Chaotic synchronization through coupling strategies

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(Received 2 February 2006; accepted 17 March 2006; published online 11 May 2006)

Usually, complete synchronization (CS) is regarded as the form of synchronization proper of identical chaotic systems, while generalized synchronization (GS) extends CS in nonidentical systems. However, this generally accepted view ignores the role that the coupling plays in determining the type of synchronization. In this work, we show that by choosing appropriate coupling strategies, CS can be observed in coupled chaotic systems with parameter mismatch, and GS can also be achieved in coupled identical systems. Numerical examples are provided to demonstrate these findings. Moreover, experimental verification based on electronic circuits has been carried out to support the numerical results. Our work provides a method to obtain robust CS in synchronization-based chaos communications. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2193684]

A vast amount of research on synchronization in chaotic systems, in both theory and experiment, has emerged over the past fifteen years. So far, various types of chaotic synchronization, including complete synchronization (CS), generalized synchronization (GS), and phase synchronization (PS) have been extensively studied. Traditionally, CS is regarded as the synchronization form realized in coupled identical chaotic systems, while GS is expected to achieved in coupled different chaotic systems as the generalization of CS. In the present work, we attack an important question; i.e., whether the identity (or nonidentity) is a necessary condition for CS (or GS). Our study shows that the coupling strategies play an important role in determining the form of chaotic synchronization. By designing appropriate coupling schemes, enables us to select the synchronization form (CS or GS) no matter whether the coupled systems are identical or different. We further present an experiment based on electronic circuits to support our findings and point out the potential application of this study in chaotic communications.

I. INTRODUCTION

In the past decade, the synchronization of chaotic systems has attracted much attention for both theoretical interest and practical applications. Theoretically, these studies extended the theory of classic synchronization of periodic oscillators, and enhanced our understanding in both the behavior of chaotic systems and the synchronization phenomenon.[1–3] On the practical side, the combination of chaos synchronization and traditional communication theory has crystallized the application of chaos-based communications.[4–6] Experimentally, chaos synchronization has been extensively observed in lasers, electric circuits, chemical reactions, and biological systems.[7–10]

So far, different forms of chaos synchronization have been investigated and classified. Mainly, these include complete synchronization,[11–14] generalized synchronization,[15–23] and phase synchronization.[24–26] Among them, CS refers to the complete coincidence of the state variables between coupled chaotic systems; GS is manifested by a functional relationship between two coupled chaotic systems. Typically, CS is expected to occur in coupled identical chaotic systems, whereas GS is understood as the generalization of CS in coupled nonidentical chaotic systems. From this commonly accepted view, the possible synchronization mode seems to be determined before two identical or nonidentical systems are coupled. In some situations, this view turns out to be true; for example, in replacement scheme[11] and feedback scheme.[12] However, there have been observations that seem to be in contradiction with this view.[27,28] So far, whether the identity (or nonidentity) is a necessary condition for CS (or GS), has not been systematically investigated. For this purpose, in this work we study the relation between identity (or nonidentity) of coupled chaotic systems and the type(s) of synchronization that can be realized. We find that the commonly accepted view ignores the role that coupling schemes play in synchronization. In fact, it is generally impossible to predict the forms of synchronization in coupled chaotic systems without considering the specific coupling scheme. Whether the CS manifold or the GS manifold exists in coupled chaotic systems (they could be identical or nonidentical) strongly depends on the coupling strategies employed. By designing simple but appropriate coupling strategies, we have shown that CS can be observed in coupled chaotic systems with parameter mismatch, and GS can be achieved in coupled identical systems. We emphasize that the proposed coupling strategies in our work do not require any special properties of chaotic systems. Therefore, they are generally suitable in coupled chaotic systems. We provide numerical examples as well as experiments to demonstrate how to select CS or GS in coupled chaotic systems by designing appropriate coupling strategies.

The paper is organized as follows. In Sec. II, we show that GS can be achieved in coupled identical chaotic systems by using two simple coupling strategies; namely, the direct driving and the generalized feedback. In Sec. III, we study the synchronization between two coupled parametrically different chaotic systems by using the hybrid coupling strategy. In this case, it is found that either CS or GS can be achieved in the same coupled systems, depending on the details of the coupling. One interesting example even shows a smooth transition from GS to CS with the increase of the coupling strength. In Sec. IV, an experiment based on electronic circuit is carried out to demonstrate how to achieve CS between two chaotic oscillators with significant parameter mismatch. In the last section, we compare the current work with several previous works and point out the relation and difference between them. Finally, we end this paper with concluding remarks.

II. GS IN COUPLED IDENTICAL SYSTEMS

A. Direct driving coupling

We first consider two identical Lorenz systems in a drive-response configuration. The drive system is described by

$$\dot{x}_d = \sigma(y_d - x_d), \quad \dot{y}_d = rx_d - y_d - x_d z_d, \quad \dot{z}_d = x_d y_d - \beta z_d, \qquad (1)$$

and the response system with the coupling is

$$\dot{x}_r = \sigma(y_r - x_r) + \epsilon x_d, \quad \dot{y}_r = rx_r - y_r - x_r z_r, \quad \dot{z}_r = x_r y_r - \beta z_r. \qquad (2)$$

Here, the subscripts d and r denote the state variables in the drive and the response systems, respectively. This convention is followed throughout this paper. The parameters in both systems are the same; i.e., $\sigma=10$, $r=28$, and $\beta=8/3$. In this model, the response system is driven by the signal from the drive system through a coupling term ϵx_d in the x variable. This kind of coupling is neither a complete replacement scheme[11] nor a feedback scheme,[12] which are frequently used in chaos synchronization. We call this coupling strategy direct driving. In Eqs. (2), direct coupling is added to the response system via its x variable. Similarly, it can also be added to the response system via the y or z variable; i.e., the coupling term ϵy_d or ϵz_d can be added to the equation of the y or z variable in the response system, respectively.

For the coupling of direct driving, it can be easily verified that the CS manifold $\mathbf{x}_d=\mathbf{x}_r$ does not exist even though the drive and the response system are identical systems. It turns out that the specific coupling destroys the CS manifold that exists between two identical systems. (In the present work, we do not consider the trivial case where $\mathbf{x}_d=\mathbf{x}_r=0$.) Since this coupling strategy requires no special properties of the dynamical system, in principle it can be applied to any coupled identical chaotic systems. Generally, it can be expected that GS rather than CS could occur.

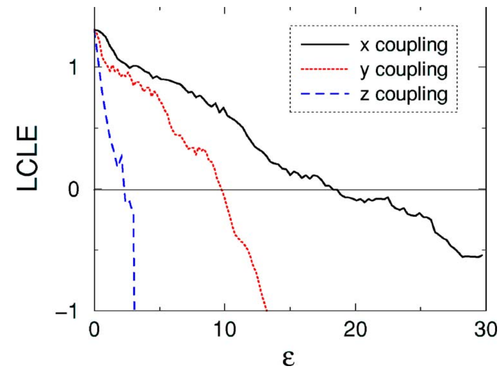

FIG. 1. The LCLEs vs the coupling strength for direct coupling in the x, y, and z variables in Eqs. (2), respectively.

Generally, in order to analyze the stability of synchronization, one should consider the Lyapunov exponents in the transverse space, which is orthogonal to the synchronization manifold; i.e., the transversal Lyapunov exponents (TLEs). The necessary condition for stable synchronization requires that all the TLEs are negative. For CS, the manifold $\mathbf{x}_d=\mathbf{x}_r$ defines the CS hyperplane and can be used as the reference manifold to compute the TLEs. It has been shown that in drive-response system, the TLEs are equivalent to the conditional Lyapunov exponents (CLEs).[1] The latter can be obtained by treating the response system as a separate dynamical system and calculating its LE as usual though it is under the driving from the drive system. For GS, usually we do not have the luxury to know or predict the GS manifold, which is generally a complicated functional relationship: $\mathbf{x}_d=\phi(\mathbf{x}_r)$. This makes it difficult to directly compute the TLEs of the coupled system when the stability of GS is concerned. However, by introducing an auxiliary system that is an exact copy of the response system, it has been shown that the stability of the GS manifold between the drive and response system is equivalent to the stability of the CS between the response and auxiliary system.[17] Since the response system and the auxiliary system are identical, the TLEs characterizing the stability of CS between them are equivalent to the CLEs for the response systems.

In the present study, two methods are used to characterize the synchronization. First, the largest conditional Lyapunov exponent (LCLE) is calculated; a negative LCLE indicates that synchronization may occur between the coupled systems.[29] Second, we directly apply the response-auxiliary system method to detect synchronization and distinguish whether it is CS or GS.[17] In Fig. 1, the synchronization between Eqs. (1) and (2) is characterized in terms of LCLE in the response system. For three cases of direct driving, it is found that the LCLE becomes negative with large enough coupling strength. Since a CS manifold does not exist between the drive and the response system, this synchronization should belong to the GS type. This is further verified by the response-auxiliary system method, as shown in Fig. 2.

In fact, the direct coupling scheme used in the above model can be easily generalized. For example, the coupling term can take the following form:

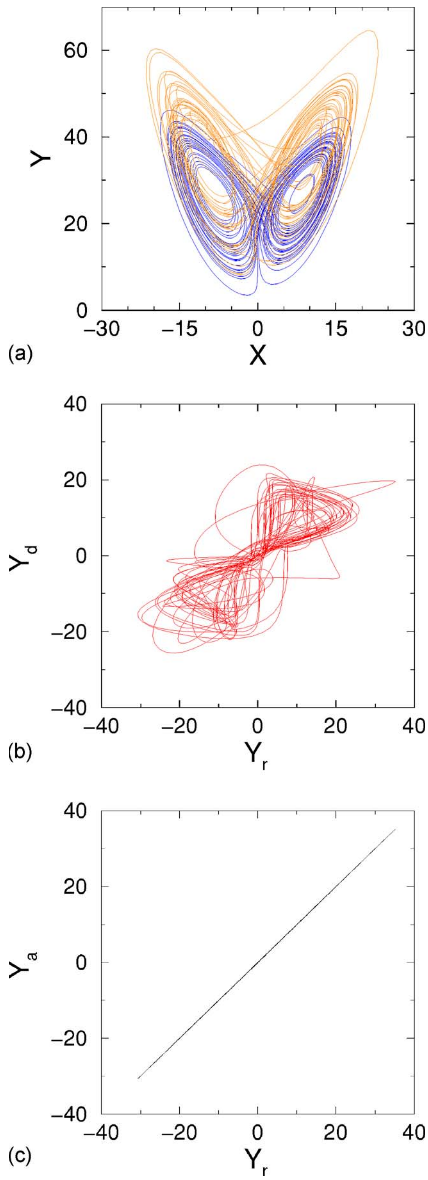

FIG. 2. GS rather than CS observed between two identical systems with direct coupling. $\epsilon=15$. (a) The attractors in the drive (black), response, and auxiliary systems (gray; they are totally overlapping). (b) y_d vs y_r. (c) y_a vs y_r. The subscripts d, r, and a denote the state variables in the drive, response, and auxiliary systems, respectively.

$$c(\mathbf{x}_d,\mathbf{x}_r) = f(\mathbf{x}_d), \tag{3}$$

where f is a usual function. If $c \neq 0$ when $\mathbf{x}_d=\mathbf{x}_r$, CS manifold cannot exist between these two identical systems coupled by such scheme. The only possible synchronization form for this coupled system, then, is GS.

B. Generalized feedback

Feedback techniques have been extensively used in chaos control and synchronization. Here we propose generalized feedback coupling schemes that can lead to GS rather than CS between two coupled identical chaotic systems. The coupled systems are still in the drive-response configuration. The drive system is the same as Eqs. (1), and the response system is

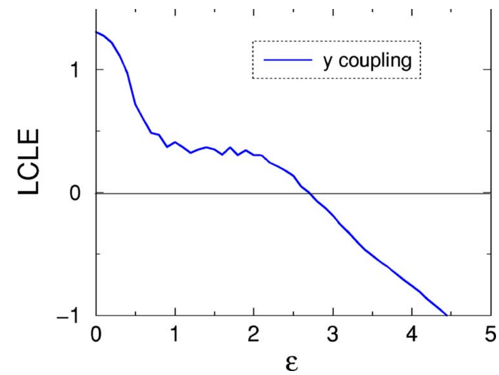

FIG. 3. The LCLE vs the coupling strength for Eqs. (4).

$$\begin{aligned} &\dot{x}_r = \sigma(y_r - x_r), \quad \dot{y}_r = r x_r - y_r - x_r z_r - \epsilon(y_r - \alpha y_d), \\ &\dot{z}_r = x_r y_r - \beta z_r, \end{aligned} \tag{4}$$

with $\sigma=10$, $r=28$, and $\beta=8/3$ for both systems. Here, α is a constant. Essentially, the coupling scheme in this set of equations is of the feedback type that includes the frequently used one, i.e., (y_r-y_d), as a special case with $\alpha=1$.

Here we consider $\alpha \neq 1$. It is easy to verify that no CS manifold $\mathbf{x}_d=\mathbf{x}_r$ exists in the coupled system. Therefore, if synchronization can be achieved, it must be GS rather than CS, although the two systems are identical without coupling. In Fig. 3, it is shown the LCLE in the response system becomes negative when the coupling strength is large enough, indicating that synchronization can be achieved. In Fig. 4, this synchronization is confirmed to be the GS type by the response-auxiliary system method.

In fact, the coupling scheme used in the above model can be easily generalized. For example, the coupling term can take the following form:

$$c(\mathbf{x}_d,\mathbf{x}_r) = f(\mathbf{x}_d,\mathbf{x}_r)[g(\mathbf{x}_r) - h(\mathbf{x}_d)], \tag{5}$$

where f, g, and h are usual functions. If $c \neq 0$ when $\mathbf{x}_d=\mathbf{x}_r$, CS is forbidden between two identical systems coupled by this generalized feedback scheme. The possible synchronization form for such a system should be GS. On the other hand, if $c=0$ when $\mathbf{x}_d=\mathbf{x}_r$, both CS and GS might be observed in this case.

C. GS between two identical spatiotemporal chaotic systems

To demonstrate that GS can be achieved between two identical spatiotemporal chaotic systems, we couple two arrays of N diffusively coupled Lorenz systems by direct driving coupling. The drive system is

$$\begin{aligned} &\dot{x}_d^i = \sigma(y_d^i - x_d^i) + D(x_d^{i+1} - 2x_d^i + x_d^{i-1}), \\ &\dot{y}_d^i = r x_d^i - y_d^i - x_d^i z_d^i, \quad \dot{z}_d^i = x_d^i y_d^i - \beta z_d^i, \end{aligned} \tag{6}$$

and the response system with driving is

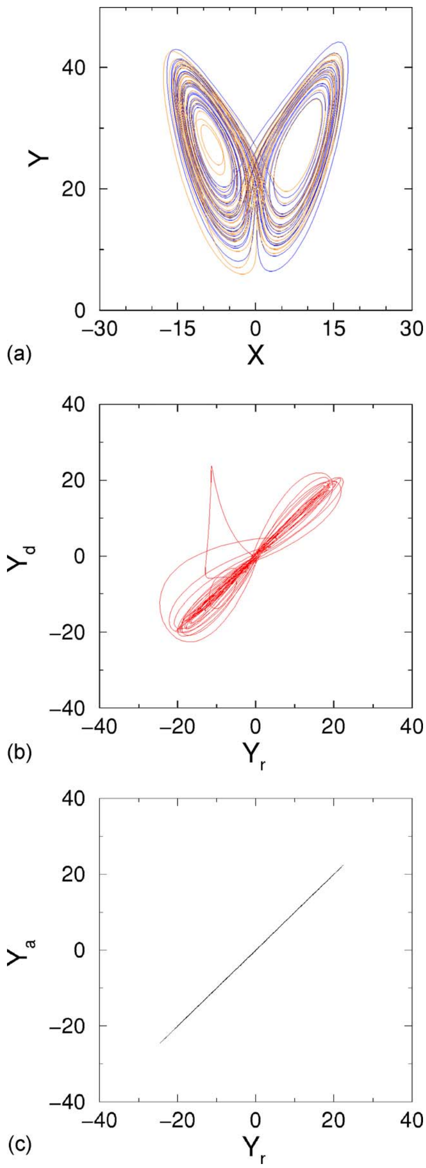

FIG. 4. GS rather than CS observed between two identical systems with generalized feedback coupling. $\epsilon=3$. (a) The attractors in the drive (black), response, and auxiliary systems (gray; they are totally overlapping). (b) y_d vs y_r. (c) y_a vs y_r.

$$\dot{x}_r^i = \sigma(y_r^i - x_r^i) + D(x_r^{i+1} - 2x_r^i + x_r^{i-1}),$$
$$\dot{y}_r^i = rx_r^i - y_r^i - x_r^i z_r^i + \epsilon y_d^i, \quad \dot{z}_r^i = x_r^i y_r^i - \beta z_r^i, \tag{7}$$

with $\sigma=10$, $r=28$, and $\beta=8/3$ for both systems. D is the coefficient of diffusive coupling inside the arrays. The response system is coupled to the drive system via direct driving in the y variable. For numerical simulations presented in this work, we assume periodic boundary conditions.

Obviously, the CS manifold cannot exist in these two spatiotemporal systems. Therefore, GS might be expected to be observed between them. In order to characterize synchronization in spatiotemporal systems, we define the time-averaged global synchronization error between the response and the drive system as

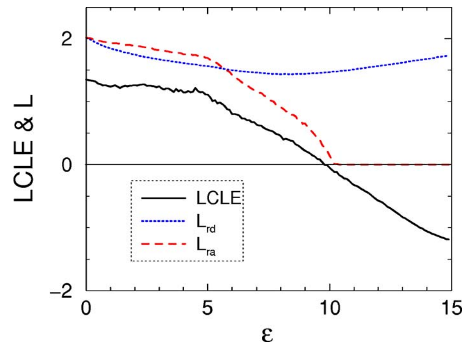

FIG. 5. GS rather than CS between two identical spatiotemporal chaotic systems using direct driving. $N=10$ and $D=0.3$.

$$L_{rd}(\epsilon) = \frac{1}{t_2 - t_1}\sum_{t_1}^{t_2}\left\{\frac{1}{N}\sum_{i=1}^{N}[(x_r^i - x_d^i)^2 + (y_r^i - y_d^i)^2 + (z_r^i - z_d^i)^2]^{1/2}\right\}. \tag{8}$$

Here, i is the index of the chaotic oscillator inside the array, and N is the total number of chaotic oscillators. The time average is carried out during a long period $[t_1, t_2]$, where t_1 should be large enough so that the system has passed the transient stage. Similarly, the synchronization error between the response and the auxiliary system L_{ra} can be defined by replacing the subscript d by a in Eq. (8). In Fig. 5, the LCLE and the global synchronization errors L_{rd} and L_{ra} are plotted versus the coupling strength. In the synchronization regime, where the LCLE is negative, it is seen that L_{ra} converges to zero, but L_{rd} does not. Obviously, the synchronization between these two identical spatiotemporal systems belongs to GS rather than CS due to the specific coupling.

III. CS IN COUPLED NONIDENTICAL SYSTEMS

A. CS between two parametrically different systems

In the previous section, we have shown that by designing appropriate coupling schemes, GS can be achieved between two identical dynamical systems. In this section, we further demonstrate that by designing flexible coupling schemes, CS can also be observed between two dynamical systems with parameter mismatch.

The model we studied consists of two Lorenz systems with parameter mismatch in drive-response configuration. The drive system reads

$$\dot{x}_d = \sigma(y_d - x_d), \quad \dot{y}_d = r_d x_d - y_d - x_d z_d, \quad \dot{z}_d = x_d y_d - \beta z_d, \tag{9}$$

and the response system with coupling is

$$\dot{x}_r = \sigma(y_r - x_r) - \epsilon(x_r - x_d),$$
$$\dot{y}_r = r_r x_r - y_r - x_r z_r - (r_r - r_d)x_d, \tag{10}$$
$$\dot{z}_r = x_r y_r - \beta z_r.$$

Here, the two systems have the same parameters $\sigma=10$, $\beta=8/3$, and the different parameters $r_d=28$ and $r_r=30$. What makes this model special is that the response system is driven by two types of driving signals from the drive system. The first coupling term is of the feedback type, which is added in the equation of the x variable. Moreover, there is a second direct driving term in the equation of the y variable, which is proportional to the parameter mismatch between the two systems. Therefore, this coupling term can be understood as the parameter mismatch compensator. In fact, the feedback coupling term can also be added to the response system through the y or z variable, while the parameter mismatch compensator keeps unchanged in the equation of the y variable. We denote these three situations as the coupling strategies A, B, and C, corresponding to the feedback coupling through the x, y, and z variables, respectively.

It is generally believed that CS is forbidden between two coupled systems with parameter mismatch. However, in the present model, due to the special hybrid coupling strategy, it can be easily verified that the CS manifold exists even though there is parameter mismatch between the two coupled systems. If this CS manifold is asymptotically stable within certain parameter range, CS could be observed between the two nonidentical systems. On the other hand, the current coupling strategy does not exclude GS manifold in the coupled systems. Therefore, there are also possibilities of the coexistence of CS and GS in the model studied. Whether CS or GS can occur in the system depends on the stability of the CS and GS manifolds.

We numerically studied the synchronization between Eqs. (9) and (10). The characterization of synchronization between Eqs. (9) and (10) with coupling strategies A, B, and C is shown in Fig. 6. For coupling strategy A, as shown in Fig. 6(a), both L_{rd} and L_{ra} converge to zero at the bifurcation point of synchronization. This implies that the synchronization achieved is the CS type. For coupling strategy B, an interesting synchronization bifurcation is observed as shown in Fig. 6(b). It is found that with the increase of coupling strength, there are three parameter regimes. In the first regime, i.e., $\epsilon < 1.5$, there is no synchronization between the coupled systems. In the second regime, i.e., $1.5 \leq \epsilon < 2.65$, the LCLE becomes negative, indicating that synchronization has achieved between the coupled systems. The synchronization type can be further identified as GS since the synchronization error $L_{ra}=0$, while $L_{rd}>0$ in this regime. A concrete example of GS at $\epsilon=2$ is shown in Figs. 7(a)–7(c). The system enters into the third regime when $\epsilon > 2.65$. In this regime, the LCLE is negative and both L_{ra} and L_{rd} converge to zero. Obviously, this belongs to the case of CS. An example of CS at $\epsilon=3$ is shown in Figs. 7(d)–7(f). Therefore, for the coupling strategy B, we have observed a smooth transition from GS to CS with the continuous increase of the coupling strength. So far, two transitions between the different synchronization forms have been found. One is the transition between GS and PS.[26,30,31] The other is the transition between PS and lag synchronization.[32] To our knowledge, the transition between GS and CS has not been reported before. We noticed that in Ref. 33, GS is observed between coupled identical system. However, in that model, GS occurs when the coupling strength is negative enough (positive feedback coupling), while CS occurs when the coupling strength is

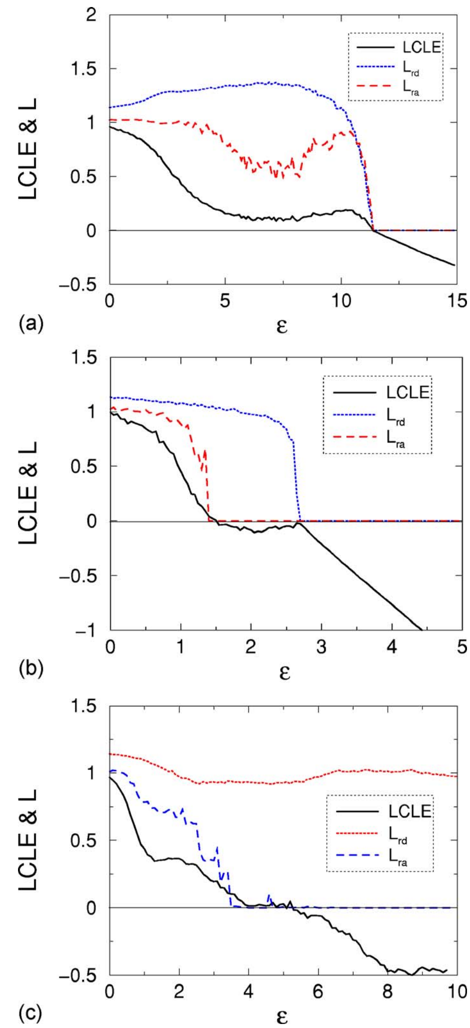

FIG. 6. Characterizing CS and GS between two coupled nonidentical systems. (a), (b), and (c) Coupling strategies A, B, and C, respectively.

positive enough (negative feedback coupling). There is no continuous transition from GS to CS. For coupling strategy C, a different synchronization scenario is observed, as shown in Fig. 6(c). In this case, the synchronization can be identified as GS, while CS does not occur.

To conclude, due to the special hybrid coupling strategy, it is possible to observe CS in coupled systems with parameter mismatch. Of course, GS may also happen in such systems. Interestingly, a smooth transition from GS to CS has been found in the current model. It should be pointed out that the coupling strategy proposed in this model has certain limitations. For some dynamical systems, it might be difficult to construct parameter mismatch compensator for certain control parameters. In experiments, it may also happen that the parameter mismatch exists in a state variable which, however, is inaccessible for feedback. Nevertheless, for a variety of nonlinear dynamical systems that are actually partially linear, for example, the Lorenz system, the Rössler system, and the Chua circuits, etc., the proposed coupling strategy can be applied in a straightforward way.

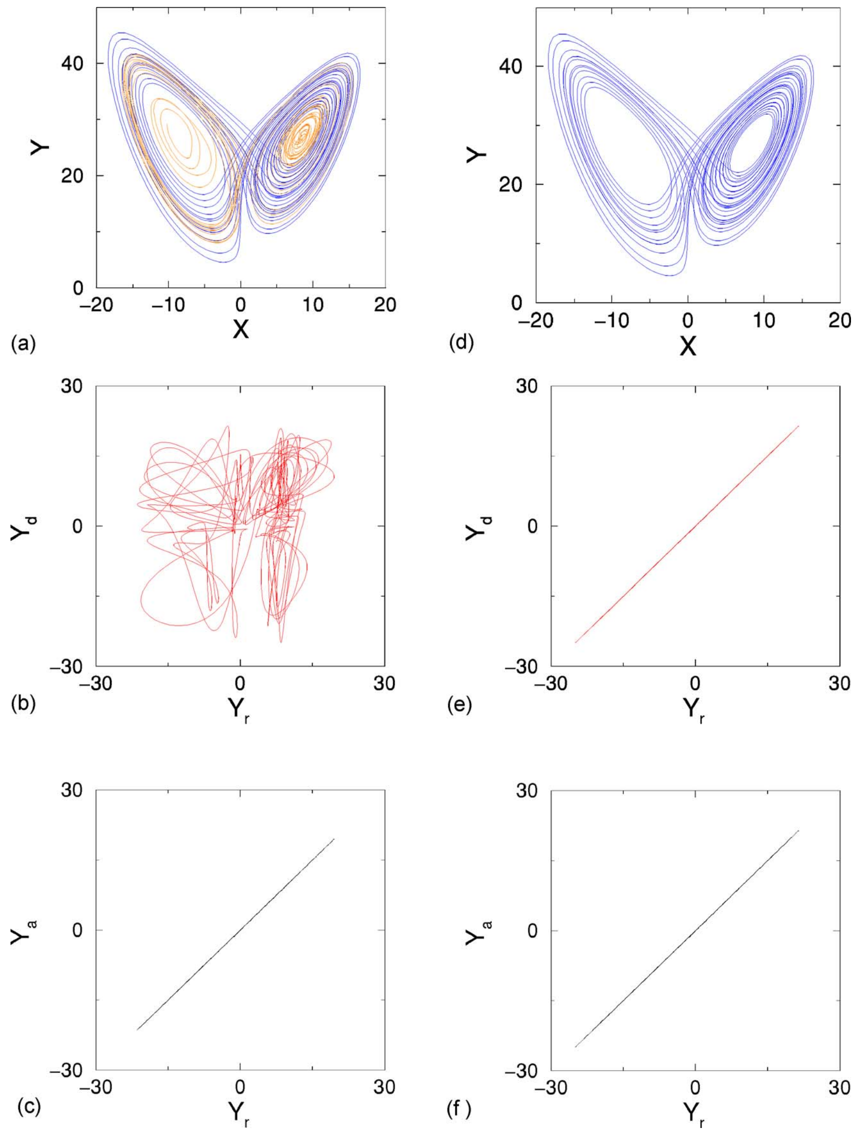

FIG. 7. Both CS and GS are observed for coupling strategy B in systems (9) and (10). $\epsilon=2$ for (a), (b), and (c); $\epsilon=3$ for (d), (e), and (f). (a) and (d) The attractors in the drive, response, and auxiliary systems. In (a), the attractors in the response and auxiliary system coincide, but they do not coincide with the attractor in the drive system. In (d), the attractors in the drive, response, and auxiliary systems all coincide with each other. (b) and (e) y_d vs y_r. (c) and (f) y_a vs y_r.

B. CS between two nonidentical spatiotemporal chaotic systems

We further demonstrate that CS can also be achieved between two nonidentical spatiotemporal chaotic systems. To this end, we consider two arrays of chaotic oscillators. Each array consists of N diffusively coupled Lorenz systems. The drive system is the same as Eqs. (6), with $r_d=28$. The response system with the driving is

$$\dot{x}_r^i = \sigma(y_r^i - x_r^i) + D(x_r^{i+1} - 2x_r^i + x_r^{i-1}),$$

$$\dot{y}_r^i = rx_r^i - y_r^i - x_r^i z_r^i - \epsilon(y_r^i - y_d^i) - (r_r - r_d)x_d^i, \tag{11}$$

$$\dot{z}_r^i = x_r^i y_r^i - \beta z_r^i,$$

with $r_r=30$. $\sigma=10$ and $\beta=8/3$ are the same for both systems. D is the coefficient of diffusive coupling inside each array. In numerical simulations, periodic boundary conditions are assumed. Similarly, the response system is driven by two signals from the drive system. One is the feedback coupling term, and the other is the parameter mismatch compensator. Due to this hybrid coupling strategy, it can be verified that the CS manifold exists between these two spatiotemporal systems. Numerically, it is found that this CS manifold is asymptotically stable when the coupling strength ϵ is large enough. As shown in Fig. 8, with the increase of ϵ, the LCLE becomes negative. In the meantime, both synchronization errors L_{rd} and L_{ra} converge to zero, showing that CS rather

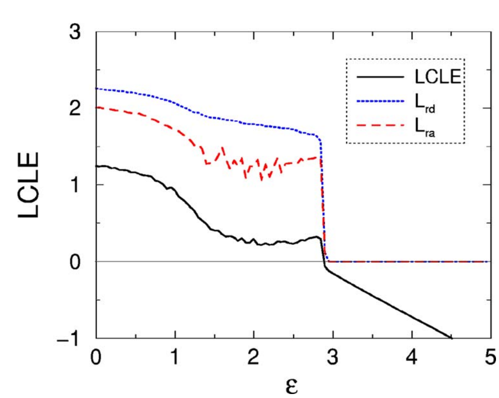

FIG. 8. CS between two nonidentical spatiotemporal chaotic systems (6) and (11). $N=10$ and $D=0.3$.

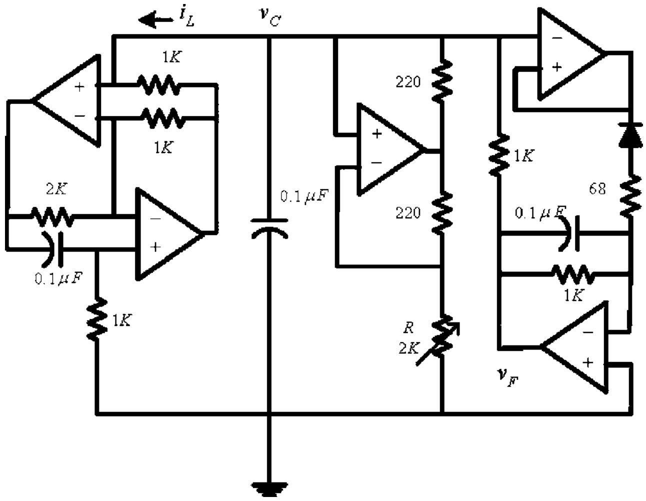

FIG. 9. The electronic circuit of the *LC* oscillator.

than GS has been achieved between these two different spatiotemporal systems.

IV. EXPERIMENTS

To verify the above numerical results, we carried out experiments based on a chaotic electronic *LC* oscillator to demonstrate how to achieve CS between two chaotic systems with parameter mismatch. The schematic of the electronic circuit is shown in Fig. 9.[34] Its dynamical equations can be modeled as

$$L\frac{di_L}{dt} = v_c, \quad C\frac{dv_c}{dt} = \frac{v_c}{R} - i_L + \frac{v_F - v_c}{1\ \mathrm{K}\Omega}, \quad C\frac{v_F}{dt} = -\frac{v_F}{R} - f(v_c), \tag{12}$$

$$f(v_c) = \begin{cases} \dfrac{V_D + v_c}{68\ \Omega}, & v_c \leqslant -V_D, \\ 0, & v_c > -V_D. \end{cases}$$

Here, i_L is the current through the tank inductor; v_c the voltage across the tank capacitor; v_F the voltage in the folding circuit; $V_D \simeq 0.7$ the voltage drop for a conducting diode; $C = 0.1\ \mu F$ the tank capacitance; and R the variable resistance for tuning the chaotic behavior of the circuit. The tank inductor is implemented using a general impedance converter, with equivalent inductance $L = 0.2$ H. At $R = 700\ \Omega$, the circuit is in the chaotic regime, exhibiting chaotic attractor-like behavior shown in Fig. 10(a).

The response system is a copy of the drive system, except that we artificially set the resistor in the folding circuit to be 50 Ω. This generates about 30% parameter mismatch between the drive and the response system. The response system with feedback coupling and parameter mismatch compensator is modeled as

$$L\frac{di'_L}{dt} = v'_c, \quad C\frac{dv'_c}{dt} = \frac{v'_c}{R} - i'_L + \frac{v'_F - v'_c}{1\ \mathrm{K}\Omega} - R_1(v_C - v'_C),$$

$$C\frac{v'_F}{dt} = -\frac{v'_F}{R} - f(v'_c), \tag{13}$$

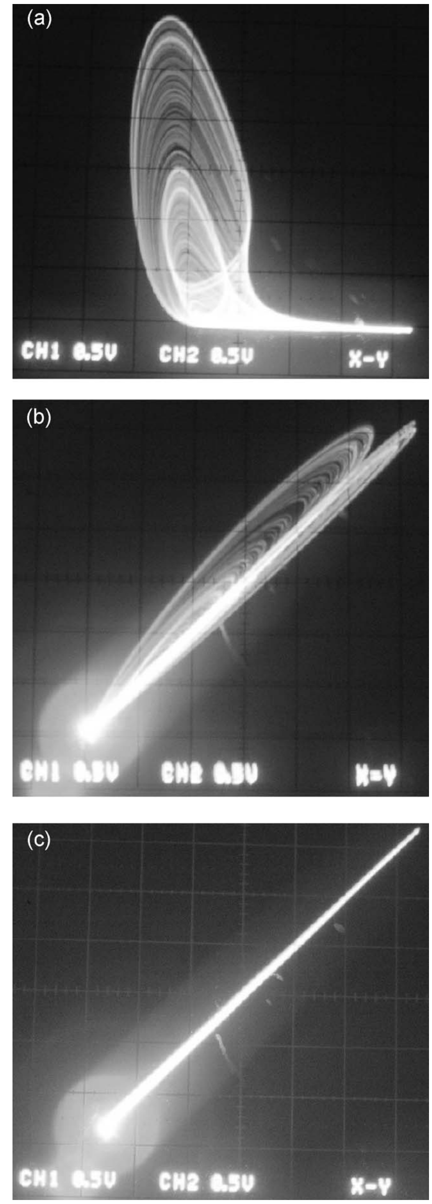

FIG. 10. (a) The chaotic attractor of the *LC* oscillator, v_C vs v_F. (b) and (c) The synchronization manifold v_F vs v'_F. (b) The response system with feedback coupling but without parameter mismatch compensator. (c) The response system with both feedback coupling and parameter mismatch compensator.

$$f(v'_c) = \begin{cases} \dfrac{V_D + v_c}{50\ \Omega} + R_2(v_F - v'_F) - \dfrac{1}{R_C}(V_D + v_c), & v_c \leqslant -V_D, \\ 0, \quad v_c > -V_D. \end{cases}$$

In the experiment, $R_1 = 12\ \mathrm{K}\Omega$ and $R_2 = 10\ \mathrm{K}\Omega$ are the coupling strengths, and R_c controls the strength of parameter mismatch compensator. First, two oscillators are coupled through the usual feedback couplings without the parameter mismatch compensator. In this case, the synchronization between the drive and the response system is of the GS type. The functional relation is shown in Fig. 10(b). We then switch on the parameter mismatch compensator and gradually increase R_c in the response system. At certain appropriate value of R_c, it can be observed that CS can be achieved between the two systems as shown in Fig. 10(c).

V. DISCUSSION AND CONCLUDING REMARKS

There are several previous works that are relevant to the current study.[27,28,32,33,35–38] Here, we discuss the relation and difference between them and our work. In Ref. 27, a model of coupled identical chaotic systems is shown to display CS combined with GS, depending on initial conditions. Let us rewrite Eqs. (2) and (5) in Ref. 27 in the following. Equation (2) in the drive system is

$$\dot{x}_2 = -x_2 - (x_3 - R)x_1, \tag{14}$$

with R a parameter. The corresponding equation (3) in the response system is

$$\dot{y}_2 = -y_2 - (x_3 - R)y_1, \tag{15}$$

which is a copy of the drive system, except that y_3 in Eq. (14) has been replaced by the driving variable x_3. The rest of the equations in the drive system are the same as that in the response system. For this replacement coupling, we can change the form of Eq. (15) into

$$\dot{y}_2 = -y_2 - (y_3 - R)y_1 + y_1(y_3 - x_3). \tag{16}$$

Now we can see that this model is equivalent to two identical systems coupled with a nonlinear feedback term $y_1(y_3-x_3)$, which can be classified into the generalized feedback couplings studied in Sec. II of this paper. Therefore, the results in Ref. 27 can be well understood in the framework of the current study. Interestingly, CS and GS are found to coexist in Ref. 27, depending on initial conditions. In this work (model A in Sec. III), we further found that CS and GS can coexist in one dynamical model, depending on parameter regimes. In another work,[28] the following coupling scheme is used. The involved equation in the drive system is [Eq. (6) in Ref. 28]

$$\dot{z} = -\beta[z - B - R(s_1 + s_2)], \tag{17}$$

with z, s_1, s_2 variables and B, R parameters. The counterpart of Eq. (17) in the response system is

$$\dot{z}' = -\beta[z' - B - R(s_1 + s_2)], \tag{18}$$

with z' a variable in the response system and s_1, s_2 the driving variables. The other equations of the drive system and the response system are the same. Similarly, Eq. (18) can be rewritten as

$$\dot{z}' = -\beta[z' - B - R(s_1' + s_2')] - \beta R[(s_1' + s_2') - (s_1 + s_2)], \tag{19}$$

where the variables with primes are in the response system. Apparently, this model can also be regarded as two identical systems coupled through a lumped feedback term. Thus, it is not strange to observe GS in this model according to the analysis of generalized feedback in Sec. II of this paper. In addition, the present study is different from previous works,[35,36] where synchronization between different systems is achieved by special chaos control techniques. Our study is also different from the work in Refs. 32, 37, and 38, where approximate CS, i.e., CS in a practical sense rather in a strict sense, is achieved between coupled systems with parameter mismatch.

Since the primary aim in this work is to illustrate unusual synchronization phenomena by adopting appropriate coupling strategies, we focus our studies on the unidirectionally coupled dynamical systems. However, the proposed coupling strategies can be easily extended in the bidirectionally coupled dynamical systems.[39] For example, the models in Sec. II can be modified to bidirectionally coupled systems by introducing couplings in the drive systems. The couplings can be constructed in flexible ways as long as they still destroy the CS manifold in these systems. For the model in Sec. III A, the modification to bidirectional coupling is straightforward. For example, we can simply add a feedback term in the equation of x variable in Eqs. (9). In this case, it is found numerically that when the coupling strength exceeds 11.5, CS occurs between the bidirectionally coupled nonidentical systems.

To summarize, in this paper we investigated the role that the coupling strategy plays in determining the synchronization type between coupled chaotic systems. By designing appropriate coupling schemes, it has been shown that GS could be observed in coupled identical systems, and CS could also be achieved in coupled nonidentical systems. Our results thus reveal that it is impossible to accurately predict the synchronization type based only on whether the uncoupled chaotic systems are identical or not. To draw a correct conclusion, the specific coupling scheme must be considered together with the dynamical equations. We emphasize that the coupling strategies used in the current work are simple and flexible; thus, they can be generally applied to many other dynamical systems.

The present finding not only deepens our understanding about synchronization between coupled chaotic systems, but also has potential application in synchronization-based chaotic communications. Currently, most of the synchronization-based communication schemes depend heavily on the CS between the dynamics of the transmitter and the receiver. However, due to the inevitable parameter mismatch in practice, the transmitter and the receiver cannot be perfectly identical. In fact, any synchronization between coupled chaotic systems in practice should belong to GS rather than CS. In certain systems, if the chaotic synchronization is highly sensitive to a parameter mismatch, the strategies demonstrated in this work can be applied so that strict and robust CS can be achieved for communication purposes.

ACKNOWLEDGMENT

This work was supported by the Temasek Laboratories at National University of Singapore.

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