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Resonance effect of direction-phase clusters in a scale-free network

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Abstract – It is known that for chaotic flows, a weak coupling does not always make the coupled systems approach synchronization but sometimes makes them become more complicated (*Phys. Rev. E*, **67** (2003) 045203(R)). We here report that a similar situation also occurs in the coupled chaotic maps, where a weak coupling will make the number of direction-phase clusters N_c increase. We find a double-resonance effect on the coupling strength ε, where the first resonance comes from the coupling-induced periodic behaviors and the second one is due to the disappearance of the disorder phase. The mechanism of the second resonance is revealed through the out-of-phase links. Moreover, we show that the critical coupling ε_c of the maximum N_c will increase rapidly with the bifurcation parameter μ but slowly with the range of the distribution of non-identical oscillators.

Introduction. – Coupled chaotic systems are ubiquitous in nature and have been extensively studied. It is found that generally, a coupling will make the coupled systems approach synchronization. Depending on the coupling strength and the intrinsic properties of the oscillators, the resulting synchronization can be classified into phase synchronization (PS), delay synchronization, partial synchronization, complete synchronization, generalized synchronization, etc. [1–4]. However, some exceptional examples exist, where a weak coupling does not make the coupled chaotic systems approach synchronization but make them become more complicated. For example, in two coupled Lorenz systems, a weak coupling will result in three positive Lyapunov exponents, in contrast to the maximum two positive Lyapunov exponents in another coupling range [3]. The reason is that the direction of flow is disturbed by the weak coupling. Then, an interesting question is: can this phenomenon be observed in the case of coupled maps where there is no direction of flow?

A map can be considered as a projection of flow on the Poincaré section and thus does not have a trajectory direction. Based on the observation that the trajectory of a Logistic map consists of a series of "up" and "down" iterations, ref. [2] introduced the concept of "direction-phase" to describe the phase of a Logistic map and afterward the concept has been widely used to characterize the PS of coupled maps [5–10], such as the PS of inhomogeneous globally coupled map lattices [5], the intermittent PS of discrete systems [6], the behavior of direction-phase at crises [7], and other coupled discrete dynamics on a variety of networks [8–10]. Recently, we have studied the influence of the network topology on the disorder phase [11] and found that the degree of heterogeneity of complex networks can influence the PS significantly.

It is well known that the property of chaotic behavior depends on the bifurcation parameter. For example, the chaotic attractor of Rössler oscillator is a single scroll for some range of parameters where there is a fixed rotation center but a funnel shape for some other range of parameters where the rotation center shifts with time [12]. For the former, the phase can be easily given in the xy-plane by $\phi = \arctan \frac{y}{x}$; while for the latter, the phase is ill-defined in the xy-plane but can be described in the $\dot{x}\dot{y}$-plane by $\phi = \arctan \frac{\dot{y}}{\dot{x}}$. For convenience, let us call the former "simple chaos" and the latter "complex chaos".

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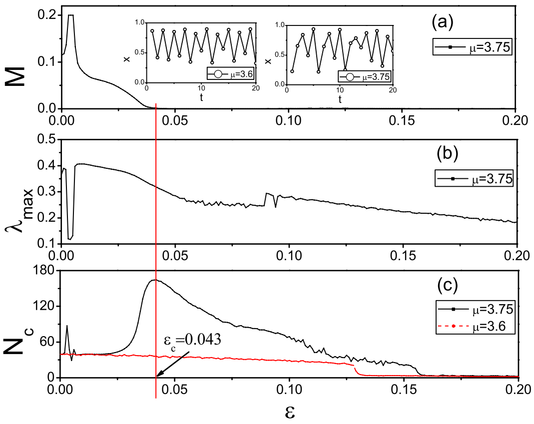

Fig. 1: (Color online) (a) M *vs.* ε for $\mu = 3.75$, where the insets represent the iterations in a short time period for $\mu = 3.6$ and $\mu = 3.75$, respectively; (b) λ_{max} *vs.* ε for $\mu = 3.75$; (c) N_c *vs.* ε where the "solid" and "dashed" lines denote the cases of complex chaos with $\mu = 3.75$ and of simple chaos with $\mu = 3.60$, respectively.

Thus, in the case of complex chaos, the description of the phase should be done in the $\dot{x}\dot{y}$-plane or by a new concept of spatial phase in the d-dimensional space, which has been presented recently for an arbitrary spatial curve [4]. A similar situation occurs in the Logistic map $x_{n+1} = \mu x_n(1 - x_n)$, whose behavior is simple chaos for $\mu_\infty < \mu < \mu_0$ and complex chaos for $\mu_0 < \mu < 4$ with $\mu_\infty \approx 3.57$ and $\mu_0 \approx 3.6786$ [2].

In the region of simple chaos of the Logistic map, the sequential iterations consist of the alternative appearance of "up" and "down" phases, see the inset of fig. 1(a) for the case of $\mu = 3.60$; while in the region of complex chaos, the sequential iterations do not always alternatively jump but sometimes have two continuous "up" phases, see the inset of fig. 1(a) for the case of $\mu = 3.75$. Comparing with the alternative up and down phases, the two continuous up phases are a kind of disorder and will cause a significant effect in the synchronization of coupled maps [11]. In this paper, we consider the coupled Logistic maps in a scale-free (SF) network and focus on the range of the weak-coupling strength. We find that there is a double-resonance effect on the coupling strength ϵ where the first resonance comes from the coupling-induced periodic behaviors and the second one from the disappearance of the disorder phase. For the second resonance, there is a critical coupling strength ε_c. The number of direction-phase clusters will increase with ε when $\varepsilon < \varepsilon_c$ and then decrease with the further increase of ε when $\varepsilon > \varepsilon_c$, which is similar to the case of two coupled Lorenz systems where the third positive Lyapunov exponent will first increase (from zero to positive) with the coupling strength and then decrease (from positive to negative) with the further increase of the coupling strength [3]. We study its mechanism through the out-of-phase links and then use two coupled maps as an example to show why the weak coupling can cause the increase of direction-phase clusters. Furthermore, we study the influence of the non-identity among oscillators and find that ε_c will increase rapidly with the parameter μ but slowly with the range of the distribution of non-identical oscillators.

Mechanism of the resonance effect on the net direction-phase. – We consider a SF network with size N and let each node of the network represent a Logistic map. The coupling between two neighboring nodes is implemented through the connected link. In detail, we construct the SF network by the Barabasi-Albert algorithm [13,14]. The constructed network has degree distribution $P(k) \sim k^{-3}$, average degree $\langle k \rangle = 6$, and size $N = 500$. The dynamics at each node can be represented by

$$x_{t+1}^i = (1 - \varepsilon) f(x_t^i) + \frac{\varepsilon}{k_i} \sum_{j \in k_i} f(x_t^j), \qquad (1)$$

where k_i denotes the degree of node i, the sum runs for all the nearest neighbors of node i, the coupling strength is ε, and the function $f(x)$ is chosen as the Logistic map

$$f(x) = \mu x(1 - x). \qquad (2)$$

The direction-phase of eq. (1) can be defined as follows. For a node i, if the value x_{t+1}^i is larger than the value x_t^i, it is in the "up" phase and signed a value "+1"; otherwise, it is in the "down" phase and signed a value "−1". Letting S_t^i denote the direction-phase, we have

$$S_t^i = \begin{cases} +1, & \text{if } x_{t+1}^i - x_t^i > 0, \\ -1, & \text{if } x_{t+1}^i - x_t^i \leqslant 0. \end{cases} \qquad (3)$$

Taking the average on all the nodes and on a time interval T, we obtain the net direction-phase

$$M = \lim_{T \to \infty} \frac{1}{TN} \sum_{t=1}^{T} \sum_{i=1}^{N} S_t^i. \qquad (4)$$

Obviously, M will be zero for the simple chaos. However, in the region of complex chaos, M may be non-zero for weak coupling because of the existence of the disorder phase. Figure 1(a) shows the result for $\mu = 3.75$. It is easy to see that M is positive for $\varepsilon < \varepsilon_c \approx 0.043$. Another point shown in fig. 1(a) is that when ε is very small, M will increase first and then decrease, *i.e.*, a resonance on ε. This is an interesting phenomenon, indicating that the coupling does not always reduce the disorder phase and that a weak coupling may even induce more the disorder phase.

Figure 1(b) shows the maximum Lyapunov exponent λ_{max}, corresponding to fig. 1(a). Comparing fig. 1(a) with fig. 1(b) one can easily see that the sudden increase of M in fig. 1(a) corresponds to the sudden decrease of λ_{max} in fig. 1(b). As the value of positive λ_{max} represents the degree of irregularity of the chaotic trajectory, the sudden decrease of λ_{max} means that the oscillators on the network become more regulated. To find the connection between

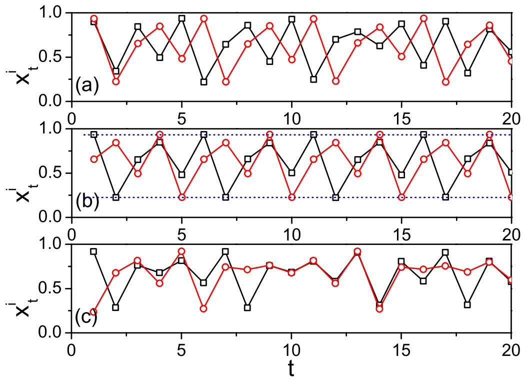

Fig. 2: (Color online) Trajectory of two arbitrary neighboring nodes in a short time interval for $\varepsilon = 0$, 0.004 and 0.01 in (a), (b) and (c), respectively. The two "dotted" straight lines in (b) are a guide for the eye.

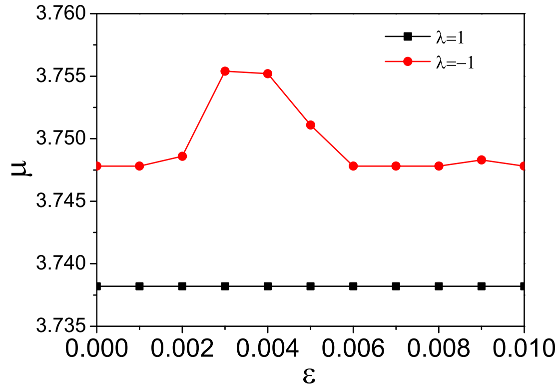

Fig. 3: (Color online) Boundaries of period-5 for $\lambda = \pm 1$. The area between the two curves has $|\lambda| < 1$.

the sudden variations of M and λ_{max}, we have checked the behaviors of the neighboring oscillators. We find that most neighboring nodes become approximately periodic and are delayed-phase–synchronized. Figure 2(b) shows the trajectory of two arbitrary neighboring nodes in a short time interval for $\varepsilon = 0.004$ in the resonance region, where the two "dotted" straight lines are a guide for the eye. From fig. 2(b) it is easy to see that both trajectories are period-5 ones. In each period-5 trajectory, there are two continuous "up" and "up" phases or a disorder phase, resulting in $M = 0.2$ in fig. 1(a). For comparison, we also plot the behaviors of the two oscillators in fig. 2(a) and (c) for the cases of $\varepsilon = 0$ and 0.01 in the non-resonance region, respectively. It is clear that there are no periodic or synchronized behaviors in both fig. 2(a) and (c). Thus, we conclude that the weak-coupling–induced period-5 trajectory makes the λ_{max} in fig. 1(b) decrease and the PS of the disorder phase in the period-5 trajectory makes the M in fig. 1(a) increase.

Our numerical simulations show that this resonance effect of M can be observed in a small range around $\mu = 3.75$ $(3.745 < \mu < 3.758)$. For revealing the mechanism of weak-coupling–induced period-5 trajectory, we suppose $x_1(i)$, $x_2(i)$, $x_3(i)$, $x_4(i)$ and $x_5(i)$ are the period-5 trajectories with relation

$$x^i_{s+1} = (1-\varepsilon)\mu x^i_s(1-x^i_s) + \frac{\varepsilon}{k_i}\sum_{j\in k_i}\mu x^j_s(1-x^j_s), \qquad (5)$$

where $s = \mathrm{mod}(t, 5)$ and $s+1 = \mathrm{mod}(t+1, 5)$ in eq. (1). Letting $x^i_{s+1} = F(x^i_s)$ and $\mathbf{J}$ be the Jacobian of F^5, we obtain the eigenvalue equation $\det|\mathbf{J} - \lambda\mathbf{I}| = 0$. In order to have the stable solutions of $x^i = F^5(x^i)$, the condition $|\lambda_{max}| \leqslant 1$ needs to be satisfied. By setting $\lambda = \pm 1$ we obtain that the boundaries of parameters μ and ε meet this condition. Figure 3 shows the boundaries on the $\mu\varepsilon$-plane for the case of two nodes. It is easy to see that there is a stable range $0.002 < \varepsilon < 0.005$ for $\mu = 3.75$. When the coupled oscillators become three or more, we observe that the stable range of ε will be slightly increased, indicating that the stable period 5 is induced by the coupling among neighboring nodes.

Mechanism of the resonance effect on the number of direction-phase clusters. – On the other hand, in the region of weak coupling, the coupling is not strong enough to induce a global PS and thus there are a lot of direction-phase clusters. In each direction-phase cluster, the oscillators have the same phase, *i.e.* their S^i_t have the same "+1" or "−1" at the same time. Generally, with the increase of coupling strength, the small clusters will merge to form larger clusters and thus the number of clusters will decrease. Finally, when the coupling strength is large enough, all the clusters will merge together to form a giant cluster and thus an in-phase synchronization will show up. We have observed this kind of phenomenon in the region of simple chaos. However, we observe a different but interesting phenomenon in the regime of complex chaos. We find that the number of direction-phase clusters will increase with ε when $\varepsilon < \varepsilon_c$ and then decrease with the further increase of ε when $\varepsilon > \varepsilon_c$, *i.e.*, a resonance on ε. The "solid" line in fig. 1(c) shows the result for $\mu = 3.75$. For comparison, we also plot the case of $\mu = 3.6$ ("dashed" line) there. Obviously, the case of $\mu = 3.6$ shows a monotonous decrease while the case of $\mu = 3.75$ shows a resonance around ε_c. We have observed this phenomenon in ref. [11], but its mechanism is still not very clear. Then an interesting question is how the resonance shows up. For better illustration, the insets of fig. 1(a) show the typical behaviors of a Logistic map at $\mu = 3.6$ and 3.75, respectively.

To reveal the mechanism of resonance of N_c, we classify the links of the network into three types. Observe a time interval such as $\tau = 10$ time steps. If two connected nodes keep the same (opposite) phase, we call it in(anti)-phase order, and the link attaching them is named in(anti)-phase link. While if the relationship of two connected nodes is not maintained as the same or opposite phase, *i.e.*, their phases are sometimes the in-phase and sometimes the anti-phase during the observed time, we name it as out-of-phase order and the corresponding link as out-of-phase link. Figures 4(a) and (b) show how the three

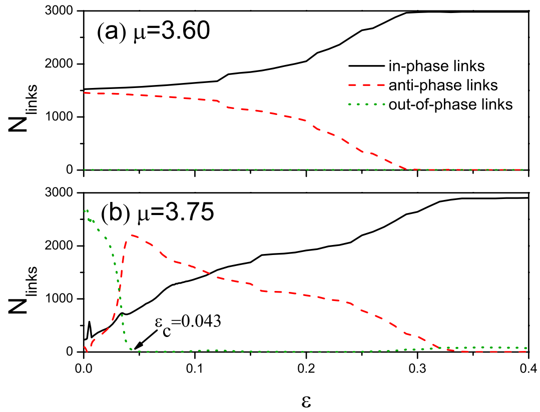

Fig. 4: (Color online) The dependence of the numbers of in-phase links ("solid" line), anti-phase links ("dashed" line), and out-of-phase links ("dotted" line) on the coupling strength ε. (a) denotes the case of $\mu = 3.60$ and (b) the case of $\mu = 3.75$.

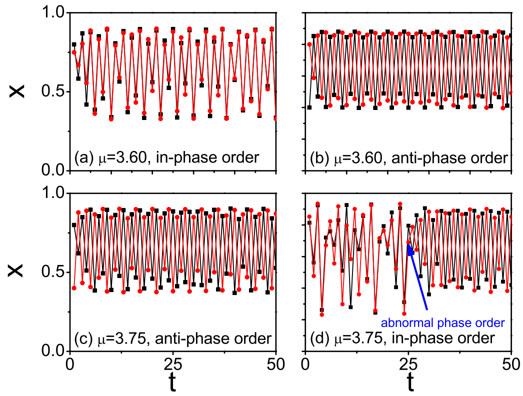

Fig. 5: (Color online) Evolutions of two coupled Logistic maps where (a) and (b) represent the case of $\mu = 3.60$ and (c) and (d) the case of $\mu = 3.75$.

numbers change with the coupling strength ε for simple and complex chaos, respectively. From fig. 4(a) it is easy to see that there are no out-of-phase links in the region of simple chaos. When $\varepsilon = 0$, the portions of the in-phase links and anti-phase links are evenly distributed. With the increase of ε, the number of in-phase links increases while the number of anti-phase links decreases, indicating the transformation from anti-phase links to in-phase links. When ε reaches 0.29, there are only in-phase links, which implies the appearance of in-phase synchronization.

However, the situation is totally different in the case of complex chaos. There is a critical coupling strength $\varepsilon_c \approx 0.043$ for $\mu = 3.75$, see fig. 4(b). When $\varepsilon < \varepsilon_c$, there exists out-of-phase links. When $\varepsilon = \varepsilon_c$, the number of out-of-phase links reaches zero and the number of anti-phase links reaches the maximum. When $\varepsilon > \varepsilon_c$, there are only in-phase links and anti-phase links in the system, and the anti-phase links transfer to in-phase links gradually with the further increasing coupling strength until there is no anti-phase links. Comparing the value of ε_c with the transition point in fig. 1 we see that they are approximately the same. Thus, we conclude that the coupling is used to suppress the disorder phase when $\varepsilon < \varepsilon_c$ and then used to enhance the synchronization when $\varepsilon > \varepsilon_c$.

In the following, we consider establishing a relationship between the number N_c in fig. 1(b) and the number of out-of-phase links in fig. 4(b). From fig. 4(b) we see that when ε increases from $\varepsilon < \varepsilon_c$, the out-of-phase links are transformed into the in-phase links and anti-phase links. The anti-phase links are dominant in all the links transformed from the out-of-phase ones. It is not difficult to see that the increase of anti-phase links results in the increase of N_c in fig. 1(b). To understand why the part of anti-phase links is larger than that of in-phase links, we consider two coupled Logistic maps with

$$x^1_{t+1} = f(x^1_t) + \varepsilon(f(x^2_t) - f(x^1_t)),$$
$$x^2_{t+1} = f(x^2_t) + \varepsilon(f(x^1_t) - f(x^2_t)). \quad (6)$$

We let $\varepsilon = 0.04$. Figure 5 shows the evolution of x^1 and x^2 with the iterations, where (a) and (b) represent the case of $\mu = 3.60$ and (c) and (d) the case of $\mu = 3.75$. From fig. 5(a) and (b) we see that the initial status of the two coupled oscillators will be conserved forever, implying that the initial conditions determine the final status. However, in fig. 5(c) and (d) we see that the situation is different from fig. 5(a) and (b). Figure 5(d) shows that if the initial condition of the two coupled oscillators is in the in-phase status, they may not be conserved. The disorder (see the arrow in fig. 5(d)) will change the in-phase status into anti-phase status. Once they become anti-phase status, the relationship will be conserved.

Why is the relationship of anti-phase status stable, while the relationship of in-phase status is not? The mechanism can be understood as follows. After the transient time, the Logistic map has different upper and lower boundaries for different μ, which can be easily calculated. The maximum is achieved when $x_t = 1/2$ and can be written as $x_{max} = x_{t+1} = \mu/4$. The minimum is the next iteration of x_{max}. Thus, the boundaries can be written as

$$\begin{cases} x_{max} = \dfrac{\mu}{4}, & \text{if} \quad x_t = \frac{1}{2}, \\ x_{min} = \frac{\mu^2}{4} - \dfrac{\mu^3}{16}, & \text{if} \quad x_t = \frac{\mu}{4}. \end{cases} \quad (7)$$

That is, the range of x is $[\frac{\mu^2}{4} - \frac{\mu^3}{16}, \frac{\mu}{4}]$. Then we calculate the range for the emergence of normal phase order, which means the alternative up and down phases. The range can be determined by the inequations

$$\begin{cases} x < \mu x(1-x), \\ \mu x(1-x) > \mu^2 x(1-x)(1-\mu x(1-x)); \end{cases} \quad (8)$$

or

$$\begin{cases} x > \mu x(1-x), \\ \mu x(1-x) < \mu^2 x(1-x)(1-\mu x(1-x)). \end{cases} \quad (9)$$

The solution of the inequations is $\frac{1}{\mu} < x < \frac{1}{2} + \frac{\sqrt{\mu^2-4}}{2\mu}$. Then we combine the two ranges and let $\frac{\mu^2}{4} - \frac{\mu^3}{16} = \frac{1}{\mu}$ or

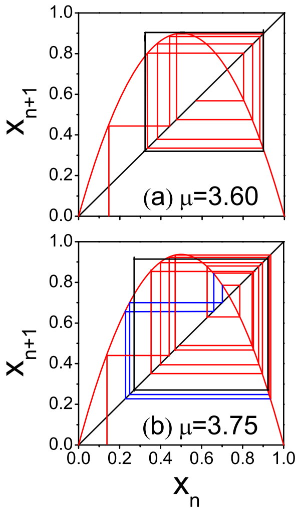

Fig. 6: (Color Online) The stationary trajectory of the Logistic map for $\mu = 3.60$ in (a) and $\mu = 3.75$ in (b).

$\frac{\mu}{4} = \frac{1}{2} + \frac{\sqrt{\mu^2-4}}{2\mu}$. It reaches $\mu \simeq 3.6786$, which is exactly the point μ_0 to separate simple chaos and complex chaos in a single Logistic map [2]. Thus, the x range of the normal phase order can be given as

$$\begin{cases} \frac{\mu^2}{4} - \frac{\mu^3}{16} < x < \frac{\mu}{4}, & \text{if} \quad \mu < \mu_0, \\ \frac{1}{\mu} < x < \frac{1}{2} + \frac{\sqrt{\mu^2-4}}{2\mu}, & \text{if} \quad \mu \geqslant \mu_0. \end{cases} \tag{10}$$

It is clear that when $\mu < \mu_0$, x is always in the normal phase order region, see fig. 6(a) where the black-frame area denotes the normal phase order region $[\frac{\mu^2}{4} - \frac{\mu^3}{16}, \frac{\mu}{4}]$. However, for $\mu \geqslant \mu_0$, the normal phase region $[\frac{1}{\mu}, \frac{1}{2} + \frac{\sqrt{\mu^2-4}}{2\mu}]$ is smaller than the available range $[\frac{\mu^2}{4} - \frac{\mu^3}{16}, \frac{\mu}{4}]$ and thus the value of x has a chance to jump out of the normal phase region and shows the disorder phase, see fig. 6(b) where the black-frame area denotes the normal phase order region. Suppose two Logistic maps with $\mu = 3.75$ are coupled and the coupling strength is 0.04, then we trace one's iteration trajectory. If their phase orders are opposite at the beginning, the coupled term is relatively large. It may prevent the trajectory jumping out of the normal phase region. However, if the initial phases of the two maps are the same, the coupled term is relatively smaller. It may be sometimes not larger enough to hold one map jumping inside the black-frame area of fig. 6(b). Once one of the two maps jumps out of that area, it will show the disorder phase, and then the maps become anti-phase related and keep stable. This is the reason why in the region of complex chaos, the system will first achieve anti-phase synchronization and then keep stable.

Moreover, we consider two coupled Logistic maps with anti-phase relation in the complex-chaos regime. The values of $f(x_t^1)$ and $f(x_t^2)$ distribute quasisymmetrically

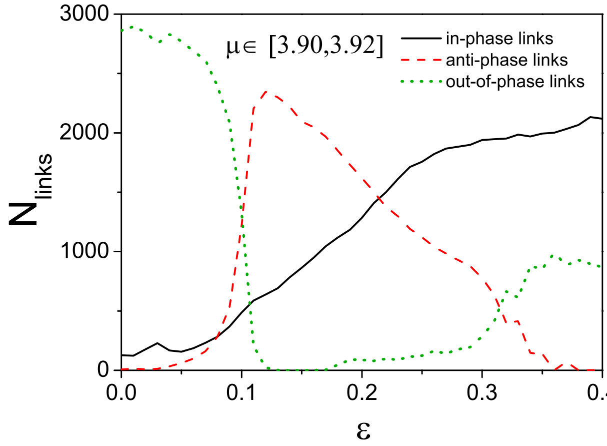

Fig. 7: (Color online) The dependence of the numbers of in-phase links ("solid" line), anti-phase links ("dashed" line), and out-of-phase links ("dotted" line) on the coupling strength ε for $\mu \in [3.90, 3.92]$.

on both sides of an average value (see figs. 5(c) and (d)), *i.e.*, $a \approx \langle \frac{f(x_t^1)+f(x_t^2)}{2} \rangle_t$, where $\langle \cdot \rangle_t$ is the average value over the iteration time for a certain μ. Then we have $f(x_t^2) \approx 2a - f(x_t^1)$. Substituting it into the first of equations (6), we obtain

$$\begin{aligned} x_{t+1}^1 &\approx 2a\varepsilon + (1-2\varepsilon) f(x_t^1) \\ &= 2a\varepsilon + (1-2\varepsilon)\mu x_t^1(1-x_t^1). \end{aligned} \tag{11}$$

Let $\mu' = (1-2\varepsilon)\mu$ and consider the weak-coupling strength, then we have $x_{t+1}^1 = \mu' x_t^1(1-x_t^1)$. With increasing ε, μ' decreases which induces the system to enter the simple-chaos regime, where the anti-phase relation between two coupled maps is stable. Therefore, the anti-phase relation is stable for coupled complex-chaos Logistic maps under weak-coupling strength.

Back to the case of network coupled Logistic maps, the anti-phase feature should be conserved between neighboring nodes. As we have mentioned in ref. [11], the topology of network will influence the anti-phase feature and result in the decrease of direction-phase clusters N_c with the increase of heterogeneity. For example, in the case of a 2D lattice, each node has the same degree and thus it is easy for the neighboring nodes to be anti-phase. However, when we change the 2D lattice to a small-world network by rewiring, the shortcuts will make it difficult for a common neighbor of two anti-phase neighbors to sustain an anti-phase relationship to both the neighbors. In this way, the increasing of heterogeneity will reduce the number N_c.

The robustness of the resonance effect to the nonidentity of coupled oscillators. – Now let us move to the case of coupled nonidentical Logistic maps, *i.e.*, different μ_i for different oscillators. We let μ_i be randomly distributed in $[3.90, 3.92]$. Doing the same numerical simulations as in the case of identical oscillators, we find that the numbers of three types of links show similar variations as in fig. 4(b) except for the transition point. Figure 7

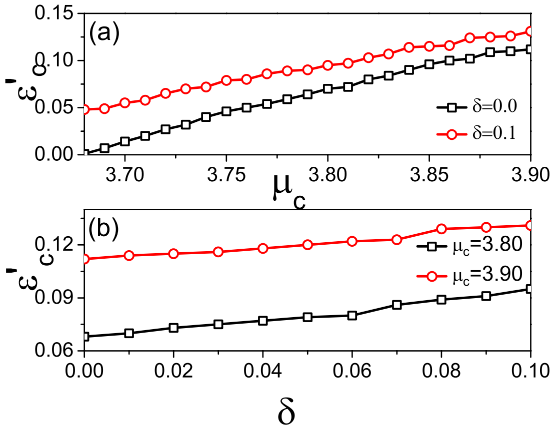

Fig. 8: (Color online) (a) The transition point ε_c' *vs.* μ_c where the "squares" and "circles" represent the cases of $\delta = 0$ and 0.1, respectively; (b) ε_c' *vs.* δ where the "squares" and "circles" represent the cases of $\mu_c = 3.8$ and 3.9, respectively.

shows the results. Therefore, the resonance effect of N_c is robust to the nonidentity of oscillators. From fig. 7 we see that the transition coupling $\varepsilon_c' \approx 0.13$ is different from the $\varepsilon_c \approx 0.04$ in fig. 4(b). Thus, it would be interesting to know how the transition point ε_c' depends on the distribution range of μ_i. To solve this problem, we let μ_i be randomly distributed in the range $[\mu_c - \delta, \mu_c + \delta]$ with μ_c in the region of complex chaos. Then we calculate ε_c' for different μ_c and δ. We find that ε_c' increases rapidly with the parameter μ_c but slowly with the range δ of the distribution of non-identical oscillators. Figure 8(a) shows the relationship between ε_c' and μ_c where the "squares" and "circles" represent the cases of $\delta = 0$ and 0.1, respectively; and fig. 8(b) shows the relationship between ε_c' and δ where the "squares" and "circles" represent the cases of $\mu_c = 3.8$ and 3.9, respectively. From fig. 8(a) we see that the ε_c' of $\delta = 0.1$ is larger than that of $\delta = 0$. This is easy to understand. As the case of $\delta = 0.1$ contains some oscillators with $\mu > \mu_c$ and their ε_c''s are proportional to μ, then the contribution from those oscillators in $[\mu_c - \delta,\ \mu_c + \delta]$ will result in a larger ε_c'. Besides, for example, when $\mu_c = 3.80$, $\delta = 0.1$, *i.e.*, $\mu \in [3.70, 3.90]$, we have $\varepsilon_c' = 0.095$, which is close to but a bit smaller than ε_c' for the identical oscillators with $\mu_c = 3.90$, $\delta = 0.0$, *i.e.*, $\varepsilon_c' = 0.112$. We can say that the critical coupling strength ε_c' is more dependent on the value of the larger bifurcation parameter $\mu_c + \delta$, which is jointly affected by the value of μ_c and δ. From fig. 8(b) we see that ε_c' increases slowly with δ, indicating the robustness to the range δ.

Discussion and conclusions. – From the above discussions we see that when ε is around ε_c, the weak coupling can be divided into two regions and their corresponding functions are different. In the region with $\varepsilon < \varepsilon_c$, the coupling is mainly used to suppress the disorder. At $\varepsilon = \varepsilon_c$, the disorder will be completely suppressed. After that, the coupling will take its normal function of increasing synchronization, *i.e.*, it will have the same function as in the case of simple chaos. The suppression of disorder is performed through the out-of-phase links. That is, in the first stage, the out-of-phase links will be mainly transformed into the anti-phase links. In the second stage with $\varepsilon > \varepsilon_c$, the anti-phase links are transformed into the in-phase links and then result in the PS.

In conclusions, we have uncovered the double-resonance effects on the net direction-phase M and the number of direction-phase clusters N_c in the region of weak coupling and explained their mechanisms. Moreover, we show that ε_c will increase with the range of the distribution of non-identical oscillators. Our finding implies that *the weak coupling making the system more complicated* is not only for the flows but it is also a general phenomenon for both the flow and map systems.

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