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### Transition to amplitude death in scale-free networks

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**Abstract.** Transition to amplitude death in scale-free networks of nonlinear oscillators is investigated both numerically and analytically. It is found that, as the coupling strength increases, the network will undergo three different stages in approaching the state of complete amplitude death (CAD). In the first stage of the transition, the amplitudes of the oscillators present a 'stair-like' arrangement, i.e. the squared amplitude of an oscillator linearly decreases with the number of links that the oscillator receives (node degree). In this stage, as the coupling strength increases, the amplitude stairs are eliminated *hierarchically* by descending order of the node degree. At the end of the first stage, except for a few synchronized oscillators, all other oscillators in the network have small amplitudes. Then, in the second stage of the transition, the synchronous clusters formed in the first stage gradually disappear and, as a consequence, the number of small-amplitude oscillators is increased. At the end of the second stage, almost all oscillators in the network have small but finite amplitudes. Finally, in the third stage of the transition, without the support of the synchronous clusters, the amplitudes of the oscillators are quickly decreased, eventually leading to the state of CAD.

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Amplitude death (AD) refers to the phenomenon of coupled oscillators ceasing their oscillations and settling on some fixed points in the phase space. Since its first discovery in an array of coupled non-identical oscillators [1], AD continues to receive research attention in both theory and experiment in the past two decades [2]–[11]. Very different from the other types of collective behaviors that emerge in coupled oscillators, e.g. the synchronization phenomenon [12], in AD the oscillator trajectory is dramatically changed, i.e. from the oscillatory to the stationary type. Because of this, to generate AD, both the oscillator dynamics and the coupling function should be set properly [2, 3, 5]. So far, three methods have been identified as workable for generating AD, which includes the parameter mismatch [1, 2], the delayed coupling [3, 4] and the conjugate coupling [5, 7]. In the first two methods, in the AD state the oscillators are ceased at the origin of the phase space. In these cases, the origin point is an unstable solution of the individual oscillator, but is stabilized by the coupling term. However, in the method of conjugate coupling, the oscillators are ceased at some fixed points away from the origin. These fixed points are not solutions of individual oscillators, but are newly created by the couplings [5]. To highlight the feature of nonzero amplitude of the ceasing state, this kind of cessation is also named the oscillation death.

Recently, AD in spatio-temporal systems has received considerable research attention, with a focus on the pattern and transition of AD in networked oscillators [13]–[16]. A chain of nonlinear oscillators of monotonic frequency distribution has been studied in [13], where partial amplitude death (PAD) (a state where only a subset of the network nodes is dead) has been observed before the state of complete amplitude death (CAD). In the same paper, the authors have also found that, by disordering the natural frequencies of the oscillators, the generation of CAD can be significantly postponed. In [14], the authors have proposed the idea of eliminating CAD by disordering the network structure and found that, compared with the regular and random networks, the small-world networks [17] are more difficult to generate CAD. In [15], the authors have studied the transition from the oscillatory state to CAD in an array of nonlinear oscillators, and found that the route to CAD can be divided into different stages.

The previous studies on the generation of AD in complex systems, however, deal with only the case of homogeneous networks, i.e. the network nodes have similar numbers of connections (node degree). As many practical systems take the form of scale-free network in which a few nodes possess a very large degree [18], it is curious to know how this topological heterogeneity will affect the generation of AD. The study of AD in scale-free networks is twofold. On the one hand, it extends our knowledge of AD to the case of heterogeneous networks, hence enhancing our understanding of the behaviors of some practical networks. On the other hand, this study can be also regarded as an important approach to revealing the interplay between the network dynamics and structure—a hot topic extensively studied in nonlinear science in the past decade [19]. Previous studies on network synchronization have shown that the collective behavior of a scale-free network can be significantly different from that of homogeneous networks, e.g. on the aspects of network synchronizability and synchronization transition [20, 21]. Here, by considering a totally different type of collective behavior, i.e. the AD, we are going to explore this interplay relationship from a new angle, and this could shed some new light on this study.

The model we have employed is a network of N Landau–Stuart oscillators, which is described by the following set of equations:

$$\dot{Z}_{i}(t) = [r^{2} + i\omega_{i} - |Z_{i}|^{2}]Z_{i}(t) + \varepsilon \sum_{j=1}^{N} a_{ij}(Z_{j}(t) - Z_{i}(t)),$$
(1)

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where i = 1, 2, ..., N represents the node index. Here,  $Z_i(t)$  is a complex number denoting the state of the *i*th oscillator at time *t*,  $\varepsilon$  is the uniform coupling strength,  $\omega_i$  is the natural frequency of node *i*, and *r* is the oscillation growth rate. The connections of the oscillators are defined by the matrix  $A = \{a_{ij}\}$ , where  $a_{ij} = 1$  if *i* and *j* are connected and  $a_{ij} = 0$  otherwise. The degree of the *i*th node is  $k_i = \sum_{j=1}^{N} a_{ij}$ . To differentiate the node dynamics,  $\omega_i$  is randomly chosen from the range  $[\omega_1, \omega_2]$ . Without coupling, the trajectory of each oscillator will settle to a limit circle of radius  $|Z_i(t)| = r$ . In our future studies, we will keep all other parameters in equation (1) fixed, while increasing  $\varepsilon$  to realize the transition of CAD.

To measure the degree of the network death, the following two macroscopic quantities are employed: the normalized network 'incoherent' energy,  $E = \langle \sum_{i=1}^{N} |Z_i(t)|^2 \rangle / N$ , and the number of 'dead' nodes,  $N_d$ , in the network. Here  $\langle \cdot \rangle$  denotes the time average over a time period T. It is important to note that, due to the possible perturbations of the external environment, in a practical situation the averaged amplitude of an oscillator,  $Z_i = \langle |Z_i(t)| \rangle$ , cannot be exactly zero, even though the conditions for generating AD are fulfilled in theory. Considering this, we define a small threshold,  $Z_0 = \langle |Z| \rangle \ll 1$ , for the averaged amplitude, and regarding nodes of  $Z_i < Z_0$  as dead. Of course, depending on the specific problems to be treated in realistic cases, the value of  $Z_0$  should be modified accordingly [14, 15]. With the above definitions, it is straightforward to recognize that a state of smaller E and larger  $N_d$ , in general, represents a higher degree of AD.

We first study the transition by means of the macroscopic quantities E and  $N_d$ . The network is generated by the standard BA model [18], which consists of N = 1000 nodes and has an average degree  $\langle k \rangle = 6$ . The degree distribution follows roughly the power-law scaling  $P(k) \sim k^{-3}$ . To facilitate the analysis, we reorder the network nodes by ascending order of their degrees. That is, we have  $k_1 = k_{\min}$  and  $k_N = k_{\max}$ . For the local dynamics, we set r = 0.5 and the natural frequency  $\omega_i$  of the oscillators is randomly chosen from the range [1, 50]. In our simulation, we initialize the states of the oscillators,  $Z_i$ , by random complex numbers and then evolve the system according to equation (1). After a transient period  $T' = 1 \times 10^3$ , we start calculating E and  $N_d$ , which are averaged over another period T = 500.

The variation of *E* as a function of  $\varepsilon$  is plotted in figure 1(a). It is seen that the functional form of *E* appears to have a natural piecewise description, with three distinct stages. In the first stage, within the range  $\varepsilon \in [0, 0.16]$ , it is found that the value of *E* is decreasing exponentially with  $\varepsilon$ , but with a relatively low speed. In the second stage, within the range  $\varepsilon \in (0.16, 0.2)$ , the decrease of *E* is quickened, and the dependence of *E* on  $\varepsilon$  has a complicated form (which cannot be fitted by a simple scaling). In the third stage, within the range  $\varepsilon \in [0.2, 0.25]$ , it is found that *E* is decreasing exponentially again with  $\varepsilon$ , which, in comparison with the previous two stages, has a much higher decrease rate. The variation of  $N_d$  as a function of  $\varepsilon$  is plotted in figure 1(b). For a small-amplitude threshold, e.g.  $Z_0 = 1 \times 10^{-10}$ , it is found that  $N_d$  is smoothly increasing from 0 to *N* within a narrow range of  $\varepsilon$ . However, if the amplitude threshold is not too small, e.g.  $Z_0 > 1 \times 10^{-2}$ , a non-smooth stair-like variation is observed.

To explore the staged transition and the stair structures shown in figure 1, we proceed to investigate the system dynamics at the microscopic level. Specifically, we calculate the empirical distribution of the averaged node amplitude,  $\{Z_i\}$ , and examine how this amplitude arrangement is changed during the transition. In the first stage, the node amplitudes are evolved as follows. At very small coupling  $\varepsilon \approx 0$ , all nodes have the same amplitude  $Z_i = r$ , regardless of the difference in the node degree. Then, as  $\varepsilon$  increases, all the amplitudes are decrease simultaneously, but with very different speeds. More specifically, the decrease rate of the



**Figure 1.** The transition to CAD for a scale-free network of N = 1000 Landau–Stuart oscillators. (a) The variation of E as a function of  $\varepsilon$ , where the transition is roughly divided into three stages by the critical couplings  $\varepsilon' \approx 0.16$  and  $\varepsilon'' \approx 0.2$ . The fitted exponents of the first and third stages are  $-9 \pm 0.1$  and  $-335 \pm 1$ , respectively. (b) The variation of  $N_d$  as a function of  $\varepsilon$ . Two different amplitude thresholds,  $Z_0 = 1 \times 10^{-2}$  (the red curve) and  $Z_0 = 1 \times 10^{-10}$  (the black curve), are used separately to characterize the transition.

squared amplitude of a node is proportional to the node degree. So, the homogeneous amplitude arrangement seen in small couplings is gradually modified and replaced by a non-smooth 'stairlike' arrangement, for instance, the one generated by  $\varepsilon = 0.05$  in figure 2(a). In this stair-like arrangement, nodes of the same degree possess the same amplitude, except some rare 'outliers,' which are caused by synchronous clusters (to be explained later). Because of the heterogeneous distribution of the node degrees, the amplitude stairs are of different widths. More interestingly, the height of the stairs is gradually stepping down with the node degree. That is, the highest stair consists of only the smallest-degree nodes and the lowest stair consists of only the largestdegree nodes. When  $\varepsilon$  is increased further, the amplitudes of the largest-degree nodes hardly decrease, while the amplitudes of the other nodes start decreasing continuously. This results in a hierarchical death of the amplitude stairs. The process of stair elimination is depicted in figure 2(a), where some typical amplitude arrangements in the first stage are presented. Finally, at about the coupling strength  $\varepsilon' \approx 0.16$ , the last stair, i.e. the stair consisting of the smallestdegree nodes, is eliminated, and the first-stage transition is completed. From the results of figure 2(a), it is straightforward to see that the stair structures appearing in the variation of  $N_{\rm d}$  (figure 1(b)) are just caused by the hierarchical death of the amplitude stairs. We would like



**Figure 2.** For the same network model as that used in figure 1, some typical amplitude arrangements of the oscillators observed in the first stage of the transition. (a) The amplitudes are generated by coupling strengths, from top to bottom,  $\varepsilon = 0.05, 0.09, 0.1, 0.15, 0.16$ . The green lines are the theoretical results predicated by equation (2). (b) For the same set of couplings as in (a), the squared stair-amplitude  $\bar{Z}_k^2$  versus the node degree k, which verifies the theoretical predication of equation (2). Inset: the inverse relationship between the critical coupling  $\varepsilon_k$  and the node degree k. The symbols are the numerical results, and the solid line is drawn according to equation (3). Each data point in (b) is an average over 50 network realizations.

to note that, at the end of the first stage of the transition, the non-synchronized oscillators have small but finite amplitudes (of the order of  $10^{-4}$ ).

The formation of the amplitude stairs in the first stage could be analyzed based on a mean-field approximation. Note that equation (1) can be rewritten as  $\dot{Z}_i(t) = [(r^2 - \varepsilon k_i) + i\omega_i - |Z_i|^2]Z_i(t) + G_i$ , where  $G_i = \varepsilon \sum_{j=1}^N a_{ij}Z_j$  is the collective coupling received by node *i*. When  $\varepsilon$  is small, most oscillators of the network will behave incoherently; therefore  $G_i$  is small and negligible compared to the node dynamics, especially for the large-degree nodes. With this concern, the node dynamics can be simplified to  $\dot{Z}_i(t) = [(r^2 - \varepsilon k_i) + i\omega_i - |Z_i|^2]Z_i(t)$ . By requiring  $d|Z_i(t)|/dt = 0$ , we obtain

$$|Z_i|^2 = r^2 - \varepsilon k_i. \tag{2}$$

Equation (2) indicates that, for a given coupling strength, the amplitude of a node is dependent only on the node degree. This explains why the height of the stairs in figure 2(a) is decreasing with the node degree in a hierarchical fashion. From equation (2) we can also estimate the

averaged amplitude for each stair,  $\overline{Z}_k = \sqrt{r^2 - \varepsilon k}$ , which is verified by the numerical data (figure 2(a)). Moreover, by setting  $|Z_i| = 0$  in equation (2), we can obtain the critical coupling  $\varepsilon_k$  where the *k*-degree stair is eliminated,

$$\varepsilon_k = r^2/k,\tag{3}$$

which, again, is consistent with the numerical results (the inset plot of figure 2(b)).

It should be mentioned that the above analysis works only for the case of weakly coupled networks. If the coupling strength is not too weak, some nodes in the network could behave coherently. In such a case, the value of  $G_i$  will deviate from 0, and the flat stairs predicted by equation (2) will be disturbed. This is just what we observed in our simulations. In figure 2(a), despite the changes of the coupling strength, there always exist some outliers in the distributions. These outliers, which have random locations and various amplitudes, are directly resulting from the synchronized nodes. The picture is the following. Since the node frequency is randomly chosen, there could be the situation that some connected nodes in the network have very small frequency mismatch. As the coupling strength increases, these nodes will be easily synchronized and form some synchronous clusters (phase synchronization [22]). Once synchronized, these nodes will be efficiently protected from AD [13, 15], showing up as the amplitude outliers. The synchronized nodes, however, may have different amplitudes, as they could be embraced by different sets of neighbors. To smooth out the amplitude outliers caused by synchronization, we have calculated the averaged stair amplitude,  $\overline{Z}_k$ , as a function of k. Now a smooth, hierarchical death of the network oscillators is evidenced (figure 2(b)).

We proceed to investigate the transition in the second stage. At the end of the first stage, a number of synchronous clusters of diverse size are formed in the network. Because of synchronization, the amplitudes of the synchronized nodes are apparently larger than the nonsynchronized ones. In the second stage, as the coupling strength increases, *both the number and the size of the synchronous clusters are decreased*. Interestingly, it is found that the robustness of a cluster is mainly determined by the frequency mismatch between the synchronized nodes, but is less affected by the node degree or the cluster size. To show this point more clearly, we have slightly modified the network by artificially adjusting the frequency mismatch between two connected nodes, (147, 232), to be  $\delta \omega = 1 \times 10^{-3}$ . This pair of nodes, both having the smallest degree  $k_{\min} = 3$ , are synchronized at a very small coupling and are kept at large amplitudes till the very end of the transition. As shown in figure 3, despite the increase of the coupling strength, the amplitudes of these two nodes are always apparently larger than the rest of the nodes. In contrast to the situation of the first stage, in the second stage the node amplitudes seem to be independent of the node degree, which can be seen from figure 3(e).

At the end of the second stage, the system comes to a state of very few synchronous clusters. Typically, each cluster consists of only a small number of nodes that have very close natural frequencies, like the pair of nodes we discussed above. The few surviving clusters, however, are extremely robust and could persist for very large coupling strengths. These robust clusters lead to the new form of transition in the third stage: *A fast death of the non-synchronized nodes accompanied by a slow death of the synchronized nodes*. This property of network transition can be partially seen from figures 3(c) and (d), where it is shown that the amplitudes of nodes 147 and 232 are kept at large values during the transition, while the amplitudes of the other nodes are extremely small (see also figure 3(e)). Like in the second stage, in the third stage the node amplitudes are also less dependent on the node degree, as can be seen from figure 3(e). With the elimination of the most robust clusters, the whole transition is completed.



**Figure 3.** For a slightly modified network model, the amplitude arrangements observed in the second and third stages of the transition. (a)  $\varepsilon = 0.18$  and (b)  $\varepsilon = 0.19$  are within the second stage. (c)  $\varepsilon = 0.21$  and (d)  $\varepsilon = 0.22$  are within the third stage. (e) For the above amplitude arrangements, the variations of the squared average amplitude  $\overline{Z}_k^2$  as a function of k. The missed data in the last distribution are due to the limited computing precision. Like in figure 2(b), each data point in (e) is an average over 50 network realizations.

It is worth discussing the difference between the second and third stages. Unlike the transition of the first stage, in the last two stages the transition is independent of the node degree, which makes the difference between the last two stages not as distinct as the first stage. While the second and third stages share the common feature of monotonically decreased amplitudes for all the oscillators, their underlying mechanisms are quite different. In the second stage, driven by a large coupling, the small-amplitude, non-synchronized oscillators are going to cease their oscillations, but this trend is prevented by the 'dotted' synchronous clusters on the network. As the coupling strength increases, the synchronous clusters are gradually eliminated, making the non-synchronized oscillators difficult to be connected to the synchronized nodes. However, as long as there are a sufficient number of synchronized nodes in the network, the cessation of the non-synchronized nodes will be slowed down. Regarding this, we say the transition of the second stage is captured by the elimination of the synchronous clusters. In contrast, in the third stage, only a few synchronized nodes survive, which give very limited support to the remaining nodes. Without the support of the synchronized nodes, the numerous non-synchronized nodes are quickly ceased, shown as a fast decrease of E in the third stage. For instance, in the  $Z_0 = 1 \times 10^{-10}$  curve plotted in figure 1(b),  $N_d$  is increased quickly from 0 to N when  $\varepsilon$  is slightly changed within the third stage. It should be mentioned that the above characteristics of the second and third stages are drawn from our numerical observations, which can be improved by further investigations, for example, a detailed study of the behavior of the synchronized nodes.

Can the above phenomena of AD transition be found for the general dynamics, for example, the chaotic oscillators? To answer this, we have checked the transition of the same network but replacing the Landau–Stuart oscillators with the chaotic Rössler oscillators. The dynamics of a



**Figure 4.** Typical amplitude arrangements  $(R_i = \langle \sqrt{x_i^2(t) + y_i^2(t)} \rangle)$  observed in the transition of a scale-free network of chaotic Rössler oscillators. (a)  $\varepsilon = 0.039$  and (b)  $\varepsilon = 0.05$  are adopted from the first stage of the transition; (c)  $\varepsilon = 0.055$  and (d)  $\varepsilon = 0.066$  are adopted from the second stage of the transition; and (e)  $\varepsilon = 0.065$  and (f)  $\varepsilon = 0.07$  are adopted from the third stage of the transition.

single Rössler oscillator is described by  $F_i(\mathbf{x}) = [-\omega_i y_i - z_i, \omega_i x_i + 0.165 y_i, z_i (x_i - 10) + 0.2]$ , where  $\omega_i$  is the natural frequency of the *i*th oscillator. In our simulations,  $\omega_i$  is chosen randomly from the range [1, 3]. The coupling function is  $H(\mathbf{x}) = x$ . Figure 4 shows some typical amplitude arrangements collected at different stages of the transition, from which we can see that the main properties of the transition observed in the Landau–Stuart oscillators are well preserved. Besides changing the oscillator dynamics, we have also checked the influence of the network structure on the AD transition, including changing the network size and the degree distribution, adopting the clustered networks and considering the property of degree assortativity. The general finding is that, provided that the degree distribution is heterogeneous, the phenomena of staged transition and stair structures will be present.

To summarize, we have studied the transition to CAD in scale-free networks of nonlinear oscillators, and found the important role the node degree plays in AD generation. Since many practical systems where AD is an important consideration possess heterogeneous degree distributions, our findings should give some new insights into the relevant studies, for example, the stability and evolution of ecological networks [23, 24].

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