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Spontaneous formation of dynamical groups in an adaptive networked system

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Abstract. In this work, we investigate a model of an adaptive networked dynamical system, where the coupling strengths among phase oscillators coevolve with the phase states. It is shown that in this model the oscillators can spontaneously differentiate into two dynamical groups after a long time evolution. Within each group, the oscillators have similar phases, while oscillators in different groups have approximately opposite phases. The network gradually converts from the initial random structure with a uniform distribution of connection strengths into a modular structure that is characterized by strong intra-connections and weak inter-connections. Furthermore, the connection strengths follow a power-law distribution, which is a natural consequence of the coevolution of the network and the dynamics. Interestingly, it is found that if the inter-connections are weaker than a certain threshold, the two dynamical groups will almost decouple and evolve independently. These results are helpful in further understanding the empirical observations in many social and biological networks.

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Modularity frequently occurs in many social and biological networked systems [1], which is generally believed to correspond to certain functional groups [2]. Usually, in modular networks, the intra-connections are stronger than the inter-connections [3]–[6]. However, they both play important roles in maintaining network structure and functions. Csermely pointed out that the strong links can define the system, while the weak links are crucial to the stabilization of the complex system [7]. Such examples can be found in many situations, such as the connectivity of social networks [3], group survival [8], social efficiency [9], firm efficiency [10] and ecosystem stability [11]. Furthermore, in many networks, such as natural food webs [12], mobile networks [3], author collaboration networks [13], metabolic networks [14] and neural networks [15], it is found that most of the interactions are weak and only a few interactions are strong, which usually leads to a power-law distribution of the connection strengths [3, 13–15].

In the past decade, there have been extensive works exploring networked complex systems. Mainly, these works focus on either the topological structures of the networks [16] or the dynamics on the networks [17]. Nevertheless, in various realistic systems, especially biological and social systems, in principle the network topology and dynamics are strongly dependent on each other. Thus any network structures and dynamical patterns that emerged are actually the results of the coevolution of the network dynamics and topology [18]. For example, the change of the synaptic coupling strength between neurons depends on the relative timing of the presynaptic and postsynaptic spikes in neural networks [19], and in mobile communication networks [3], the connection strengths are determined by the dynamical behavior of the mobile agents.

Recently, attention has been paid to adaptive coevolutionary networks [18]. These include adaptive rewiring links [20, 21] and adaptive altering connection strength [22]–[25] based on the states of local dynamics. However, previous studies still focus mainly on the topological properties of the networks, while neglecting the dynamical evolution and characteristics, which are actually a very important aspect of networked dynamical systems. We noticed that in many social and biological networked systems, with the evolution of network topology, dynamically the system may form different functional groups corresponding to different dynamical states. One such example is the mammalian brain, in which the connections are plastic [19]. It is known that the mammalian brain is composed of a number of functional groups, within which the nodes can be regarded as sharing similar dynamical states. However, so far, how the dynamical groups are generated during the coevolution of network structure and dynamics has not been investigated from the point of view of complex networks.

Motivated by this idea, in the present work, we set up a toy adaptive network model consisting of phase oscillators. Due to the simplicity of the dynamics, phase oscillators have been frequently used to describe many simplified real dynamical systems, such as biological networks, chemical oscillators and so on [26]. In our model, the coupling functions adopt the higher-order Fourier modes, and the connection strengths are coupled with the local dynamical states following the plasticity function. Particularly, we investigate what kinds of dynamical states and network structures can be formed as a result of the coevolution of both network dynamics and topology. Mainly, our study presents three new results. (i) The dynamical groups can be spontaneously formed in our model, i.e. in-phase and anti-phase synchronized states simultaneously exist in our system. In the previous work [27], although the desynchronized states and the synchronized states coexist and are both stable, the network only tends to be one of two states, depending on its initial mean coupling. However, in our model, the oscillators within (between) groups tend to in-phase (anti-phase) synchronization. (ii) The connection strengths in the network can self-organize into a power-law distribution from the initial random distribution. In addition, communities, which correspond to the dynamical groups in our model, can also be spontaneously formed. The community structure and the power-law distribution of the connection strengths are common in many empirical networked systems. (iii) The resource constraint can significantly affect the formation of dynamical groups. If the total connection strength is a finite constant, the network tends to split into two dynamical groups: within each group the oscillators are in-phase synchronized, while the oscillators in different groups are anti-phase synchronized. However, if there is no resource constraint, the two groups finally merge into one.

In our model, the dynamical equations for the networked phase oscillators read

$$\dot{\theta}_m = \omega_m + \gamma \sum_{k=1}^{N} w_{mk}\Gamma(\theta_k - \theta_m). \tag{1}$$

Here, $m, k = 1, 2, \ldots, N$ are the oscillator (node) indices and γ is the uniform coupling strength. θ_m and ω_m are the instantaneous phase and intrinsic frequency of the mth oscillator, respectively. $W = \{w_{mk}\}$ $(w_{mk} = w_{km})$ is the weighted coupling matrix, where $w_{mk} > 0$ is the coupling strength if nodes m and k are directly connected, and $w_{mk} = 0$ otherwise. In order to generate possible dynamical groups in our model, we tentatively choose the coupling function $\Gamma(\phi)$ as the higher order of Fourier modes, i.e. $\Gamma(\phi) = \sin(h\phi)$ $(h = 2, 3, 4, \ldots)$ [28], where the parameter h can control the number of groups. Without losing generality, we set $h = 2$ throughout this paper.

In the coevolutionary networked system, how the network topology couples with the dynamics is crucial to both the dynamical pattern and topological structure that result. In our model, we propose a coupling rule for the connection strength w_{mk} based on the following hypothesis: w_{mk} is a finite real number, and the connections will be strengthened (weakened) if the phase differences are smaller (bigger) than some threshold α. Actually, this can be regarded as an extension of the spike-timing-dependent plasticity (STDP) rule [19]. In fact, in many realistic networked systems, individuals with similar states usually tend to form the group that has relatively stronger intra-connections inside. For instance, in human society, individuals with similar attributes are easily organized into the same communities [4, 29, 30]. Meanwhile, similarity will breed connection [30], indicating that relations among individuals with similar attributes may be constantly strengthened, whereas those among individuals with dissimilar attributes may be gradually weakened. Based on the above consideration, the change of the connection strength is assumed to satisfy the following equation:

$$\frac{\mathrm{d}w_{mk}}{\mathrm{d}t} = \epsilon w_{mk}\Theta(\Delta\theta_{mk}, \alpha)\Lambda(\Delta\theta_{mk}), \tag{2}$$

where $\Delta\theta_{mk} = |\theta_k - \theta_m|$ $(0 \leqslant \Delta\theta_{mk} \leqslant \pi)$ is the phase difference between oscillators m and k. w_{mk} on the right-hand side of the equation ensures that the rate of change of the link weight is proportional to itself, and $w_{mk} \geqslant 0$ always. ϵ is a constant that can be chosen to make the time scale of the network topology evolution much longer than that of the local dynamics of the oscillators. The function $\Theta(\phi, \alpha)$ determines how the coupling strength evolves according to the phase difference between oscillators. In this study, we set $\Theta(\phi, \alpha) = \mathrm{e}^{-2|\phi-\pi/2|}$.

The function $\Lambda(\phi)$, which is similar to the sign function, controls either the strengthening or weakening of the connections based on the phase differences. For simplicity, we assume the form $\Lambda(\phi) = \Gamma(\phi)$. The form $\Theta(\phi, \alpha)\Lambda(\phi)$ is similar to the STDP rule, which has been widely used in neural network studies [19, 27]. The difference is that the STDP rule [19, 27] depends on the relative timing Δt of presynaptic and post-synaptic spikes and the critical window τ, while the plasticity function in our model depends on the phase difference $\Delta\theta$ and the connection strength itself. In addition, the exponential function $\Theta(\phi, \alpha) = \mathrm{e}^{-2|\phi-\pi/2|}$ is modulated by the sine function $\Lambda(\phi) = \sin(2\phi)$, which makes the plasticity function not a monotone function on the same side of the threshold value, e.g. $\Delta\theta < \pi/2$.

With the above assumptions, the model is fully described by

$$\begin{aligned} \dot{\theta}_m &= \omega_m + \gamma \textstyle\sum_{k=1}^{N} w_{mk} \sin[2(\theta_k - \theta_m)], \\ \frac{\mathrm{d}w_{mk}}{\mathrm{d}t} &= \epsilon w_{mk} \mathrm{e}^{-2|\Delta\theta_{mk}-\pi/2|} \sin(2\Delta\theta_{mk}). \end{aligned} \tag{3}$$

In this study, the natural frequencies and initial conditions of the oscillators are chosen randomly from the range [0, 1] and $[-\pi, \pi]$, respectively. It is known that in many realistic adaptive networks, the 'resource', which can be represented by the summation of all connection strengths in the network, is usually limited. Consequently, all connections will compete for this resource. Therefore, in our model we define the 'resource' as $M = L\langle w\rangle$, where L is the number of total connections and $\langle w\rangle$ is the average connection strength. In our simulation, we use the normalization $\langle w\rangle = 1$ during the evolution in order to make the 'resource' $M = L$, i.e. the total 'resource' to be allocated is a constant during evolution.

The collective behavior of the dynamical system can be conveniently described by two order parameters R and F. The order parameter R, which characterizes whether global coherence occurs or not, is defined as

$$R = \frac{|\sum_{m=1}^{N} s_m \mathrm{e}^{\mathrm{i}\theta_m}|}{\sum_{q=1}^{N} s_q}, \tag{4}$$

where s_m is the strength of node m, i.e. $s_m = \sum_k w_{mk}$. This type of order parameter has been widely used to characterize phase synchronization in complex networks [31]. From the definition in papers [31], it seems natural to use equation (4) as the order parameter in weighted networks. The order parameter F, which measures the fraction of all link weights synchronized in networks [32], is defined as

$$F = \frac{|\sum_{mk} w_{mk} \mathrm{e}^{\mathrm{i}(\theta_m-\theta_k)}|}{\sum_{lq} w_{lq}}. \tag{5}$$

In adaptive oscillator networks where connections are coupled with dynamical states, the order parameters R and F can be jointly used to characterize whether local coherence within the subnetwork takes place. For example, if $R \approx 0$ and $F \gg R$ after a long time evolution from random initial phases on random networks, this indicates that local synchronization (rather than global synchronization) emerges within subnetworks, i.e. dynamical groups have been generated in the system.

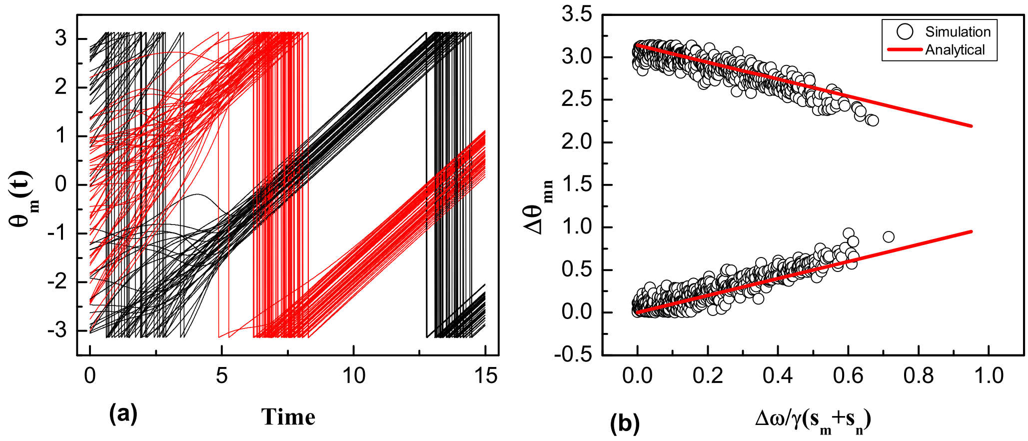

Figure 1. (a) Time evolution of the oscillator phases. After transients, oscillators spontaneously differentiate into two dynamical groups with different states. (b) Comparison between the analysis and the simulation results of phase differences among oscillators. The network parameters are $N = 100$, $\langle k \rangle = 20$, $\gamma = 0.04$ and $\epsilon = 0.01$. Initially $\omega_m \in [0, 1]$ and $w_{mn} = 1$.

First, we consider a simplified situation: a two-oscillator system. In this case, the dynamics can be rewritten in terms of two variables, $\Delta\theta = (\theta_1 - \theta_2)$ and w, as

$$\begin{aligned} \frac{\mathrm{d}\Delta\theta}{\mathrm{d}t} &= \omega_1 - \omega_2 - 2\gamma w \sin(2\Delta\theta), \\ \frac{\mathrm{d}w}{\mathrm{d}t} &= \epsilon w \mathrm{e}^{-2|\Delta\theta - \pi/2|} \sin(2\Delta\theta). \end{aligned} \tag{6}$$

From the above equations, we can see that if $|\Delta\omega| = |\omega_1 - \omega_2| = 0$, the system will have stable equilibrium states $\Delta\theta^* = 0$ or π, and the final connection strength w^* is a finite constant. These two states correspond to the in-phase synchronization and anti-phase synchronization of the two oscillators, respectively. If $|\Delta\omega| \neq 0$, strictly speaking the two-oscillator system does not have any equilibrium states. This implies that the coupling strength will always be varying during the evolution. Nevertheless, if the rate of change of the connection strength is much slower than the phase dynamics, we can approximately consider w as a constant. In this case, we can obtain the stable equilibrium states of $\Delta\theta$ provided that $|\Delta\omega| \leqslant 2\gamma w$, i.e.

$$\Delta\theta^* = \begin{cases} \dfrac{1}{2}\arcsin\left|\dfrac{\Delta\omega}{2\gamma w}\right|, \\ \pi - \dfrac{1}{2}\arcsin\left|\dfrac{\Delta\omega}{2\gamma w}\right|. \end{cases} \tag{7}$$

In our numerical simulations, the above analysis has been verified.

Next, we consider the case of a many-oscillator system. Without losing generality, the initial network topology is chosen as a random structure, and the initial connection strengths are chosen uniformly from the range (0, 2]. To monitor the evolution, we record the instantaneous phases of all the oscillators ($\{\theta_m(t)\}$). It is found that after the transient period, the oscillators can spontaneously separate into two dynamical groups. Within each group, all oscillators have similar phases. Meanwhile, the two dynamical groups as a whole tend to approximate anti-phase synchronization, as shown in figure 1(a). Through extensive numerical simulations, we found that the sizes of the two groups depend on the initial conditions. In general, they are almost equal to each other when the initial phases are chosen uniformly. Of course, if all the oscillators are identical, the coevolution can still generate two dynamical groups as a nonidentical system. In this case, the phase states within each group are strictly identical.

The collective behavior of the dynamical system with multiple dynamical groups can also be described by the following parameter [24]:

$$R' = \frac{|\sum_{m=1}^{N} s_m \mathrm{e}^{\mathrm{i}2\theta_m}|}{\sum_{q=1}^{N} s_q}. \tag{8}$$

If the order parameter R' converges to 1 and the order parameter R converges to 0, this also implies that dynamical groups have formed. The difference between F and R' is that F can characterize the properties of the dynamical states and the topology of weighted networks simultaneously, while R' can mainly characterize the properties of the dynamical states. In order to explain the formation of different dynamical groups in our model equation (1), we rewrite it in a more convenient form by defining the local order parameter according to equation (8):

$$r'_m \mathrm{e}^{\mathrm{i}2\psi_m} = \frac{1}{s_m} \sum_{k=1}^{N} w_{mk} \mathrm{e}^{\mathrm{i}2\theta_k}. \tag{9}$$

Here r'_m with $0 < r'_m < 1$ measures the local coherence among neighbors of oscillator m. ψ_m is the average phase and s_m is the strength of node m, i.e. $s_m = \sum_k w_{mk}$. With this definition, equation (1) becomes

$$\dot{\theta}_m = \omega_m + \gamma r'_m s_m \sin[2(\psi_m - \theta_m)]. \tag{10}$$

When $\gamma \to 0$, equation (10) yields $\theta_m \approx \omega t + \theta_m(0)$, that is, the oscillators evolve according to their own natural frequencies. The oscillators are neither in-phase nor anti-phase synchronized, i.e. $r'_m \to 0$ as $t \to \infty$. On the other hand, in the limit of strong coupling, the oscillators tend to anti-phase synchronization, $r'_m \to 1$ and $2\psi_m - 2\theta_m \approx 2q_m\pi (q_m = 0, \pm 1)$, i.e. $2\psi_m - 2\theta_m - 2q_m\pi \approx 0$. Consequently, equation (10) can be rewritten as

$$\dot{\theta}_m = \omega_m + 2\gamma s_m (\psi'_m - \theta_m), \tag{11}$$

where $\psi'_m = \psi_m - q_m\pi$. Thus, the phase difference $\Delta\theta_{mn} = \theta_m - \theta_n$ between m and n becomes

$$\frac{\mathrm{d}\Delta\theta_{mn}}{\mathrm{d}t} = \omega_m - \omega_n + 2\gamma[s_m(\psi'_m - \theta_m) - s_n(\psi'_n - \theta_n)]. \tag{12}$$

From $\frac{\mathrm{d}\Delta\theta_{mn}}{\mathrm{d}t} = 0$, we can obtain the equilibrium value $\Delta\theta_{mn}$, i.e.

$$\Delta\theta_{mn} = \frac{\Delta\omega_{mn}}{\gamma(s_m + s_n)} + \psi'_m - \psi'_n + \frac{s_m - s_n}{s_n + s_m}(\psi'_m - \theta_m + \psi'_n - \theta_n), \tag{13}$$

where $\Delta\omega_{mn} = \omega_m - \omega_n$, and $\frac{s_m - s_n}{s_n + s_m}(\psi'_m - \theta_m + \psi'_n - \theta_n)$ is the high-order infinitesimal, which can be neglected. When oscillators m and n tend to in-phase (anti-phase) synchronization,

$\psi'_m - \psi'_n \approx q\pi$ $(q = 0, \pm 1)$, so the equilibrium values of the phase difference $\Delta\theta^*_{mn}$ are

$$\Delta\theta^*_{mn} = \begin{cases} \dfrac{|\Delta\omega_{mn}|}{\gamma(s_m + s_n)}, \\ \pi - \dfrac{|\Delta\omega_{mn}|}{\gamma(s_m + s_n)}. \end{cases} \tag{14}$$

As shown in figure 1(b), our numerical simulations of the phase differences are consistent with the analytical results.

Physically, the spontaneous formation of two different dynamical groups in our model can be attributed to the adaptive evolution rule described by equation (2). Based on this equation, the connection strength among oscillators with initially close phases will be enhanced. Meanwhile, if two oscillators initially have large phase difference (e.g. $\Delta\theta > \pi/2$), the connection strength between them will be weakened during evolution. As a combined effect of these two 'forces', the networked oscillators self-organize into two dynamical groups after a long time evolution. Within the same group, the oscillators have similar states, while oscillators in different groups have approximate anti-phases. Interestingly, in many social and biological systems, we often find that two groups are formed with opposite states. For instance, in human society, individuals with homogeneous character, e.g. the same generation or living in the same neighborhood, are disposed to associate [4], and conflicting (accordant) characters could weaken (strengthen) the social contacts. In food webs, if the living habits of predator and prey are similar (different), the predator–prey relationships are strong (weak) [6]. Our model can thus shed light on the origin of the formation of such dynamical groups.

With the formation of dynamical groups, how the network structure evolves is another important question. In this work, we do not consider the rewiring of network connections. Instead, we fix the network topology and focus on how the network connections compete for the limited 'resource', i.e. the reallocation of the connection strength. At every time step, we normalize $\langle w \rangle = 1$, i.e. $w^*_{mn} = \frac{M w_{mn}}{\sum_{j>i} w_{ij}}$, in order to make the 'resource' $M = L$. In figure 2, we illustrate, using one typical example, the properties of the network structure. As shown in figure 2(a), the oscillators in the network self-organize into two dynamical groups with different phase states, i.e. oscillators within the same group have similar but nonidentical states, while oscillators in different groups have approximate anti-phases. The formation of the dynamical groups can be characterized by the two order parameters R and F. As shown in figure 2(b), F keeps increasing during the evolution, but R always maintains very small values. This suggests that local dynamical patterns (rather than global ones) gradually form in the system. To characterize the emerging network structure, we show the average strength of the inter-connections $\langle w_{\text{inter}} \rangle$ and the intra-connections $\langle w_{\text{intra}} \rangle$ as a function of time in figure 2(c). It is evident that average strength of the inter-connections $\langle w_{\text{inter}} \rangle$ decreases, while the intra-connection strength $\langle w_{\text{intra}} \rangle$ keeps increasing with time. These results indicate that with the appearance of dynamical groups, the distribution of connection strengths in the network also changes. From the initial random distribution, the connection strengths within the groups are gradually strengthened, while the connection strengths between the two groups are weakened simultaneously. In this way, after a long time evolution, the topological structure of the networked system has the following characteristics, as shown in figures 2(d) and (e). Firstly, the network evolves into a modular structure with the formation of dynamical groups. Secondly, the network consists of many weak connections and a few strong connections.

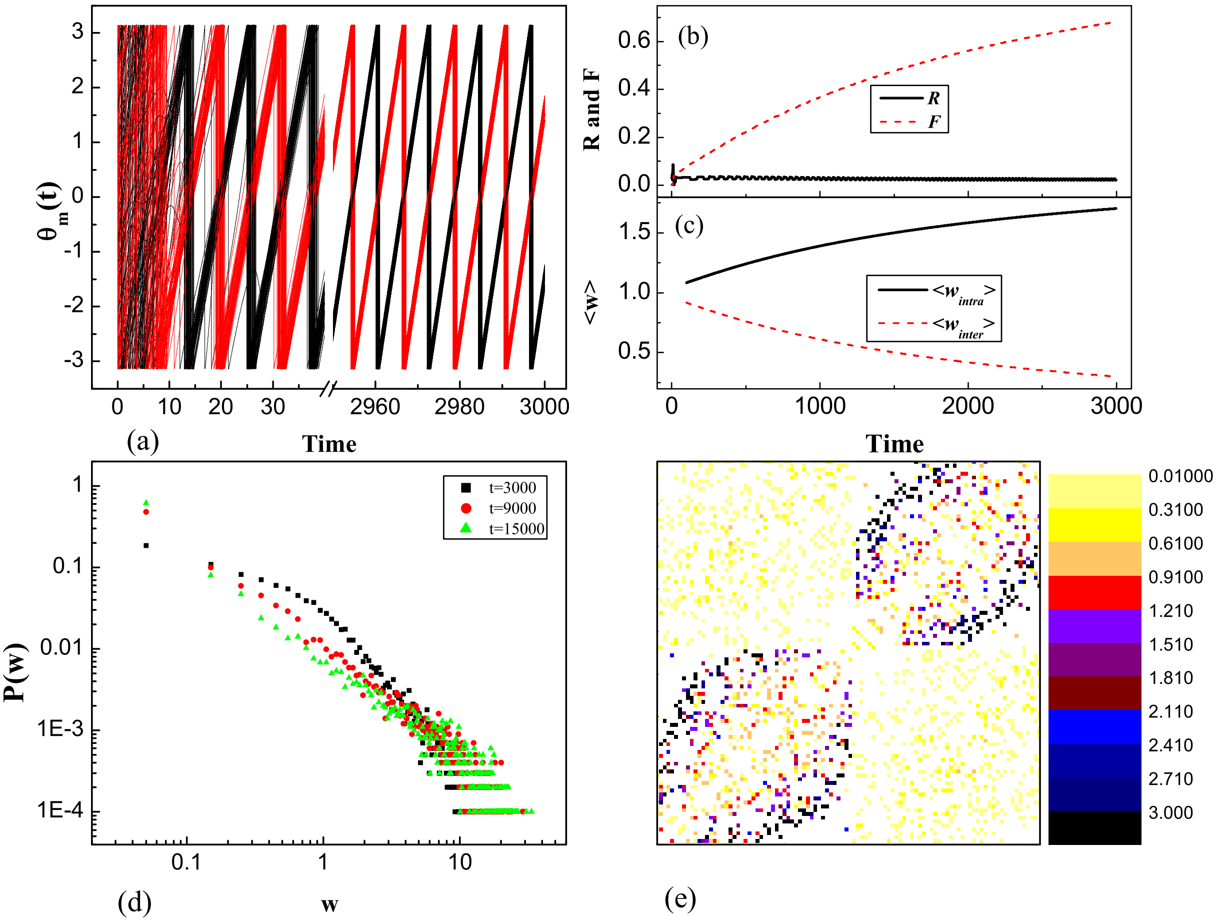

Figure 2. Characterization of the formation of the dynamical groups and modular structure of network. (a) Evolution of the oscillator states. (b) Evolution of the order parameters, where F keeps increasing, but R always maintains very small values, indicating that the dynamical groups have formed. (c) Evolution of the average connection strength, where the average strength of the inter-connections ($\langle w_{\rm inter} \rangle$) decreases all the time, while the intra connection strength ($\langle w_{\rm intra} \rangle$) keeps increasing. (d) Distribution of the connection strength for the network at $t = 3000$, 9000 and 15 000. The longer the time t, the more obvious the power-law distribution of connection strength. This result is the average of 20 runs with different initial conditions. (e) Snapshot of weight matrix w_{mk} at $t = 3000$, where modular structure occurs simultaneously with the formation of dynamical groups. The indices of the oscillators have been rearranged according to the phase. The parameters are the same as those of figure 1, except for $w_{mn} \in (0, 2]$ initially.

Thirdly, to be specific, we have verified that the distribution of the connection strengths follows a power law, as compared to the initial random distribution. It should be pointed out that this power-law distribution of the link weights in the present model is a natural consequence of the coevolution of the network topology and dynamics. These results are consistent with the empirical observations of social systems [3, 13], biological systems [12, 14]

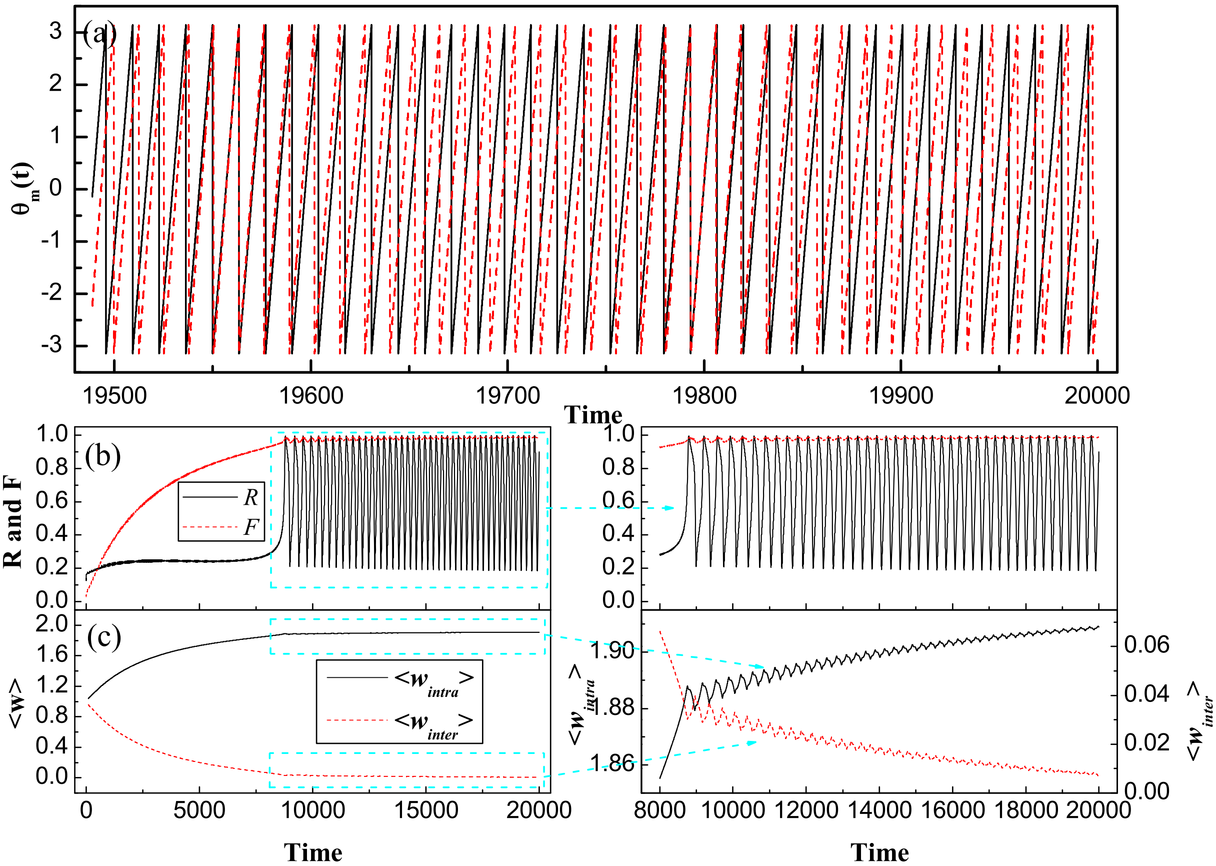

Figure 3. Characterization of the dynamical and topological properties of the network after an extremely long time evolution. (a) Schematic diagram of the evolution of the two groups of oscillators as a whole, where each group of oscillators behaves like an individual oscillator. (b, c) Evolution of order parameters and the average inter- and intra-connection strength, showing that when the inter-connections become very small, the two groups of oscillators almost decouple. All the network parameters are the same as in figure 2.

and neural networks [15, 25]. For instance, in neural networks [15, 25], the synaptic strengths of experimental data follow a power-law distribution.

As shown by figure 2, with the evolution of networked dynamics, the oscillators begin to separate into two groups with different states. Figure 2(c) shows that during the evolution process the average intra-connection strength is gradually enhanced while the average inter-connection strength is always weakened. Here, the question is: how would the two groups behave when the connections between them become weak enough? In figure 3, we further explore this situation. Interestingly, it is found that when the inter-connections between the two groups are too weak, e.g. $\langle w_{\rm inter}\rangle < 0.1$[6], the two dynamical groups effectively decouple and evolve independently according to their own frequencies. As shown in figure 3(a), the frequencies of the two groups are almost equal to each other, and during the evolution their phases will slowly approach the same value and then begin to separate. This occurs

[6] This is an approximate value, which depends on the initial condition of the system.

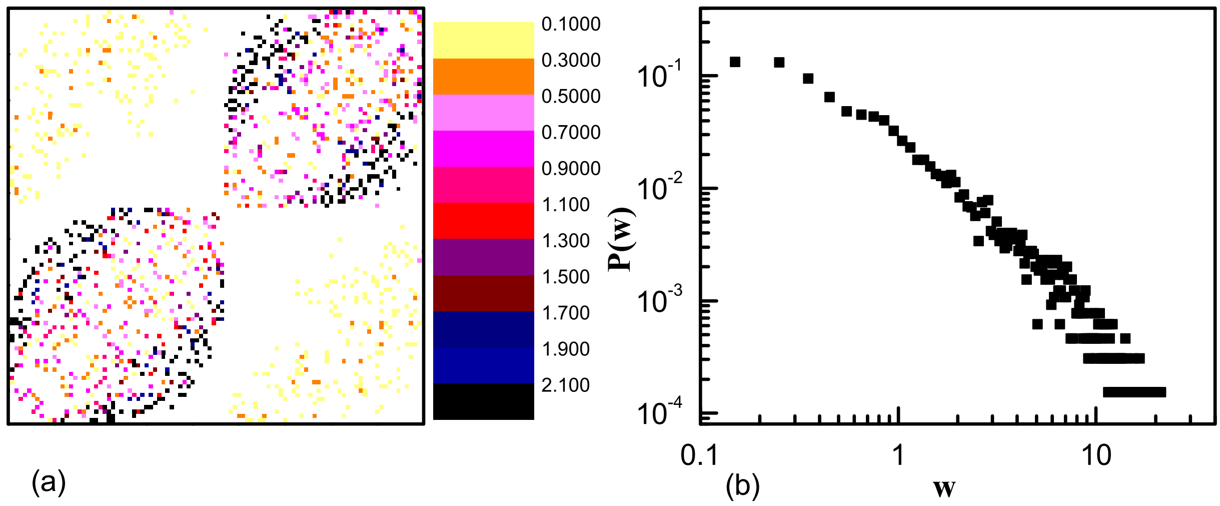

Figure 4. Characterization of the properties of observable networks consisting of active connections. (a) Weighted matrix of observable networks, where the modularity is more distinct when compared with figure 2(e). (b) Distribution of the active connection strengths, which follows a power law. This result is the average of 20 runs with different initial conditions. All network parameters are the same as in figure 2.

repeatedly, which leads to regular oscillation of the global order parameter R as shown in figure 3(b). Meanwhile, when the phase differences between the two groups become small enough, according to equation (3), the inter-connection strength will be enhanced. However, this trend will not last long because the phase differences of the two groups will begin to be significant soon. As shown in figure 3(c), although both the average inter-connection strength and the average intra-connection strength oscillate with a small range, the trends are an overall decrease for the average inter-connection strength and an increase for the average intra-connection strength. This implies that these two dynamical groups will become more and more independent after a long time evolution. Moreover, even in the collective oscillatory regime, the distribution of the link weights still follows a power law (as shown in figure 2(d)).

In realistic networked systems, if the connections are extremely weak, it may be impossible to measure them. As a consequence, any observed real network should consist of connections whose strengths are strong enough to be measured. In our model, we found that there exists a large number of weak links and many of these have no opportunity to be enhanced again. Therefore, from a practical point of view they may not be observable at all after a long time evolution. To distinguish them, we can define the active connections as follows: if w_{mk} exceeds a threshold value, the connection between oscillator m and k is regarded as 'active'; otherwise it is 'inactive'. The threshold can be reasonably taken as the average of the inter-connections, i.e. $\langle w_{\mathrm{inter}} \rangle$. Using this criterion, we obtain observable networks after a long time evolution based on our model. Numerically, we let the networked system evolve from many different initial conditions. After $t = 5000$, we start taking snapshots of w_{mk}. After discarding the 'inactive' connections, we obtain the observable networks consisting of only the 'active' connections. It is found that in these observable networks, the modular property becomes even more distinct. As shown in figure 4, the oscillators can be reasonably partitioned into two communities, and the

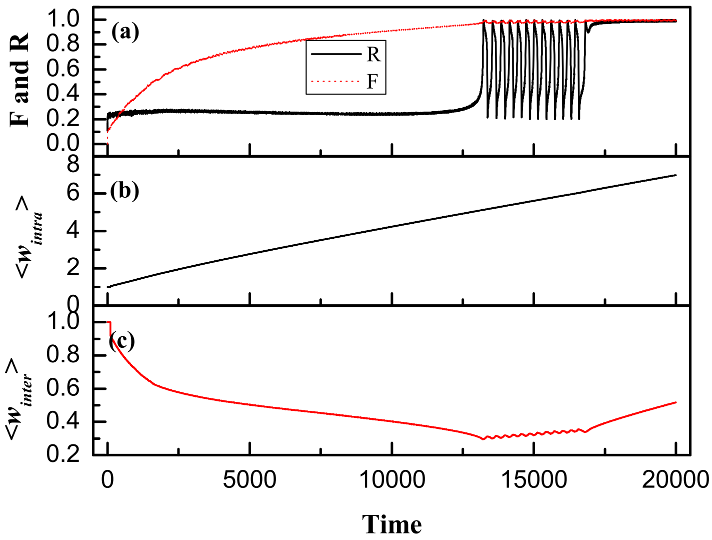

Figure 5. Characterization of the dynamical and topological properties of the network after an extremely long time evolution, where the total connection strength is not limited. (a) Order parameters F and R characterize three distinct stages of network evolution: first the dynamical groups form; then the two groups of oscillators almost decouple; and finally all oscillators achieve in-phase synchronization. (b) Evolution of the average intra-connection strength, keeps increasing. (c) Evolution of the average inter-connection strength, which first decreases and then increases. All the network parameters are the same as in figure 2.

distribution of connection strengths still approximately satisfies the power-law relation. These results suggest that the widely observed community structure and the power-law distribution of link weights in complex networks could emerge simultaneously from the coevolution of the network topology and the dynamics.

In the above studies, we have limited the total connection strength as a constant in the network. This consideration makes sense in certain practical circumstances. For example, the bandwidth of a local area network in a university is always limited. However, in other networks, e.g. the social acquaintance network, there is no need to limit the total connection strength during the network evolution. In this case, how would the dynamics and the network structure coevolve? In the following, we investigate one such example. It is found that the initial stage of the network evolution is quite similar to the case when the total network connection strength is limited. As shown in figure 5(a), the global order parameter R remains small, while the local order parameter F keeps increasing. This indicates that the two dynamical groups have been generated. In figures 5(b) and (c), it is shown that the average intra-connection strength continues to increase, while the average inter-connection strength keeps decreasing. This leads to the formation of the dynamical groups. When the inter-connection strength among the groups is small enough, the two groups almost decouple and behave just like two independent oscillators. However, with the further increase of time, contrary to the previous situations, the inter-connection strength starts to gradually increase as shown in figure 5(c). Due to this strengthening of inter-connections, the two dynamical groups eventually merge into one, and all oscillators achieve in-phase synchronization. Therefore, our results suggest that during the evolution of a network, the limitation of the total connection strength is in favor of the formation of stable dynamical groups.

In summary, we have investigated a coevolutionary networked model. In this model, the node dynamics are described by phase oscillators, and the connections among oscillators are coupled with the dynamical states. By adopting a simple evolution rule, it is shown that the evolution of the networked system naturally leads to two dynamical groups with different phase states. Simultaneously, with the formation of the dynamical pattern, the network also converts from the initial random structure with a uniform distribution of connection strengths to the final modular network with a power-law distribution of the connection strengths. Interestingly, it is found that if the total connection strength is limited as a constant, the two dynamical groups will almost decouple eventually when the inter-connection is too weak. In contrast, if the total connection strength does not have an imposed limit, the two dynamical groups will finally merge into one with all the oscillators achieving in-phase synchronization. In our numerical simulations, the above results were qualitatively verified on networks with sizes up to $N = 1000$. Although the model studied is simple, it essentially captures the interplay between network topology and dynamics. Thus, it can exhibit reasonable results that are useful for us to better understand the behavior of many real networked dynamical systems, such as the evolution of social networks [4] and the evolution of food webs [6].

In this paper, we only investigate the particular case with two groups, i.e. $h = 2$. In fact, the above analysis can be conveniently generalized to a general case with h groups if we replace 2 by h in the sine function and the exponential function of equations (6), (8), (9) and (10). For the case of a two-oscillator system, the stable equilibrium states of the phase difference are $\Delta\theta^* = 2q\pi/h \pm \arcsin|\frac{\Delta\omega}{2\gamma w}|/h$ $(q = 0, 1, 2, \ldots, [h/2])$, where the symbol $[x]$ means taking the integer part of the real number x. For the case of a many-oscillator system, the equilibrium values of the phase difference are $\Delta\theta^*_{mn} = \frac{2q\pi}{h} \pm \frac{2|\Delta\omega_{mn}|}{h\gamma(s_m+s_n)}$ $(q = 0, 1, 2, \ldots, [h/2])$. Of course, the mechanism of changing the connection strengths in equation (2) should be modified accordingly for the case of $h > 2$, e.g. $\frac{\mathrm{d}w_{mk}}{\mathrm{d}t} = S(\beta)\epsilon w_{mk} \mathrm{e}^{-h|\Delta\theta_{mk}-\alpha|}|\sin(h\Delta\theta_{mk})|$, where $\beta = [h\Delta\theta_{mk}/\pi]$, $\alpha = \{\beta + [1 + (-1)^\beta]/2\}\pi/h$, and $S(0) = 1$ or $S(\beta > 0) = -1$. Our numerical simulations have verified the analysis.

In our model, the connection strengths are assumed to respond immediately to the change of phase difference. Nevertheless, time delay inevitably exists in realistic networked systems. For example, electric signals can only propagate along neural axons at a finite speed in neural networks. Recently, time delays have been investigated in some theoretical models of neural networks [33] and networked oscillator systems [26]. Interestingly, it is shown that these models can present very rich dynamical behavior. We believe that the extension of our current model to the delay-coupling case may provide more helpful insights in understanding the coevolution of realistic networked systems. We keep this problem as our future research topic.

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References

[1] Girvan M and Newman M E J 2002 *Proc. Natl Acad. Sci. USA* **99** 7821–26
Ravasz E, Somera A L, Mongru D A, Olvai Z N and Barabási A L 2002 *Science* **297** 1551
Guimerà R and Amaral L A N 2005 *Nature* **433** 895–900
Gleiser P and Danon L 2003 *Adv. Complex Syst.* **6** 565–73
[2] Holme P, Huss M and Jeong H 2003 *Bioinformatics* **19** 532
[3] Onnela J-P, Saramäki J, Hyvönen J, Szabó G, Lazer D, Kaski K, Kertész J and Barabási A-L 2007 *Proc. Natl Acad. Sci. USA* **104** 7332–6
Onnela J-P, Saramäki J, Hyvönen J, Szabó G, Argollo de Menezes M, Kaski K, Barabási A-L and Kertész J 2007 *New J. Phys.* **9** 179
[4] Palla G, Barabási A-L and Vicsek T 2007 *Nature* **446** 664
[5] May R M 1973 *Stability and Complexity in Model Ecosystems* (Princeton, NJ: Princeton University Press)
[6] Krause A E, Frank K A, Mason D M, Ulanowicz R E and Taylor W W 2003 *Nature* **426** 282
[7] Csermely P 2004 *Trends Biochem. Sci.* **29** 331–4
[8] Silk J B, Alberts S C and Altmann J 2003 *Science* **302** 1231–4
[9] Granovetter M 1973 *Am. J. Sociol.* **78** 1360–80
Degenne A and Forse M 1999 *Introducing Social Networks* (London, UK: SAGE Publications)
[10] Dodds P S, Watts D J and Sabel C F 2003 *Proc. Natl Acad. Sci. USA* **100** 12516–21
[11] McCann K S, Hastings A and Huxel G R 1998 *Nature* **395** 794–8
Berlow E L 1999 *Nature* **398** 330–4
[12] Paine R T 1992 *Nature* **355** 73–5
Fagan W F and Hurd L E 1994 *Ecology* **75** 2022–32
Wootton J T 1997 *Ecol. Monogr.* **67** 45–64
Goldwasser L and Roughgarden J 1993 *Ecology* **74** 1216–33
[13] Barrat A, Barthélemy M, Pastor-Satorras R and Vespig-Nani A 2004 *Proc. Natl Acad. Sci. USA* **101** 3747–52
Li M, Wu J, Wang D, Zhou T, Di Z and Fan Y 2007 *Physica* A **375** 355–64
[14] Almaas E, Kovács B, Vicsek T, Oltvai Z N and Barabási A-L 2004 *Nature* **427** 839
[15] Barbour B, Brunel N, Hakim V and Nadal J-P 2007 *Trends Neurosci.* **30** 622–9
[16] Albert R and Barabási A-L 2002 *Rev. Mod. Phys.* **74** 47
Newman M E J 2003 *SIAM Rev.* **45** 167
Boccaletti S, Latora V, Moreno Y, Chavez M and Hwang D-U 2006 *Phys. Rep.* **424** 175
[17] Dorogovtsev S N, Goltsev A V and Mendes J F F 2008 *Rev. Mod. Phys.* **80** 1275
Arenas A, Díaz-Guilera A, Kurths J, Moreno Y and Zhou C 2008 *Phys. Rep.* **469** 93
[18] Gross T and Blasius B 2008 *J. R. Soc. Interface* **5** 259–71
[19] Bi G-Q and Poo M M 1998 *J. Neurosci.* **18** 10464–72
Dan Y and Poo M M 2006 *Physiol. Rev.* **86** 1033–48
[20] Gong P and Leeuwen C V 2004 *Europhys. Lett.* **67** 328
Zimmermann M G, Eguíluz V M and San Miguel M 2004 *Phys. Rev.* E **69** 065102
Gross T, Dommar D'Lima C J and Blasius B 2006 *Phys. Rev. Lett.* **96** 208701
Pacheco J M, Traulsen A and Nowak M A 2006 *Phys. Rev. Lett.* **97** 258103
Vazquez F, Eguíluz V M and Miguel M S 2008 *Phys. Rev. Lett.* **100** 108702
Nardini C, Kozma B and Barrat A 2008 *Phys. Rev. Lett.* **100** 158701
Kozma B and Barrat A 2008 *Phys. Rev.* E **77** 016102
[21] Sun J and Deem M W 2007 *Phys. Rev. Lett.* **99** 228107
[22] Ito J and Kaneko K 2001 *Phys. Rev. Lett.* **88** 028701
Ito J and Kaneko K 2003 *Phys. Rev.* E **67** 046226
[23] Zhou C and Kurths J 2006 *Phys. Rev. Lett.* **96** 164102
[24] Aoki T and Aoyagi T 2009 *Phys. Rev. Lett.* **102** 034101

[25] Meisel C and Gross T 2009 *Phys. Rev.* E **80** 061917
[26] Acebrón J A, Bonilla L L, Pérez Vicente C J, Ritort F and Spigler R 2005 *Rev. Mod. Phys.* **77** 137
Strogatz S H 2000 *Physica* D **143** 1
[27] Masuda N and Kori H 2007 *J. Comput. Neurosci.* **22** 327–45
Tass P A and Majtanik M 2006 *Biol. Cybern.* **94** 58–66
Kepecs A, van Rossum M C W, Song S and Tagner J 2002 *Biol. Cybern.* **87** 446–58
[28] Tass P 1997 *Phys. Rev.* E **56** 2043
[29] Kossinets G and Watts D J 2006 *Science* **311** 88
[30] McPherson M, Smith-Lovin L and Cook J 2001 *Ann. Rev. Sociol.* **27** 415
[31] Ichinomiya T 2004 *Phys. Rev.* E **70** 026116
Restrepo J G, Ott E and Hunt B R 2005 *Phys. Rev.* E **71** 036151
Lee D-S 2005 *Phys. Rev.* E **72** 026208
Restrepo J G, Ott E and Hunt B R 2005 *Chaos* **16** 015107
Wang X, Huang L, Guan S, Lai Y-C and Lai C H 2008 *Chaos* **18** 037117
[32] Gómez-Gardeñes J, Moreno Y and Arenas A 2007 *Phys. Rev. Lett.* **98** 034101
Gómez-Gardeñes J, Moreno Y and Arenas A 2007 *Phys. Rev.* E **75** 066106
[33] Karimi H R and Maass P 2009 *Chaos Solitons Fractals* **41** 1125
Karimi H R and Gao H 2010 *IEEE Trans. Syst. Man Cybernet.* B **40** 173