Phase synchronization between two essentially different chaotic systems

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In this paper, we numerically investigate phase synchronization between two coupled essentially different chaotic oscillators in drive-response configuration. It is shown that phase synchronization can be observed between two coupled systems despite the difference and the large frequency detuning between them. Moreover, the relation between phase synchronization and generalized synchronization is compared with that in coupled parametrically different systems. In the systems studied, it is found that phase synchronization occurs after generalized synchronization in coupled essentially different chaotic systems.

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I. INTRODUCTION

Chaos synchronization has been extensively studied since the pioneer work by Pecora and Carroll in 1990 [1]. So far, various types of synchronization in chaotic systems have been classified [2,3], such as complete synchronization (CS), generalized synchronization (GS), lag synchronization (LS), and phase synchronization (PS). Among them, PS refers to the condition where the phases between two chaotic oscillators are locked, or the weaker condition where the mean frequencies between two chaotic oscillators are locked. This interesting synchronization phenomenon was first reported by Rosenblum *et al.* [4,5]. They found that a suitable definition of phase $\phi(t)$ can be introduced for certain chaotic systems, such as the Rössler system and the Lorenz system. Due to the fluctuation in the amplitude of the chaotic oscillators, the defined phase is generally diffusive and the phase diffusion can be characterized by

$$\langle[\phi(t) - \langle\phi(t)\rangle]^2\rangle = 2D_\phi t, \quad (1)$$

where $\langle\cdots\rangle$ denotes ensemble average and a very small D_ϕ indicates a well-defined phase [2,5]. It has been shown that when the parameter mismatch between the coupled chaotic oscillators is small, phase locking between two oscillators can be achieved while their amplitudes may remain chaotic and be uncorrelated. In such autonomous chaotic flow systems, the defined phase variable corresponds to the null Lyapunov exponent (LE) of the system, which generally becomes negative when PS occurs [4]. Apart from the theoretical studies, experimentally PS has already been demonstrated in various fields, such as electrical circuits [6–8], lasers [9,10], fluids [11], and biological systems [12,13]. These important works reveal that PS is the key to understand the dynamics of many chaotic systems. The finding and investigation of PS in coupled chaotic systems have greatly enriched the field of traditional synchronization of periodic oscillators.

So far, the research on PS in chaotic systems mainly concentrates on the following three domains. In the first domain, the chaotic oscillator entrained by the external periodic force is investigated [5,6,8,14]. In the second domain, PS between two coupled chaotic oscillators with different natural frequencies [4,7,16–19] is studied. In the third domain, PS in the array of coupled chaotic oscillators has been investigated [15]. For PS in coupled chaotic oscillators, most of the theoretical works [4,16–21] deal with two chaotic systems with parameter mismatch, i.e., coupled parametrically different systems.

We notice that PS extensively exists between essentially different chaotic systems in nature, for example, the synchronization between cardiac and respiratory system [22], biological clocks synchronized by day and night rhythm, and ecological systems entrained by seasonal cycles, just to name a few. Also, there are experiments where PS between coupled essentially different chaotic systems are reported [7,9]. By essentially different chaotic systems we refer to systems with different dynamical equations. In our opinion, PS between these essentially different chaotic systems with characteristic internal time scales is very interesting and important. However, the existing studies on PS in chaotic systems almost all concentrate on coupled parametrically chaotic oscillators. Motivated by this, in the present paper we aim at investigating PS between coupled essentially different chaotic systems. We believe that for certain coupled essentially different chaotic oscillators, PS may be achieved provided phases can be well defined in such systems. Nevertheless, due to the physical difference, it can be expected that PS between two essentially different chaotic oscillators may exhibit different features from that in coupled parametrically different systems. In addition, the transition between PS and GS in coupled essentially different chaotic systems deserves further comparison with the previous results [4,16–18]. To this end, three physically different chaotic oscillators have been considered in the present work, including a nonlinear electric circuit system, the Rössler system and the Lorenz system. Our results show that PS can be achieved even when there is large difference between the natural frequencies of the coupled oscillators. We further analyze the relation between PS and GS by computing the conditional Lyapunov exponents (CLEs). Finally, the relation between PS and GS in essentially different systems is compared with that in parametrically different systems. We emphasize that the unidirectional coupling, i.e., the drive-response configuration, is considered in the present work, which is different from Ref. [23] where special mutual phase coupling is used.

This paper is organized as follows. In Sec. II, we present the results of PS between the electric circuit oscillator and the Rössler oscillator. Their attractors have similar topological structures. In Sec. III, PS between the Rössler oscillator and the Lorenz oscillator is investigated. These two attractors have different topological structures. At the end of this paper, there is the conclusion followed by some discussion.

II. PS BETWEEN THE ELECTRIC CIRCUIT SYSTEM AND THE RÖSSLER SYSTEM

It has been shown that for many chaotic systems, a phase can be suitably defined [5]. The typical examples are those attractors which behave like smeared limit cycles in phase space. Usually, the rotation center is one of the unstable fixed points of the system. As the first attempt to study PS between coupled essentially different systems, we couple two such chaotic oscillators in our model. The drive system is a nonlinear electric circuit [24] whose dynamical equations are

$$\dot{x}_1 = y_1,$$

$$\dot{y}_1 = z_1,$$

$$\dot{z}_1 = \mu x_1(1 - x_1) - y_1 - \gamma z_1, \quad (2)$$

with $\mu=1$ and $\gamma=0.5$. The Rössler oscillator

$$\dot{x}_2 = -\alpha(y_2 + z_2),$$

$$\dot{y}_2 = \alpha(x_2 + a y_2) - \epsilon(y_2 - y_1),$$

$$\dot{z}_2 = \alpha[b + z_2(x_2 - c)], \quad (3)$$

is used as the response system with $a=b=0.2$ and $c=5.7$. Here the response system is coupled with the drive system through the y variable. The time scale parameter α is introduced to adjust the natural frequency of the Rössler oscillator [28]. By continuously varying α, we can change the frequency detuning, i.e., the difference between the natural frequencies of the uncoupled oscillators, which usually plays an important role in PS problems.

The attractors of the drive and the response systems without coupling are shown in Figs. 1(a) and 1(b), respectively. For these one-center rotation chaotic oscillators, the phase can be straightforwardly defined as the angle in the xy plane [5], i.e.,

$$\phi(t) = \arctan\frac{y(t) - y_0}{x(t) - x_0}, \quad (4)$$

where (x_0, y_0) is the rotation center. The mean frequency (winding number) of the chaotic oscillator is [15,17]

$$\Omega = \langle d\phi(t)/dt \rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T \dot{\phi}(t)dt. \quad (5)$$

According to these definitions, the 1:1 PS between the drive and response systems can be characterized as phase locking

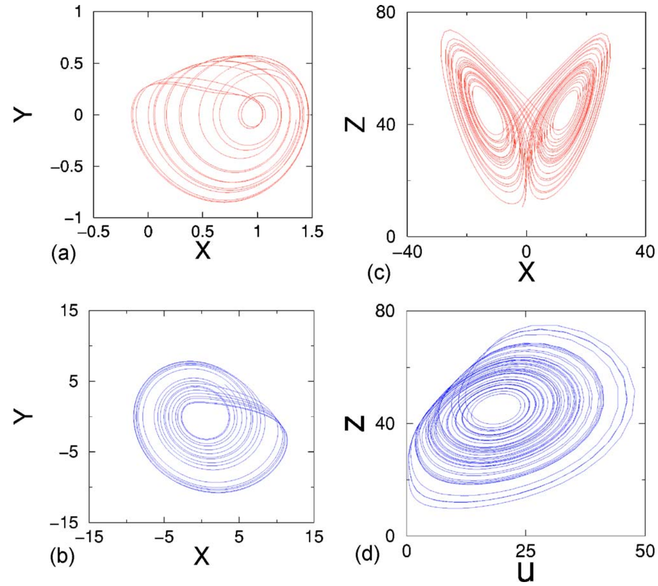

FIG. 1. (Color online) Typical attractors of (a) the electric circuit system; (b) the Rössler system; (c) the Lorenz system; (d) the Lorenz system on the uz plane.

$$|\phi_d(t) - \phi_r(t)| < \text{const} \quad (6)$$

or a weaker condition of frequency locking, i.e., the equality of their mean frequencies $\Omega_r=\Omega_d$. Throughout this paper the subscripts d and r denote the drive system and the response system, respectively. We emphasize that strictly phase locking and mean frequency locking are two independent criteria to characterize PS [2,5]. For example, in the case of noise in the phase dynamics, due to the phase slips the phase locking relation is difficult to be satisfied, while the weaker condition of frequency locking might be applicable. However, for deterministic dynamical systems, if the phase of a chaotic oscillator is well defined, usually these two criteria are equivalent to characterize PS in chaotic oscillators.

Without coupling, the natural frequencies of systems (2) and (3) are different, i.e., $\Omega_d=0.94$ and $\Omega_r\simeq\alpha$, respectively. When coupled, despite the different dynamical equations and the different natural frequencies, it is found that PS can occur between these two coupled oscillators with appropriate α and ϵ values. Figure 2 gives one example illustrating PS at $\alpha=1$ and $\epsilon=2$. It is seen that under the driving the attractor of the response system is still like a smeared limit cycle. What has been changed is its size, which shrinks and becomes comparable with that of the drive system. The phase locking between the two oscillators is demonstrated in Fig. 2(b), where the phase difference between the drive and the response systems does not grow with time but remain bounded. Figure 2(c) shows the frequency difference $\Delta\Omega=\Omega_d-\Omega_r$ versus the coupling strength ϵ. There it is found that PS begins at $\epsilon_p=1.75$. In the CLE spectrum of the response system shown in Fig. 2(d), it is seen that the largest CLE becomes negative at about $\epsilon_g=0.3$, which actually is the bifurcation point of GS in the coupled systems. Therefore, in this case, $\epsilon_g<\epsilon_p$, i.e., PS occurs after GS. In the drive-response configuration GS usually can be characterized by

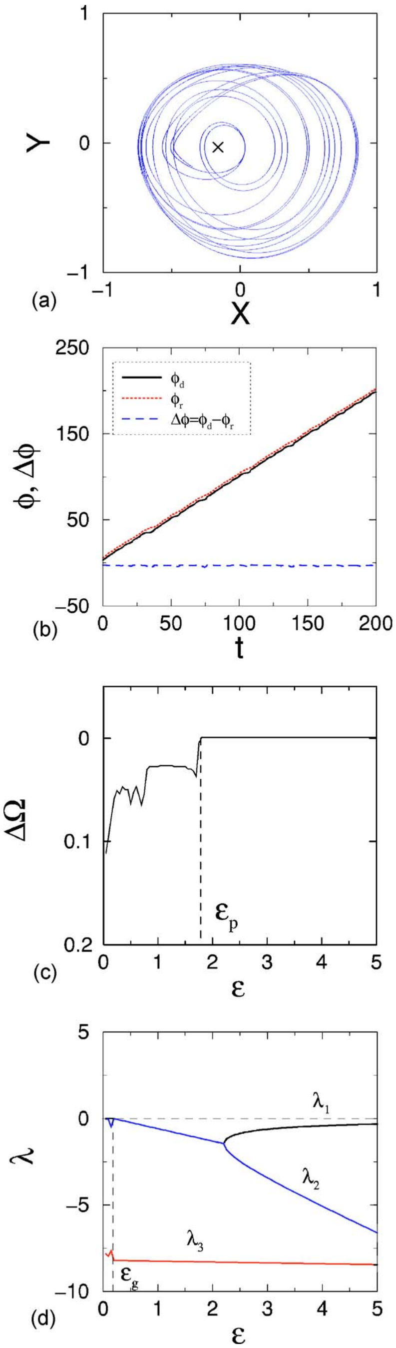

FIG. 2. (Color online) PS between systems (2) and (3). $\alpha=1, \epsilon=2$ for (a) and (b); $\alpha=1$ for (c) and (d). (a) The attractor of the response system under driving in the xy plane. The cross denotes the unstable fixed point which is used as the rotation center to define the phase. (b) The phases of the drive system (ϕ_d), the response system (ϕ_r), and their difference ($\Delta\phi$). (c) The frequency difference $\Delta\Omega$ versus the coupling strength ϵ. (d) The CLEs of the response system versus the coupling strength ϵ.

the negativity of the largest CLE in the response system [26–28]. However, we notice that the negativity of the largest CLE is only a necessary but not the sufficient condition for GS [25]. Therefore, in the present study the response-auxiliary system approach [27] has also been applied to confirm the occurrence of GS. Generally, the results from these two detecting methods are consistent.

One important factor in PS is the frequency detuning between the coupled oscillators, which usually determines whether PS can occur or not. Of course, another important parameter is the coupling strength. We study how these two parameters affect PS. The result is shown in Fig. 3(a) where parameter α ranges from 0.5 to 1.5. It is found that within this interval, PS can always be achieved as long as the coupling strength is large enough. Since within this α interval the relative frequency detuning $|\Omega_d-\Omega_r|/\Omega_d$ ranges from 0 to 0.68, it implies that PS between two essentially different chaotic oscillators can be achieved even when the frequency detuning of the coupled oscillators is large. Moreover, by exploring the bifurcations in the model, it is found that in general PS occurs after GS in the present model. This result is illustrated in Fig. 3(b). Also in Fig. 3(b), it is found that the PS bifurcation curve l_3 does not change continuously with respect to parameter α. At $\alpha=0.9$ there is a discontinuous point. For the response system (3), its frequency increases linearly with α. Numerically we found this relation is $\Omega_r=1.05\alpha$. From this relation it is found that $\alpha=0.9$ actually corresponds to the critical point where the natural frequencies of two oscillators are approximately equal to each other. Therefore, in the present model, if the natural frequency of the drive system is larger than that of the response system, relatively smaller driving is needed to achieve PS. Otherwise, relatively larger driving is needed as shown in Fig. 3(b). Recently, three types of transitions to PS in coupled parametrically different chaotic oscillators have been reported [18]. Our result is similar to the third type of transition defined in Ref. [18], i.e., the phases cannot be locking until a strong correlation of the amplitudes is established.

III. PS BETWEEN THE RÖSSLER SYSTEM AND THE LORENZ SYSTEM

So far, as theoretical models, two coupled Rössler oscillators with parameter mismatch and two coupled Lorenz oscillators with parameter mismatch have been extensively investigated in PS [4,16–19]. In this section, we further couple the Rössler system and the Lorenz system to study whether PS could occur between these two essentially different systems. The dynamical equations for the drive system are

$$\dot{x}_2=-\alpha(y_2+z_2),$$

$$\dot{y}_2=\alpha(x_2+ay_2),$$

$$\dot{z}_2=\alpha[b+z_2(x_2-c)], \tag{7}$$

with the same parameter settings as in the previous section. The response system is governed by

$$\dot{x}_3=\sigma(y_3-x_3)-\epsilon(x_3-x_2),$$

$$\dot{y}_3=rx_3-y_3-x_3z_3,$$

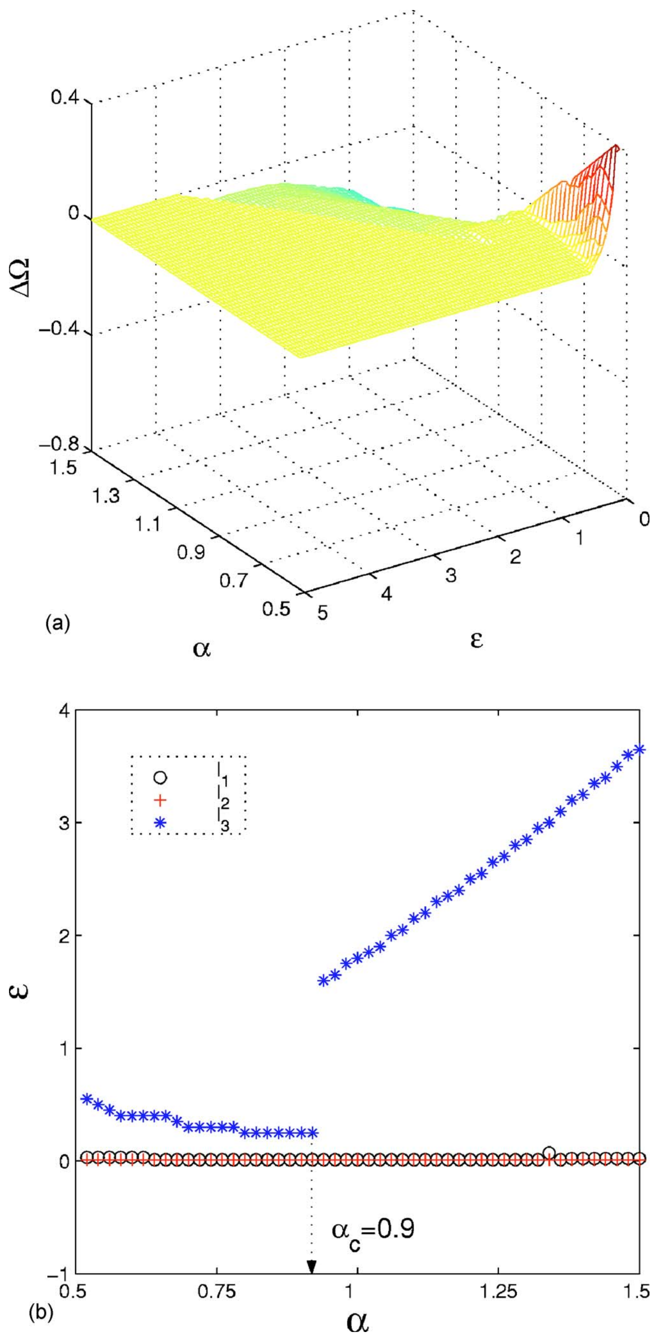

FIG. 3. (Color online) (a) The PS platform on the $\alpha-\epsilon$ parameter plane for system (3) driven by system (2). (b) Bifurcation curves for Eqs. (3) driven by Eqs. (2). l_1 corresponds to the GS bifurcation, i.e., above this line, the largest CLE of the response system becomes negative; l_2 corresponds to transition where the null LE of the response system becomes negative; and l_3 corresponds to the PS bifurcation, i.e., above this line, the mean frequencies of the drive and the response systems are locking.

$$\dot{z}_3 = x_3 y_3 - \beta z_3, \tag{8}$$

with $\sigma=16, r=45.92$ and $\beta=4$. Here the driving is added through the x variable. As shown in Fig. 1(c), the Lorenz attractor has two rotation centers. However, due to the symmetry, the phase in the Lorenz oscillator still can be defined in the uz plane with $u=\sqrt{x^2+y^2}$ [5], as shown in Fig. 1(d). Without coupling, the natural frequency of the Lorenz oscillator is about $\Omega_r=13.95$, which is much larger than that of the Rössler oscillator at $\alpha=1$, i.e., $\Omega_d=1.05$.

When $\alpha=1$, which corresponds to the case where the usual Rössler oscillator drives the Lorenz oscillator, unfortunately no PS is observed. We attribute this failure to the fact that the frequency detuning between two coupled oscillators is too large. With the increase of α, as expected, PS can be successfully observed in the present model. Interestingly, it takes place even between two chaotic oscillators with one rotation center and two rotation centers respectively when uncoupled. Figures 4(a) and 4(b) present an example when $\alpha=13$ and $\epsilon=10$. For $\alpha=13$, the natural frequency of the drive system becomes 13.75, which approaches the natural frequency of the response system. It is seen that the attractor of the response system has changed significantly: now it has only one rotation center under driving, rather than two rotation centers in the absence of driving. The rotation center is numerically found to be one of the unstable fixed points of the response system under driving. Although the attractor of the response system shrinks, it is still much larger than that of the drive system.

As shown in Fig. 4(a), the phase in the response system actually can be defined in both the xz and the yz planes. However, in our computation we still define the phase of system (8) in the uz plane. The reason for this is that the present model, i.e., Eqs. (7) and (8), can exhibit bistability [29]. Depending on the initial conditions, there are two different attractors in the response system, which are confined in the first quadrant and the third quadrant in the xy plane, respectively. But in the uz plane, we found that the rotation centers of the bistable attractors almost coincide, therefore defining phase in the uz plane can avoid the inconvenience in computation caused by the bistability in the present model. We emphasize that the rotation centers of the bistable attractors, which are the two unstable fixed points of the system, are only approximately symmetric with respect to the origin in the xy plane, not strictly, since the inversion symmetry of the Lorenz system has been destroyed by the driving term in Eqs. (8).

The phase-locking of the two chaotic oscillators is illustrated in Fig. 4(b), showing PS can be achieved. In order to investigate how PS is affected by the frequency detuning of the two coupled oscillators, we compute the frequency difference $\Delta\Omega=\Omega_d-\Omega_r$ on the α-ϵ parameter plane. The result is shown in Fig. 4(c). It is seen that a platform for PS exists on the α-ϵ parameter plane. From $\alpha=11$ to $\alpha=15$, PS can be achieved for sufficient large coupling strength. Within this interval, the relative frequency detuning $|\Omega_d-\Omega_r|/\Omega_r$ ranges from 0 to 0.21. This is still a relatively large range of frequency detuning.

It is known that the present model is a prototype demonstrating GS [26–28], which is another important type of chaotic synchronization between different dynamical systems. Now we have shown that PS can also be observed in this model. Naturally, we are interested in the relation between PS and GS when both of them can occur in one model. For this purpose, we explore the CLEs of the response system. Without coupling, the response system has three LEs: one

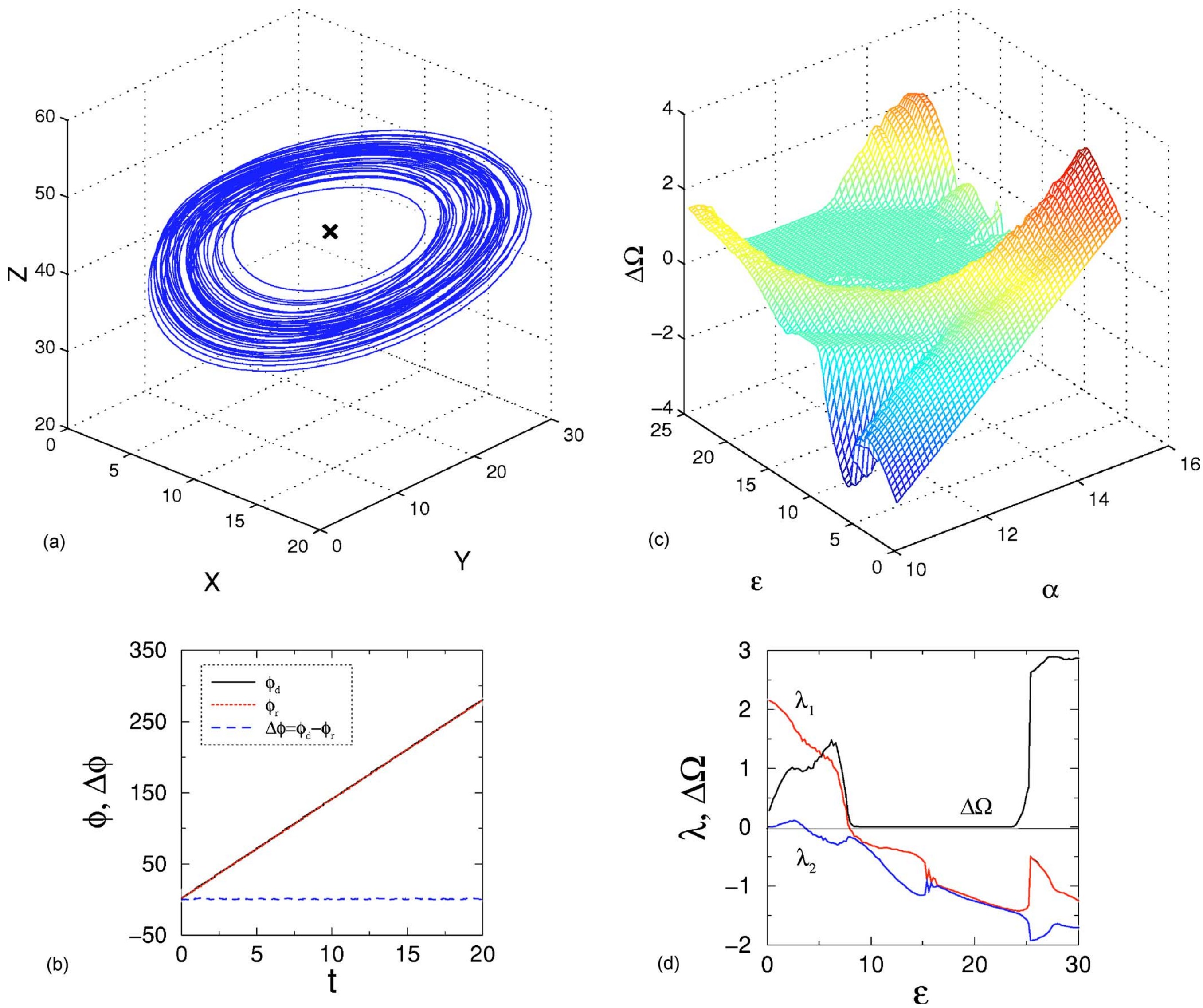

FIG. 4. (Color online) PS between systems (7) and (8). $\alpha=13, \epsilon=10$ for (a) and (b); $\alpha=13$ for (d). (a) The attractor of the response system under driving. The cross denotes the unstable fixed point which is used as the rotation center to define the phase. (b) The phases of the drive system (ϕ_d), the response system (ϕ_r), and their difference ($\Delta\phi$). (c) The PS platform on the $\alpha-\epsilon$ parameter plane. (d) The frequency difference $\Delta\Omega$, the largest two CLEs λ_1, λ_2 versus the coupling strength ϵ.

positive, one null and one negative. In Fig. 4(d), the two largest CLEs λ_1, λ_2 and the frequency difference between two oscillators $\Delta\Omega$ are plotted versus the coupling strength. From Fig. 4(d), it is found that in the current parameter setting PS occurs immediately after GS. In order to verify whether this is the general case, we intensively explore the largest CLEs of the response system with other α values. It is found that in the present model GS always can be observed, but PS cannot always be achieved. Whenever PS is achieved, it occurs after GS. These results are confirmed in Fig. 5. Qualitatively, they are the same as that in the previous section. But this time, there seems no obvious discontinuous point on PS bifurcation curve l_3.

In Fig. 4(d), we observe two types of anomalous behavior. The first one corresponds to the behavior of $\Delta\Omega$ before PS. It is found that with the increase of the coupling strength, the frequency difference is first amplified leading to maximal phase decoherence between two systems before PS is achieved. Therefore, what we observed in Fig. 4(d) is a kind of anomalous PS. Very recently, such phenomenon has been reported in coupled parametrically chaotic oscillators [30]. We found that it could be generally observed between two essentially different chaotic systems. The second anomalous behavior corresponds to λ_2 with small coupling strength. It is found that under weak coupling, λ_2 first becomes positive; then it becomes negative with the further increasing of coupling. This is different from Fig. 2(d) and the results in Ref. [4], where no positive regime of the second largest CLE is found with the increase of coupling strength. Since with weak coupling the largest CLE λ_1 is usually positive, the response system actually exhibits hyperchaos within the regime where λ_2 is positive. Physically, the null LE corresponds to the phase variable along the trajectory in phase space [2]. Its null value implies that the perturbation along

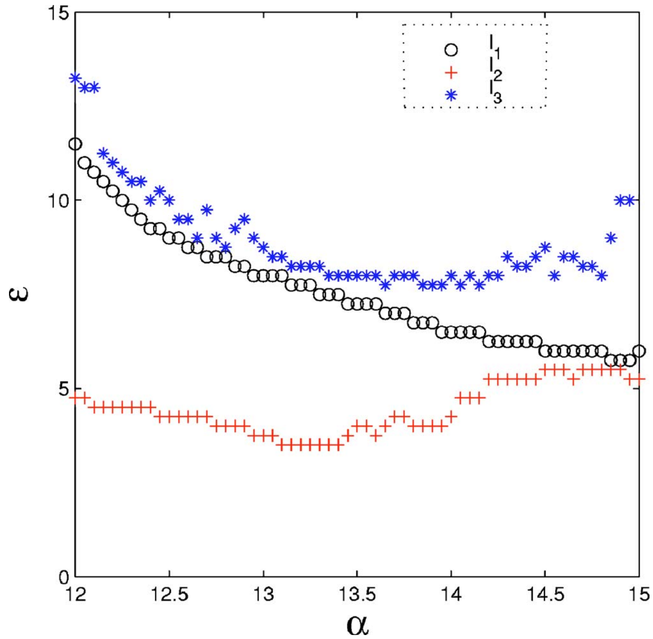

FIG. 5. (Color online) Bifurcation curves for system (8) driven by system (7). The meaning of curves l_1, l_2, and l_3 are the same as that in Fig. 3(b).

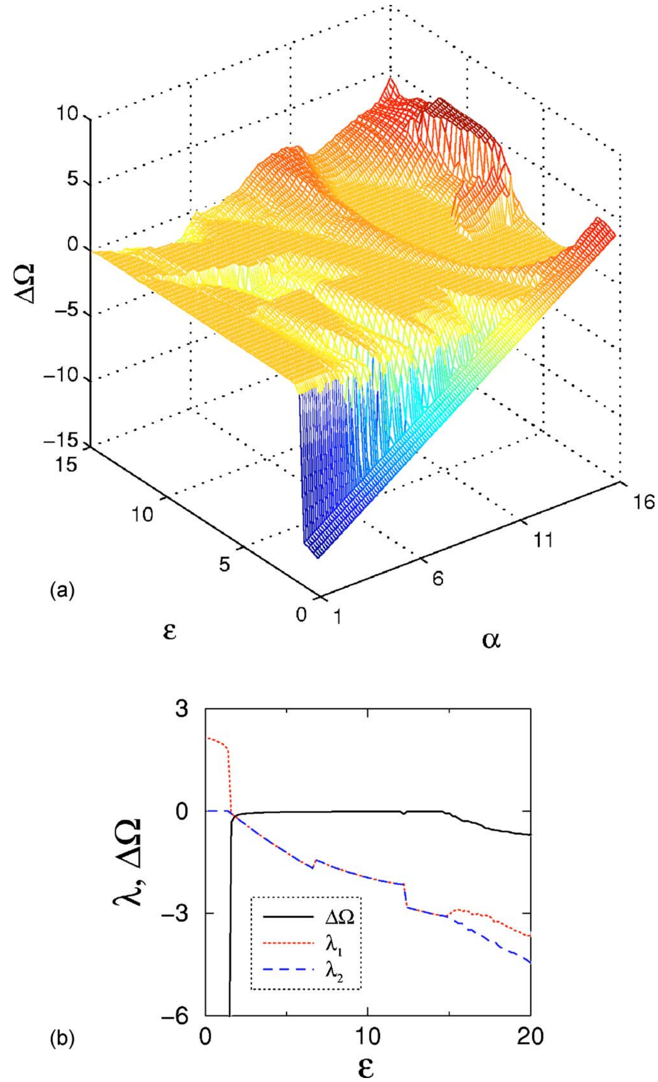

FIG. 6. (Color online) (a) The PS platform on the the $\alpha-\epsilon$ parameter plane for system (7) coupling system (8) via the y variable. (b) The frequency difference $\Delta\Omega$, the largest two CLEs λ_1, λ_2 versus ϵ. $\alpha=1$.

the trajectory neither grows nor decays. The positive λ_2 implies that the neutrality of the phase variable in the response system has been broken under the chaotic driving. Similar anomalous behavior of λ_2 has already been found in Ref. [31]. On the other hand, the second largest LE can also become locally negative under chaotic driving, which has been discussed in Ref. [32].

Finally the Lorenz oscillator driven by the Rössler oscillator through the y variable has been investigated. The results are shown in Fig. 6. In Fig. 6(a), it is found that PS can be observed for almost the whole range of parameter α studied, namely, from $\alpha=1$ to $\alpha=16$. This result is quite remarkable since it implies that PS can be achieved even when the natural frequency of the response oscillator is one order larger than that of the driving oscillator. One such example is illustrated in Fig. 6(b), where PS can even be achieved with $\Omega_d=1.05$ and $\Omega_d=13.95$, respectively. In Ref. [23], a special method called mutual phase coupling has been developed to achieve PS between the Rössler oscillator and the Lorenz oscillator. Here we show actually the usual one-way state coupling also works effectively. Moreover, as shown in Fig. 6(b), PS can only happens after GS is achieved. The examination of the CLEs for different α values once again confirms that in this case PS still cannot occur before GS.

IV. CONCLUSION AND DISCUSSION

In this work, PS between two essentially different chaotic oscillators has been investigated. Two models in drive-response configuration are considered. In the first situation, two one-center rotation chaotic oscillators, i.e., a nonlinear electric circuit system and the Rössler system, are unidirectionally coupled; in the second situation the one-center Rössler oscillator drives the two-center Lorenz oscillator. PS is characterized by the phase locking and frequency locking between two coupled oscillators. To explore the relation between PS and GS, the CLEs in the response system have been computed. Our numerical results show that PS does occur between two essentially different chaotic oscillators in spite of their physical difference as well as the large frequency detuning between them.

In the first work of PS [4], two bidirectionally coupled Rössler oscillators with very small parameter mismatch is studied. Due to the very small parameter mismatch, it is not strange that small coupling is enough to entrain the two oscillators to achieve phase locking. In the meantime, this coupling is weak enough not to cause the amplitudes of the two oscillators to be correlated. Therefore, in this case PS manifests itself as phase locking or frequency locking between two chaotic oscillators while their amplitudes can remain uncorrelated. On the other hand, for coupled chaotic oscillators with large parameter mismatch [17,18], or for coupled essentially different chaotic oscillators, PS still can be observed in terms of phase locking or frequency locking. Nevertheless, in these cases, the amplitudes of the coupled chaotic oscillators is found to be correlated, i.e., the amplitudes of two systems have functional relation in terms of GS. Recently, a mathematical theory has been developed for PS in phase coherent systems [33,34]. It reveals that whether PS could achieve between a dynamical system and a periodic driving signal (or other phase coherent chaotic signal) depends on the property of the unperturbed system, i.e., the dependence of the phase dynamics on the amplitude dynamics. If such dependence is weak, the phase variable corresponds to the neutral direction in the attractor which is characterized by the null Lyapunov exponent. In this case, it is likely for the phase of the system to be locked by the driving signals. However, if the dependence is strong, PS may occur after the correlation has been achieved between the amplitudes, or may not occur at all.

It is known that PS and GS both take place between different chaotic systems. The relation between these two kinds of synchronization has been discussed in several publications [4,16–18]. In Ref. [16], two coupled Rössler oscillators with small parameter mismatch were studied. It is found that PS occurs before GS is achieved in the coupled systems. Moreover, since the parameter mismatch is small, the coupling needed to achieve PS is also very small. Later, the study in Ref. [17], which still considers two parametrically different chaotic oscillators, demonstrates that this is not necessarily the case. If the parameter mismatch is large, PS may occur after GS. In addition, for large parameter mismatch, a large coupling should be applied for the system to achieve PS. Very recently, three types of transition to PS have been identified in coupled parametrically different chaotic oscillators. The transition depends on the coherence properties of motions measured by diffusion of the phase [18]. For small diffusion, PS occurs before GS; while for strong diffusion PS sets in only after the onset of GS. Once again, stronger coupling strength is needed to achieve PS in the latter case. For PS between two essentially different chaotic oscillators, the present results seems to suggest that only when their amplitudes are correlated through GS, can their phases be locked to achieve PS. In order to set up functional relation between the amplitudes of two oscillators, usually a relatively large coupling is required, especially in the unidirectional coupling scheme as in the current study.

Physically, for essentially different chaotic systems, or for two systems with large parameter mismatch, there exist many distinct characteristic time scales, or the mean periods of the unstable periodic orbits (UPOs) are broadly distributed. Sometimes even the topological structures of the drive and the response systems are different. Under these circumstances, it is not surprising that relatively strong coupling is need to synchronize UPOs with large difference in periods, or to drive the attractor in the response system away from its original attractor to become topologically equivalent with the driving attractor. This is the reason why in this work the phases between two essentially different chaotic systems become locking only after their amplitudes are correlated through GS. We believe this characteristic should accompany PS in many coupled essentially different chaotic systems.

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