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Synchronization stability of general complex dynamical networks with time-varying delays

Kun Li [a,b,*], Shuguang Guan [a,b], Xiaofeng Gong [a,b], C.-H. Lai [b,c]

[a] *Temasek Laboratories, National University of Singapore, Singapore 117508, Singapore*
[b] *Beijing-Hong Kong-Singapore Joint Center of Nonlinear and Complex systems (Singapore), National University of Singapore, Singapore*
[c] *Department of Physics, National University of Singapore, Singapore*

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ABSTRACT

The synchronization problem of some general complex dynamical networks with time-varying delays is investigated. Both time-varying delays in the network couplings and time-varying delays in the dynamical nodes are considered. The novel delay-dependent criteria in terms of linear matrix inequalities (LMI) are derived based on free-weighting matrices technique and appropriate Lyapunov functional proposed recently. Numerical examples are given to illustrate the effectiveness and advantage of the proposed synchronization criteria.

1. Introduction

Complex dynamical networks have received a great deal of attention since they are shown to widely exist in various fields of real world [1]. A complex network is a large set of interconnected nodes, in which a node is a fundamental unit with specific contents. Examples of complex networks include the Internet, which is a network of routers or domains; the World Wide Web (WWW), a network of web-site; the brain, a network of neurons; food webs; telephone cell graphs and electricity distribution networks, etc. Many of these networks exhibit complexity in the overall topological properties and dynamical properties of the network nodes and the coupled unites. The complex nature of complex networks has results in a series of important research problems. In particular, one significant and interesting phenomenon is the synchronization of all its dynamical nodes.

Synchronization is a basic motion in nature. It has been demonstrated that many real-world problems have close relationship with network synchronization [1,2]. Synchronization of complex dynamical networks has been studied in various fields of science and engineering [3–16]. Recently, Wang and Chen introduced a uniform dynamical network model and investigated its synchronization and control [7,8]. In [9], Li and Chen further extended the uniform dynamical network model to include coupling delays among the network nodes and studied its synchronization. Gao et al. [10] considered the synchronization stability of both continuous- and discrete-time networks with coupling delays. Lü and Chen [11] studied the synchronization of time-varying complex dynamical networks in which the inner-couplings are time-varying. Zhou et al. [12] studied the adaptive synchronization of uncertain complex dynamical network. Sorrentino et al. [13] investigated the controllability of complex networks with pinning controllers. Zhang et al. [14] studied the synchronization of a general complex dynamical network with delayed nodes with adaptive feedback control. However, it is worth noting that the time-delays of system dynamical states considered in these works are assumed to be constant. Time-varying delay case, which is more general than the constant one should be considered. Moreover, in much of the literature, time delays in the couplings are considered; however, the time delays in the dynamical nodes, which are more complex, are still relatively unexplored. In this Letter, we further investigate the syn-

* Corresponding author at: Temasek Laboratories, National University of Singapore, Singapore 117508, Singapore. Tel.: +65 65166604.
E-mail address: tsllk@nus.edu.sg (K. Li).

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chronization stability of some general complex dynamical network models with time-varying coupling delays and time-varying delayed nodes, respectively.

The phenomena of time delays are common in various systems. Due to the finite speeds of transmission and spreading as well as traffic congestion, a signal or influence traveling through a network often is associated with time delays. Real-world complex networks, particularly in biological and physical networks, are time-varying networks. The stability criteria for system with time delays can be classified into two categories, namely, delay-independent and delay-dependent. Since delay-independent criteria tend to be conservative, especially when the delay is small, considerable attention has been paid to the delay-dependent type. For system with time-varying delay, fixed model transformations are the main method to deal with delay-dependent stability problems [17], in which some inequalities such as Park and Moon et al.'s inequalities [18,19], were used to estimate the upper bound of cross product terms. Recently, in order to reduce the conservatism, a free-weighting matrix method was proposed to estimate the upper bound of the derivative of Lyapunov functional [20–24]. Therefore, in this Letter, we attempt to introduce some more general time-varying dynamical network models, and based on the free-weighting matrix method, derive synchronization conditions for delay-dependent stabilities in terms of linear matrix inequality (LMIs). Numerical examples are given to demonstrate the effectiveness and the advantage of the proposed method.

The rest of this Letter is organized as follows. A complex dynamical network model with coupling time-varying delays and its synchronization theorem are presented in Section 2. Section 3 introduces a complex dynamical network model with time-varying delayed nodes and derives its synchronization theorems. Section 4 uses two numerical examples for illustration. Conclusions are finally given in Section 5.

2. Complex dynamical network with coupling delay

Consider the following continuous-time complex dynamical network with time-varying coupling delays:

$$\begin{cases} \dot{x}_i = f(x_i) + c\sum_{j=1}^{N} G_{ij}Ax_j(t-\tau(t)), & t>0,\ i=1,2,\ldots,N, \\ x(t)=\phi(t), & t\in[-h,0], \end{cases} \tag{1}$$

where N is the number of coupled nodes; each node is a n-dimensional dynamical system with node dynamics $\dot{x}=f(x,t)$; $f:R^n\to R^n$ is continuously differentiable; $x_i=(x_{i1},x_{i2},\ldots,x_{in})^T\in R^n$ are the state variables of node i; the constant $c>0$ is the coupling strength; $A=(a_{ij})_{n\times n}\in R^{n\times n}$ is the constant inner-coupling matrix of the nodes; $G=(g_{ij})_{N\times N}$ is the outer-coupling matrix of the network, in which g_{ij} is defined as follows: if there is a connection between node i and node j $(j\neq i)$, then $g_{ij}=g_{ji}=1$; otherwise, $g_{ij}=g_{ji}=0$ $(j\neq i)$, and the diagonal elements of matrix G are defined by

$$g_{ii} = -\sum_{j=1,\, i\neq j}^{N} g_{ij} = -\sum_{j=1,\quad i\neq j}^{N} g_{ji}, \quad i=1,\ldots,N. \tag{2}$$

Suppose that the network (1) is connected in the sense that there are no isolated clusters, that is, G is an irreducible matrix.

The time delay, $\tau(t)$, is a time-varying differentiable function that satisfies

$$0\leqslant \tau(t)\leqslant h \tag{3}$$

and

$$\dot{\tau}(t)\leqslant \mu, \tag{4}$$

where $h>0$ and μ are constants. The initial condition, $\phi(t)$, is a continuous and differentiable vector-valued function of $t\in[-h,0]$.

The network (1) is synchronized if

$$x_1(t)=x_2(t)=\cdots=x_N(t)=s(t), \quad \text{as } t\to\infty, \tag{5}$$

where the synchronization manifold $s(t)\in R^n$ is a solution of the local dynamics of an isolate node of the network, satisfying $\dot{s}(t)=f(s(t))$. $s(t)$ may be a limit cycle, or a chaotic orbit in the phase space. Apparently, the stability of the synchronized state (5) of network (1) is determined by the dynamics of the isolate node, the coupling strength c, the inner-coupling matrix A, the outer-coupling matrix G, and the time delay $\tau(t)$. According to the result established in [9], we have the following lemma.

Lemma 1. *Consider the coupling delayed dynamical network in (1). Let $0=\lambda_1>\lambda_2\geqslant\lambda_3\geqslant\cdots\geqslant\lambda_N$ be the eigenvalues of the outer-coupling matrix G. If the following $N-1$ liner time-varying delayed differential equations are asymptotically stable about their zero solutions*

$$\dot{w}_k(t)=J(t)w_k(t)+c\lambda_k A w_k\big(t-\tau(t)\big), \quad k=2,\ldots,N, \tag{6}$$

where $J(t):=D(f(s(t)))$ is the Jacobian of $f(s(t))$ at $s(t)$, then the synchronized states (5) are asymptotically stable.

Consider a Lyapunov functional candidate to be

$$V_k(t)=w_k^T(t)P_k w_k(t)+\int_{t-\tau(t)}^{t} w_k^T(\alpha)Q_k w_k(\alpha)\,d\alpha+\int_{t-h}^{t} w_k^T(\alpha)R_k w_k(\alpha)\,d\alpha+\int_{-h}^{0}\int_{t+\theta}^{t} \dot{w}_k^T(\alpha)(Z_{k1}+Z_{k2})\dot{w}_k(\alpha)\,d\alpha\,d\theta,$$

where $P_k=P_k^T>0$, $Q_k=Q_k^T\geqslant 0$, $R_k=R_k^T\geqslant 0$, and $Z_{ki}=Z_{ki}^T>0$, $i=1,2$ are to be determined. Then, we have the following theorem.

Theorem 1. *Consider the coupling delayed dynamical network in (1) with a time-varying delay $\tau(t)$ satisfy (3) and (4). Given scalars $h>0$ and μ, the asymptotically synchronization in (5) is achieved if there exist matrices*

$$P_k = P_k^T > 0, \quad Q_k = Q_k^T \geqslant 0, \quad R_k = R_k^T \geqslant 0, \quad Z_{ki} = Z_{ki}^T > 0, \quad i = 1, 2,$$
$$N_k = \begin{bmatrix} N_{k1} \\ N_{k2} \\ N_{k3} \end{bmatrix}, \quad S_k = \begin{bmatrix} S_{k1} \\ S_{k2} \\ S_{k3} \end{bmatrix} \quad \text{and} \quad M_k = \begin{bmatrix} M_{k1} \\ M_{k2} \\ M_{k3} \end{bmatrix}$$

for $k = 2, 3, \ldots, N$, such that the following LMI holds:

$$\begin{bmatrix} \Phi_k & hN_k & hS_k & hM_k & hA_k^T(Z_{k1}+Z_{k2}) \\ * & -hZ_{k1} & 0 & 0 & 0 \\ * & * & -hZ_{k1} & 0 & 0 \\ * & * & * & -hZ_{k2} & 0 \\ * & * & * & * & -h(Z_{k1}+Z_{k2}) \end{bmatrix} < 0, \tag{7}$$

where

$$\Phi_k = \Phi_{k1} + \Phi_{k2} + \Phi_{k2}^T,$$
$$\Phi_{k1} = \begin{bmatrix} P_kJ(t) + J^T(t)P_k + Q_k + R_k & c\lambda_k P_k A & 0 \\ * & -(1-\mu)Q_k & 0 \\ * & * & -R_k \end{bmatrix},$$
$$\Phi_{k2} = [N_k + M_k \quad -N_k + S_k \quad -M_k - S_k],$$
$$A_k = \begin{bmatrix} J(t) & c\lambda_k & 0 \end{bmatrix}$$

and $*$ denotes the symmetric terms in a symmetric matrix.

Proof. Calculating the derivative of $V_k(t)$ along the solution of system (6) yields

$$\begin{aligned}
\dot V_k(t) &= 2w_k^T(t)P_k\dot w_k(t) + w_k^T(t)Q_kw_k(t) - \big(1-\dot\tau(t)\big)w_k^T\big(t-\tau(t)\big)Q_kw_k\big(t-\tau(t)\big) \\
&\quad + w_k^T(t)R_kw_k(t) - w_k^T(t-h)R_k\dot w_k(t-h) + h\dot w_k^T(t)(Z_{k1}+Z_{k2})\dot w_k(t) - \int_{t-h}^{t} \dot w_k^T(\alpha)(Z_{k1}+Z_{k2})\dot w_k(\alpha)\,d\alpha \\
&\leqslant 2w_k^T(t)P_k\dot w_k(t) + w_k^T(t)(Q_k+R_k)w_k(t) - (1-\mu)w_k^T\big(t-\tau(t)\big)Q_kw_k\big(t-\tau(t)\big) - w_k^T(t-h)R_k\dot w_k(t-h) \\
&\quad + h\dot w_k^T(t)(Z_{k1}+Z_{k2})\dot w_k(t) - \int_{t-\tau(t)}^{t} \dot w_k^T(\alpha)Z_{k1}\dot w_k(\alpha)\,d\alpha - \int_{t-h}^{t-\tau(t)} \dot w_k^T(\alpha)Z_{k1}\dot w_k(\alpha)\,d\alpha - \int_{t-h}^{t} \dot w_k^T(\alpha)Z_{k2}\dot w_k(\alpha)\,d\alpha \\
&\quad + 2\varsigma_k^T(t)N_k\left[w_k(t) - w_k\big(t-\tau(t)\big) - \int_{t-\tau(t)}^{t}\dot w_k(\alpha)\,d\alpha\right] + 2\varsigma_k^T(t)S_k\left[w_k\big(t-\tau(t)\big) - w_k(t-h) - \int_{t-h}^{t-\tau(t)}\dot w_k(\alpha)\,d\alpha\right] \\
&\quad + 2\varsigma_k^T(t)M_k\left[w_k(t) - w_k(t-h) - \int_{t-h}^{t}\dot w_k(\alpha)\,d\alpha\right] \\
&\leqslant \varsigma_k^T(t)\big[\Phi_k + hA_k^T(Z_{k1}+Z_{k2})A_k + hN_kZ_{k1}^{-1}N_k^T + hS_kZ_{k1}^{-1}S_k^T + hM_kZ_{k2}^{-1}M_k^T\big]\varsigma_k(t) \\
&\quad - \int_{t-\tau(t)}^{t}\big[\varsigma_k^T(t)N_k + \dot w_k^T(\alpha)Z_{k1}\big]Z_{k1}^{-1}\big[N_k^T\varsigma_k(t) + Z_{k1}\dot w_k(\alpha)\big]\,d\alpha - \int_{t-h}^{t-\tau(t)}\big[\varsigma_k^T(t)S_k + \dot w_k^T(\alpha)Z_{k1}\big]Z_{k1}^{-1}\big[S_k^T\varsigma_k(t) + Z_{k1}\dot w_k(\alpha)\big]\,d\alpha \\
&\quad - \int_{t-h}^{t}\big[\varsigma_k^T(t)M_k + \dot w_k^T(\alpha)Z_{k2}\big]Z_{k2}^{-1}\big[M_k^T\varsigma_k(t) + Z_{k2}\dot w_k(\alpha)\big]\,d\alpha,
\end{aligned}$$

where $\varsigma_k(t) = [w_k^T(t)\ w_k^T(t-\tau(t))\ w_k^T(t-h)]^T$. Since $Z_{ki} > 0$, $i = 1, 2$, then the last three parts are all less than 0. So if $\Phi_k + hA_k^T(Z_{k1} + Z_{k2})A_k + hN_kZ_{k1}^{-1}N_k^T + hS_kZ_{k1}^{-1}S_k^T + hM_kZ_{k2}^{-1}M_k^T < 0$, which is equivalent to (8) by Schur complements, $\dot V_k < -\varepsilon\|w(t)\|^2$ for a sufficiently small $\varepsilon > 0$ such that the subsystem in (1) are asymptotically stable. □

3. Complex dynamical network with delayed nodes

In this section, we will study the synchronization of dynamical nodes delayed complex network. Unless otherwise defined, we endorse the same notations used in the above section.

Consider a general complex network consisting of N delayed dynamical nodes. Each node of the network is an n-dimensional dynamical system with time-varying delay, which is described by

$$\begin{cases} \dot x_i = f(x_i(t), x_i(t-\tau(t))) + c\sum_{j=1}^{N} G_{ij}Ax_j(t), & t > 0,\ i = 1, 2, \ldots, N, \\ x(t) = \phi(t), & t \in [-h, 0], \end{cases} \tag{8}$$

where $f: R^n \to R^n$ is continuous differentiable function, the time-varying delay $\tau(t)$ is in the dynamical nodes, which is satisfies (3) and (4). It is assumed that network (8) is connected in the sense that there are no isolate clusters, that is, G is an irreducible matrix. The dynamical nodes delayed network (8) is said to be achieve asymptotic synchronization if

$$x_1(t) = x_2(t) = \cdots = x_N(t) = s(t), \quad \text{as } t \to \infty, \tag{9}$$

where $s(t) \in R^n$ is a solution of an isolate node of the network, satisfying $\dot{s}(t) = f(s(t), s(t - \tau(t)))$.

Theorem 2. *Consider the dynamical nodes delayed network (8), Let $0 = \lambda_1 > \lambda_2 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_N$ denote the eigenvalues of the outer-coupling matrix G. If the following $N-1$ liner time-varying delayed differential equations are asymptotically stable about their zero solutions*

$$\dot{w}_k(t) = R(t)w_k(t) + S(t)w_k\big(t - \tau(t)\big) + c\lambda_k A w_k(t), \quad k = 2, \ldots, N, \tag{10}$$

where $R(t) := \delta_{s(t)}(f(s(t), s(t - \tau(t)))$ is the Jacobian of $f(s(t), s(t - \tau(t)))$ at $s(t)$, and $S(t) := \delta_{s(t-\tau(t))}(f(s(t), s(t - \tau(t)))$ is the Jacobian of $f(s(t), s(t - \tau(t)))$ at $s(t - \tau(t))$. Then the synchronization states (9) are asymptotically stable.

Proof. Define the error vectors as

$$e_i(t) = x_i(t) - s(t), \quad i = 1, 2, \ldots, N. \tag{11}$$

We have the error system

$$\dot{e}_i(t) = \dot{x}_i(t) - \dot{s}(t) = f\big(x_i(t), x_i\big(t - \tau(t)\big)\big) - f\big(s(t), s\big(t - \tau(t)\big)\big) + c\sum_{j=1}^{N} G_{ij} A e_j(t), \quad i = 1, \ldots, N. \tag{12}$$

Since f is continuous differentiable, it is easy to know that the origin of the nonlinear system (12) is an asymptotically stable equilibrium point if it is an asymptotically stable equilibrium point of the following liner time-varying system:

$$\dot{e}_i(t) = R(t)e_i(t) + S(t)e_i\big(t - \tau(t)\big) + c\sum_{j=1}^{N} G_{ij} A e_j(t) = R(t)e_i(t) + S(t)e_i\big(t - \tau(t)\big) + cA\big(e_1(t), \ldots, e_N(t)\big)(G_{i1}, \ldots, G_{iN})^T. \tag{13}$$

Letting $e(t) = (e_1(t), \ldots, e_N(t)) \in R^{n\times N}$, we have

$$\dot{e}(t) = R(t)e(t) + S(t)e\big(t - \tau(t)\big) + cAe(t)G^T. \tag{14}$$

There exists an nonsingular matrix, $\Phi = (\phi_1, \ldots, \phi_N) \in R^{N\times N}$, such that $G^T\Phi = \Phi\Lambda$, with $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. Using the nonsingular transform $e(t)\Phi = v(t) = (v_i(t), \ldots, v_N(t)) \in R^{n\times N}$, from (14), we have the following matrix equation:

$$\dot{v}(t) = R(t)v(t) + S(t)v\big(t - \tau(t)\big) + cAv(t)\Lambda. \tag{15}$$

Namely,

$$\dot{v}_i(t) = R(t)v_i(t) + S(t)v_i\big(t - \tau(t)\big) + c\lambda_i A v_i(t), \quad i = 1, \ldots, N. \tag{16}$$

Thus, we have transformed the stability problem of the synchronized states (9) to the stability problem of the N pieces of n-dimensional linear time-varying delayed differential equations (16). Note that $\lambda_1 = 0$ corresponding to the synchronization of the system states (9), where $s(t)$ is an orbitally stable solution of the isolate node. If the following $N-1$ pieces of n-dimensional linear time-varying delayed differential equations

$$\dot{v}_i(t) = R(t)v_i(t) + S(t)v_i\big(t - \tau(t)\big) + c\lambda_i A v_i(t), \quad i = 2, \ldots, N \tag{17}$$

are asymptotically stable, then $e(t)$ will tend to the zero, which implies that the synchronized states (9) are asymptotically stable. □

Our new delay-dependent condition for ensuring the stability of the synchronized states of the dynamical nodes delayed network in (8) is presented in the following theorem.

Theorem 3. *Consider the dynamical nodes delayed network (8), the time-varying delay $\tau(t)$ satisfies (3) and (4). Given scalar $h > 0$ and μ, the asymptotically synchronization in (9) is achieved if there exists matrices $P_k = P_k^T > 0$, $Q_k = Q_k^T \geqslant 0$, $R_k = R_k^T \geqslant 0$, $Z_{ki} = Z_{ki}^T > 0$, $i = 1, 2$,*

$$N_k = \begin{bmatrix} N_{k1} \\ N_{k2} \\ N_{k3} \end{bmatrix}, \qquad S_k = \begin{bmatrix} S_{k1} \\ S_{k2} \\ S_{k3} \end{bmatrix} \quad \text{and} \quad M_k = \begin{bmatrix} M_{k1} \\ M_{k2} \\ M_{k3} \end{bmatrix}$$

for $k = 2, 3, \ldots, N$, such that the following LMI holds:

$$\begin{bmatrix} \Phi_k & hN_k & hS_k & hM_k & hA_k^T(Z_{k1} + Z_{k2}) \\ * & -hZ_{k1} & 0 & 0 & 0 \\ * & * & -hZ_{k1} & 0 & 0 \\ * & * & * & -hZ_{k2} & 0 \\ * & * & * & * & -h(Z_{k1} + Z_{k2}) \end{bmatrix} < 0, \tag{18}$$

where

Table 1
Upper bounds of h for different c and μ (Example 1).

Coupling strength c	0.3	0.4	0.5	0.6
$\mu = 0$	1.935	1.130	0.813	0.638
$\mu = 0.5$	1.275	0.854	0.646	0.520
$\mu = 0.9$	0.970	0.711	0.562	0.464
any μ	0.960	0.710	0.562	0.464

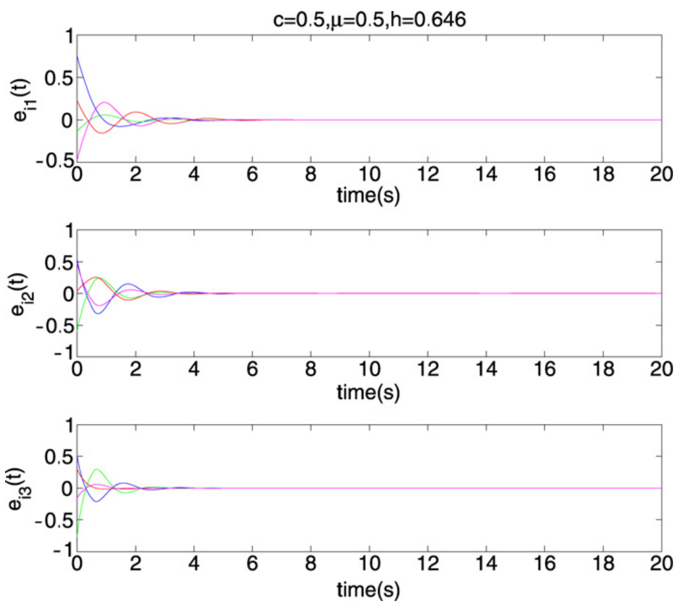

Fig. 1. Synchronization errors for the coupling delayed network with $c = 0.5$, $\mu = 0.5$ and $h = 0.646$ (Example 1).

$$\Phi_k = \Phi_{k1} + \Phi_{k2} + \Phi_{k2}^T,$$

$$\Phi_{k1} = \begin{bmatrix} P_k(R(t) + c\lambda_k A) + (R(t) + c\lambda_k A)^T P_k + Q_k + R_k & P_k S(t) & 0 \\ * & -(1-\mu)Q_k & 0 \\ * & * & -R_k \end{bmatrix},$$

$$\Phi_{k2} = [N_k + M_k \quad -N_k + S_k \quad -M_k - S_k],$$

$$A_k = \begin{bmatrix} R(t) + c\lambda_k A & S(t) & 0 \end{bmatrix}$$

and $*$ denotes the symmetric terms in a symmetric matrix.

Proof. the proof flows a similar line to that of Theorem 1. □

4. Numerical example

In this section, we use three examples (for respectively dynamical networks with coupling time-varying delays and dynamical nodes time-varying delays) to illustrate the results derived in this work. The above synchronization conditions can be applied to networks with different topologies and different sizes. We first consider a lower-dimensional network model.

Example 1. Consider a network model with 5 nodes, where each node is a three-dimensional stable linear system described by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -x_1 \\ -2x_2 \\ -3x_3 \end{bmatrix},$$

which is asymptotically stable at $s(t) = 0$, with Jacobian given by $J = \mathrm{diag}\{-1, -2, -3\}$. Assume that the inner-coupling matrix is $A = \mathrm{diag}\{1, 1, 1\}$, and the outer-coupling matrix is given by the following irreducible symmetric matrix satisfying condition (2)

$$G = \begin{bmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -3 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & -3 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix}.$$

The eigenvalues of G are $\lambda_i = \{0, -1.382, -2.382, -3.618, -4.618\}$. According to Lemma 1, if the delayed subsystem in (6) are asymptotically stable, then the asymptotic synchronization is achieved.

Table 2
Upper bounds of h for different c and μ (Example 2).

Coupling strength c	0.3	0.4	0.5	0.6
$\mu = 0$	1.160	1.218	1.286	1.366
$\mu = 0.5$	1.115	1.148	1.186	1.232
$\mu = 0.9$	1.115	1.147	1.179	1.209
any μ	1.115	1.147	1.179	1.209

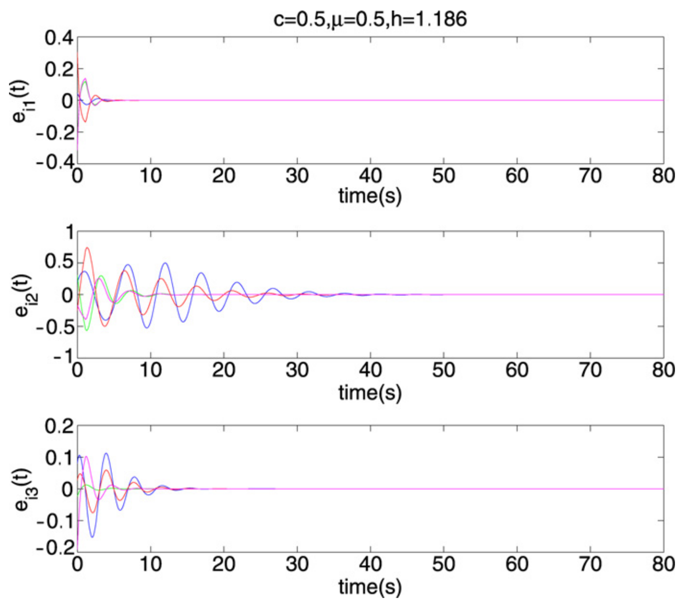

Fig. 2. Synchronization errors for the dynamical nodes delayed network with $c = 0.5$, $\mu = 0.5$ and $h = 1.186$ (Example 2).

The upper bounds on the time delay for different values of the coupling strength c obtained from Theorem 1 are listed in Table 1. These results are obtained by using MATLAB LMI toolbox. For example, for the coupling strength $c = 0.6$, it is found that the maximum delay bound for the synchronized states to be asymptotically stable is $h = 0.638$ for $\mu = 0$, constant delay. The result is same as that obtained by Theorem 1 in [10], much larger than that obtained in [9]. However, theorem obtained in these references only discussed constant delay; Theorem 1 in this Letter has time-varying delay.

Fig. 1 shows the synchronization errors between the states of node i and node $i + 1$ for random initial conditions for the case with $c = 0.5$, $\mu = 0.5$ and $h = 0.646$, where $e_{ij}(t) = x_{ij}(t) - x_{(i+1)j}(t)$ for $i = 1, \ldots, 4$, $j = 1, 2, 3$. We see that the synchronization errors converge to zero under the above conditions.

Example 2. Next, we consider a network model with 5 nodes, where each node is the following dynamical delayed system:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -2x_1 \\ 0.9x_2 \\ -0.1x_3 \end{bmatrix} + \begin{bmatrix} -x_1(t-\tau(t)) \\ x_1(t-\tau(t)) - x_2(t-\tau(t)) \\ -x_3(t-\tau(t)) \end{bmatrix},$$

which is asymptotically stable at $s(t) = 0$, with Jacobian given by $R = \mathrm{diag}\{-2, 0.9, -0.1\}$ and $S = \{-1, 0, 0; 1, -1, 0; 0, 0, -1\}$. Assume that the network coupling is same as that in Example 1. The upper bounds on the time delay obtained from Theorem 3 are listed in Table 2. As a example, the synchronization errors between the states node i and node $i + 1$ for random initial conditions with $c = 0.5$, $\mu = 0.5$ and $h = 1.186$ are shown in Fig. 2, where $e_{ij}(t) = x_{ij}(t) - x_{(i+1)j}(t)$ for $i = 1, \ldots, 4$, $j = 1, 2, 3$. Clearly, all the synchronization errors are converging to zero.

Example 3. Finally, we consider a scale-free network with 50 dynamical nodes. We here take the parameters $N = 50$, $m = 4$ and $m_0 = 3$, and then the coupling matrix G of the network can be randomly generated by the BA scale-free model [25]. In the simulation, each node is the following dynamical delayed system:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -2x_1 \\ 0.9x_2 \end{bmatrix} + \begin{bmatrix} -x_1(t-\tau(t)) \\ x_1(t-\tau(t)) - x_2(t-\tau(t)) \end{bmatrix}.$$

For the coupling strength $c = 0.3$ and $\mu = 0, 0.5$ and any μ, it is easy to obtain that the upper bounds on the time delay are $h = 1.17, 1.12$ and 1.12 according to Theorem 3, respectively. Fig. 3 is the simulation results of the synchronization errors between the states node i and node $i + 1$ with $c = 0.3$, $\mu = 0.5$ and $h = 1.12$, which shows that the synchronization can be achieved for the scale-free network.

5. Conclusion

We have introduced some general complex dynamical network models with time-varying delays in this Letter. Both time-varying delays in network couplings and time-varying delays in dynamical nodes have been considered. And further investigated their synchronization

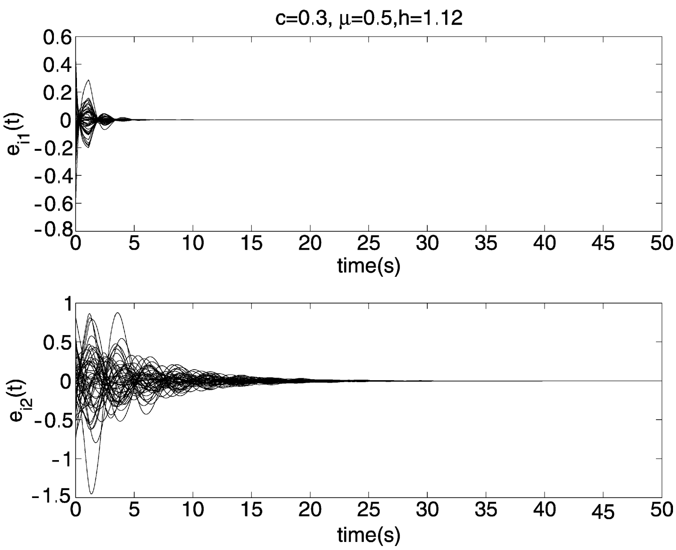

Fig. 3. Synchronization errors for the scale-free network with $c = 0.3$, $\mu = 0.5$ and $h = 1.12$ (Example 3).

criteria respectively. New delay-dependent synchronization criteria in terms of LMIs have been derived based on free-weighting matrices technique and appropriate Lyapunov functional. These synchronization conditions are applicable to networks with different topologies and different sizes. Numerical examples have been presented to illustrate the effectiveness and advantage of the proposed synchronization criteria.

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