-y) \phi(y)dy \int_{\mathbb{R}^d} p_u(x-z) \phi(z)dz,
\]
which, via the inverse of Fourier transform, equals
\[
\frac{L(n)}{(2\pi)^{2d}} n^\alpha \int_{0}^{1} ds \int_{\mathbb{R}^d} \tilde{h}(s+u)du \int_{0}^{1-s} \tilde{h}(s+v) dv
\]
\[
\times \int_{\mathbb{R}^d} 1_{\hat{B}_n}(z+y) \hat{\phi}(z)\hat{\phi}(y)\exp\{-nu \sum_{i=1}^{d} |z_i|^\alpha_i - nv \sum_{i=1}^{d} |y_i|^\alpha_i\}dzdy
\]
\[
\leq \frac{K^2 L(n)}{(2\pi)^{2d}} n^{\alpha-2} \int_{\mathbb{R}^d} \frac{\hat{\phi}(z)}{\sum_{i=1}^{d} |z_i|^\alpha_i} dz \int_{\mathbb{R}^d} 1_{\hat{B}_n}(z+y) \frac{\hat{\phi}(y)}{\sum_{i=1}^{d} |y_i|^\alpha_i} dy,
\]
where \( K \) is an upper bound of \(|h(t)|, t \in [0,1]\) and \( \hat{B}_n = \{n^{H(\vec{a})}x : x \in B_n\} \). Since
\[
\left| \int_{\mathbb{R}^d} e^{-izx} 1_{\hat{B}_n}(x)dx \right| \leq \lambda(\hat{B}_n) = n^\alpha \lambda(B_n)
\]
for any \( z \in \mathbb{R}^d \), from (3.17) and (3.18), it follows that
\[
\int_{\Omega} L(n) 1_{B_n}(x) \Delta_n(s, u, v, x, y, z) ds du vdxdydz \leq \frac{K^2 L(n)}{(2\pi)^{2d}} n^{\alpha-2} \lambda(B_n) \left( \int_{\mathbb{R}^d} \frac{\hat{\phi}(z)}{\sum_{i=1}^{d} |z_i|^\alpha_i} dz \right)^2.
\]
By Lemma 2.1 in Li [9], \( \int_{\mathbb{R}^d} \frac{\hat{\phi}(z)}{\sum_{i=1}^{d} |z_i|^\alpha_i} dz \) is finite. Therefore, the assumption (1.7) indicates that
\[
\lim_{n \to \infty} \int_{\Omega} L(n) 1_{B_n}(x) \Delta_n(s, u, v, x, y, z) ds du vdxdydz = 0.
\]
From (3.10)-(3.13), (3.16) and (3.20) and the dominated convergence theorem, we obtain (3.12).

Now we are at the place to prove the second statement. As we mentioned in Section 2, we still need to prove the tightness of \((X_n, \phi); n \geq 1\) in \(C([0, 1], \mathbb{R})\) for all \(\phi \in \mathcal{S} (\mathbb{R}^d)\), which, according to Billingsley [1, Theorem 12.3] and the fact \((X_n(0), \phi) = 0\), suffices to prove that for all non-negative \(\phi \in \mathcal{S} (\mathbb{R}^d)\), there exist a constant \(h > 0\), such that for all \(n \geq 1\) and \(0 \leq s < t \leq 1\)

\[
\mathbb{E} (\langle X_n(t), \phi \rangle - \langle X_n(s), \phi \rangle)^2 \leq K (t - s)^{1+h}, \quad (3.21)
\]

where \(K\) is a constant independent of \(n\), \(s\) and \(t\). From (4.3) in Li [10], one has that

\[
\mathbb{E} (\langle X_n(t), \phi \rangle - \langle X_n(s), \phi \rangle)^2 = I_4(n, \phi) + I_5(n, \phi), \quad (3.22)
\]

where

\[
I_4(n, \phi) = \frac{n^2}{F_n^2} \int_s^t \int_s^t \int_{\mathbb{R}^d} \phi(x) L_{n[u-v]} \phi(x) dx du dv,
\]

\[
I_5(n, \phi) = \frac{\gamma n^3}{F_n^2} \int_s^t \int_s^t \int_{\mathbb{R}^d} \sigma(x) \int_0^{u \wedge v} L_{n[u-r]} \phi(x) L_{n[v-r]} \phi(x) dr dx du dv.
\]

By the Plancherel formula,

\[
I_4(n, \phi) = \frac{n^2}{F_n^2} \int_s^t \int_s^t \int_{\mathbb{R}^d} |\hat{\phi}(z)|^2 e^{-|n[u-v]\sum_{k=1}^{d} |z_k|^{\alpha_k}} dx du dv
\]

\[
= \frac{2n^2}{F_n^2} \int_s^t du \int_{u}^{t} dv \int_{\mathbb{R}^d} |\hat{\phi}(z)|^2 e^{-n(v-u) \sum_{k=1}^{d} |z_k|^{\alpha_k}} dz
\]

\[
= \frac{2n}{F_n^2} \int_{\mathbb{R}^d} \frac{|\hat{\phi}(z)|^2}{\sum_{k=1}^{d} |z_k|^{\alpha_k}} dz \int_s^t \left(1 - e^{-n(t-u) \sum_{k=1}^{d} |z_k|^{\alpha_k}}\right) du. \quad (3.23)
\]

Using the inequality \(1 - e^{-x} \leq x^\eta\) for all \(x \geq 0\) and \(0 < \eta < 1\) and substituting \(F_n^2 = n^{3-\alpha}/L(n)\) into (3.23), we get that,

\[
I_4(n, \phi) \leq \frac{2n^{1+\eta} L(n)}{n^{3-\alpha}} \int_{\mathbb{R}^d} \frac{|\hat{\phi}(z)|^2}{\left(\sum_{k=1}^{d} |z_k|^{\alpha_k}\right)^{1-\eta}} dz \int_s^t \left(t-u\right)^\eta du.
\]

Take \(\eta_1 \in (0, 2 - \bar{\alpha}) \subset (0, 1)\). Then \(1 - \eta_1 \in (0, 1)\) and \(1 + \eta_1 \leq 3 - \bar{\alpha}\). Lemma 2.1 in Li [9] ensures that

\[
\int_{\mathbb{R}^d} \frac{|\hat{\phi}(z)|^2}{\left(\sum_{k=1}^{d} |z_k|^{\alpha_k}\right)^{1-\eta}} dz < \infty.
\]

Therefore Assumption (A) implies that

\[
I_4(n, \phi) \leq K (t - s)^{1+\eta_1}, \quad (3.24)
\]

for some constant \(K > 0\) which only depends on \(\phi, d\) and \(\bar{\alpha}\).

Substituting \(F_n^2 = n^{3-\alpha}/L(n)\) into \(I_5(n, \phi)\) and then letting \(w = n^{-H(\bar{\alpha})} x\), we get that \(I_5(n, \phi)\) equals

\[
\gamma n^{2\bar{\alpha}} L(n) \int_s^t \int_s^t \int_{\mathbb{R}^d} \sigma(n^{H(\bar{\alpha})} w) \int_0^{u \wedge v} L_{n[u-r]} \phi(n^{H(\bar{\alpha})} w) L_{n[v-r]} \phi(n^{H(\bar{\alpha})} w) dr dx du dv.
\]
The assumption (1.6) and the fact \( \sigma \in [0,1] \) yield that there is a constant \( M \) independent of \( \phi \), \( n \), \( s \) and \( t \) such that

\[
I_5(n, \phi) \leq 2\gamma [I_{51}(n, \phi) + I_{52}(n, \phi)]
\]

where

\[
I_{51}(n, \phi) = Mn^{\alpha} \int_s^t \int_s^u du \int_s^v \int_\mathbb{R}^d L_n(u-r) \phi(x) L_n(v-r) \phi(x) dx dr dx,
\]

\[
I_{52}(n, \phi) = n^{\alpha} \int_s^t \int_s^u du \int_\mathbb{R}^d L(n) 1_{B_n(n^{-H(\tilde{\alpha})}x)} \int_0^v L_n(u-r) \phi(x) L_n(v-r) \phi(x) dx dr dx.
\]

By the Plancherel formula,

\[
I_{51}(n, \phi) = Mn^{\alpha} \int_s^t \int_s^u du \int_\mathbb{R}^d \int_0^v |\hat{\phi}(z)|^2 e^{-n(u+v-2r)} \sum_{k=1}^d |z_k|^{\alpha_k} d\mathbf{z} dr.
\]

Therefore,

\[
I_{51}(n, \phi) \leq K n^{\alpha} \int_{\mathbb{R}^d} |\hat{\phi}(z)|^2 dz \int_s^t \int_s^u e^{-n(u+v)} \sum_{k=1}^d |z_k|^{\alpha_k} du \int_0^v e^{2nt} \sum_{k=1}^d |z_k|^{\alpha_k} dr
\]

\[
\leq \frac{K n^{\alpha}}{n} \int_{\mathbb{R}^d} |\hat{\phi}(z)|^2 \sum_{k=1}^d |z_k|^{\alpha_k} dz \int_s^t du \int_s^u e^{-n(u-v)} \sum_{k=1}^d |z_k|^{\alpha_k} dv
\]

\[
= \frac{K n^{\alpha}}{n^2} \int_{\mathbb{R}^d} \sum_{k=1}^d |z_k|^{\alpha_k} dz \int_s^t 1 - e^{-n(u-s)} \sum_{k=1}^d |z_k|^{\alpha_k} du.
\]

Substituting and \( y = n^{H(\tilde{\alpha})} z \) into (3.26), we get that

\[
I_{51}(n, \psi_n) \leq K \int_{\mathbb{R}^d} |\hat{\phi}(n^{-H(\alpha)} y)|^2 dy \int_{\mathbb{R}^d} \sum_{k=1}^d |y_k|^{\alpha_k} dz \int_s^t \frac{1 - e^{-n(u-s)} \sum_{k=1}^d |y_k|^{\alpha_k}}{\sum_{k=1}^d |y_k|^{\alpha_k}} du.
\]

Since \( |\hat{\phi}| \) is bounded, using the inequality \( 1 - e^{-x} \leq x^\eta \) for all \( x \geq 0 \) and \( 1 > \eta > 0 \) again, we obtain that for any \( \eta \in (2 - \tilde{\alpha}, 1) \) and \( \eta' \in (0, 2 - \tilde{\alpha}) \),

\[
I_{51}(n, \phi) \leq M |\hat{\phi}|^2 \left[ \int_{[0,1]^d} \sum_{k=1}^d |z_k|^{\alpha_k} \right]^{2-\eta} dz \int_s^t (u-s)^{\eta'} du + \int_{\mathbb{R}^d \setminus [0,1]^d} \sum_{k=1}^d |z_k|^{\alpha_k} \right]^{2-\eta} dz \int_s^t (u-s)^{\eta'} du,
\]

where \( \|\hat{\phi}\| := \sup_{z \in \mathbb{R}^d} |\hat{\phi}(z)| \). Since \( \tilde{\alpha} \in (1,2) \), Lemma 2.1 in Li [9] ensures that

\[
\int_{[0,1]^d} \sum_{k=1}^d |z_k|^{\alpha_k} \right]^{2-\eta} dz + \int_{\mathbb{R}^d \setminus [0,1]^d} \sum_{k=1}^d |z_k|^{\alpha_k} \right]^{2-\eta} dz < \infty.
\]

Therefore, there exist constants \( \eta_2 \in (0,1) \) and \( K > 0 \) independent on \( n, s \), and \( t \) such that

\[
I_{51}(n, \phi) \leq K (t-s)^{1+\eta_2}.
\]
Furthermore, using the inverse Fourier formula, we obtain that
\[
I_{52}(n, \phi) = \frac{L(n)}{(2\pi)^d} n^\alpha \int_{\mathbb{R}^d} |\hat{\phi}(z)| \, dz \int_{\mathbb{R}^d} |\hat{1_{B_n}}(z + y)| \, dy \\
\times \int_s^t e^{-nu} \sum_{k=1}^d |z_i|^\alpha_i \, du \int_s^u e^{-nv} \sum_{k=1}^d |y_i|^\alpha_i \, dv \int_0^u e^{nr} \sum_{k=1}^d (|y_i|^\alpha_i + |z_i|^\alpha_i) \, dr
\]
\[
\leq \frac{L(n)}{(2\pi)^d} n^\alpha \int_{\mathbb{R}^d} \frac{|\hat{\phi}(z)||\hat{\phi}(y)||1_{B_n}(z + y)|}{\sum_{k=1}^d (|y_i|^\alpha_i + |z_i|^\alpha_i) \, dz} \, dy \\
\times \int_s^t \left( 1 - a^{-n(s-t)} \sum_{k=1}^d |z_i|^\alpha_i \right) \, du.
\]
Note that for all \(\alpha \in (0, 2)\),
\[
|a + b|^\alpha \leq 2(|a|^\alpha + |b|^\alpha).
\]
using the inequality \(1 - e^{-x} \leq x^\eta\) for all \(x \geq 0\) and \(1 > \eta > 0\) again, we further get that
\[
I_{52}(n, \phi) \leq \frac{2L(n)}{(2\pi)^d} n^{\alpha + \eta_0} \int_{\mathbb{R}^d} \frac{|\hat{\phi}(z)||\hat{\phi}(y)||1_{B_n}(z + y)|}{\sum_{k=1}^d (|y_i|^\alpha_i + |z_i|^\alpha_i) \, dz} \, dy \int_s^t (u - s)^{\eta_0} \, du.
\]
Note that Lemma 2.1 in Li [9] implies that
\[
\int \frac{1}{\sum_{k=1}^d |w_i|^\alpha_i \leq 1} \, dw \int_{\mathbb{R}^d} \frac{|\hat{\phi}(z)|}{\sum_{k=1}^d |z_i|^\alpha_i \, dz} < \infty.
\]
We can readily get positive \(K\)'s independent of \(n\), \(s\) and \(t\) such that
\[
\int \frac{|\hat{\phi}(z)||\hat{\phi}(y)||1_{B_n}(z + y)|}{\sum_{k=1}^d |y_i|^\alpha_i + |z_i|^\alpha_i} \, dz \, dy \\
\leq K \int_{\mathbb{R}^d} \left( \sum_{k=1}^d |w_i|^\alpha_i \sum_{k=1}^d |z_i|^\alpha_i \right) \, dw \, dz \leq K \sup_w |1_{B_n}(w)|.
\]
In addition, by the similar reason, we have that
\[
\int \frac{|\hat{\phi}(z)||\hat{\phi}(y)||1_{B_n}(z + y)|}{\sum_{k=1}^d |y_i|^\alpha_i + |z_i|^\alpha_i} \, dz \, dy \\
\leq K \sup_w |1_{B_n}(w)| \int \left( \sum_{k=1}^d |w_i|^\alpha_i \sum_{k=1}^d |z_i|^\alpha_i \right) \, dw \, dz \leq K \sup_w |1_{B_n}(w)|.
\]
Consequently, there is a constant \(K > 0\) independent of \(n\), \(s\), and \(t\) such that
\[
I_{52}(n, \phi) \leq KL(n) n^{\alpha - 2 + \eta_0} \sup_w |1_{B_n}(w)| (t - s)^{1 + \eta_0}.
\]
Condition (1.9) plus (3.19) yields that there is a constant \(K > 0\) independent of \(n\), \(s\), and \(t\) such that
\[
I_{52}(n, \phi) \leq K(t - s)^{1 + \eta_0}. \tag{3.28}
\]
Hence, combining (3.22) with (3.24), (3.25), (3.27) and (3.28) leads to (3.21) with \(h = \min\{\eta_1, \eta_2, \eta_0\} > 0\). The proof of Theorem 1.2 is complete. \(\square\)
Remark 3.1 We mentioned before that if \( \lim_{x \to \infty} \sigma(x) = \sigma > 0 \), the assumption (A) holds naturally. The following example shows that if \( \lim_{x \to \infty} \sigma(x) = 0 \), the assumption (A) and the condition (1.9) are also meaningful.

Example 3.1 Let \( d = 2 \), \( \vec{\alpha} = (1, 2) \), \( \sigma(x) = \ln^{-1}(|x_1|^{3/2} + |x_2|^{3} + e^2) \). Then from

\[
\sigma(t^{H(\vec{\alpha})} x) = \frac{1}{\ln(t^{3/2}(|x_1|^{3/2} + |x_2|^{3} + e^2/t^{3/2}))} = \frac{1}{\ln t(\frac{3}{2} + \ln(|x_1|^{3/2} + |x_2|^{3} + e^2/t^{3/2})/\ln t)},
\]

it is easy to see that if we take \( L(t) = \ln t \), then

\[
L(t)\sigma(t^{H(\vec{\alpha})} x) = \frac{1}{\frac{3}{2} + \ln(|x_1|^{3/2} + |x_2|^{3} + e^2/t^{3/2})/\ln t)} \to \frac{2}{3}, \text{ a.s.}
\]

Furthermore, let

\[
B_t = \{(x_1, x_2) : |x_1| \leq \frac{1}{t^{3/4}}, |x_2| \leq \frac{1}{t^{3/8}}\}.
\]

Then on \( \mathbb{R}^2 \setminus B_t \),

\[
\sigma(t^{H(\vec{\alpha})} x)L(t) = \frac{1}{\frac{3}{2} + \ln(|x_1|^{3/2} + |x_2|^{3} + e^2/t^{3/2})/\ln t)} \leq \frac{1}{\frac{3}{2} + \ln(t^{-9/8})/\ln t} = \frac{8}{3}, \quad (3.29)
\]

and \( \lambda(B_t) = 4t^{-9/8} \) which and \( \vec{\alpha} = 3/2 \) result in

\[
L(t)t^{2(\vec{\alpha}-1)}\lambda(B_t) = 4 \ln t/t^{1/8} \to 0
\]

as \( t \to \infty \). Hence the assumption (A) is true. Furthermore, it is easy to see (1.9) holds for any \( \eta_0 \in (0, 1/8) \).

References


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