The development of generalized synchronization on complex networks

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In this paper, we numerically investigate the development of generalized synchronization (GS) on typical complex networks, such as scale-free networks, small-world networks, random networks, and modular networks. By adopting the auxiliary-system approach to networks, we observe that GS generally takes place in oscillator networks with both heterogeneous and homogeneous degree distributions, regardless of whether the coupled chaotic oscillators are identical or nonidentical. We show that several factors, such as the network topology, the local dynamics, and the specific coupling strategies, can affect the development of GS on complex networks. © *2009 American Institute of Physics*. [DOI: 10.1063/1.3087531]

The investigation of synchronization dates back to the 17th century when Huygens first discovered synchronization between two pendulum clocks. Since then, the ubiquitous phenomena of synchronization in both natural and artificial systems have attracted great research efforts. The early theoretical studies on synchronization are mainly limited in the cases of couple periodic oscillators. In the 90s of last century, breakthrough has been made when synchronization between chaotic oscillators was discovered and extensively studied later. Recently, with the booming in the field of complex networks, synchronization among interacting oscillatory elements on networks has once again become a hot topic. So far, much effort has been given to the study of complete synchronization and phase synchronization in complex networks. However, one important synchronization forms, i.e., the generalized synchronization has seldom been addressed in complex networks. In this paper, we present a numerical investigation on the occurrence and development of generalized synchronization on various complex networks, including scale-free networks, small-world networks, random networks, and modular networks. Hopefully, this work will provide us further understanding and new perspective in the field of network synchronization.

I. INTRODUCTION

Synchronization in coupled chaotic oscillators has been extensively studied in the past 20 years,[1] including complete synchronization (CS),[2] generalized synchronization (GS),[3–11] phase synchronization (PS),[12] lag synchronization,[13] partial synchronization or clustering,[14] practical (approximation) synchronization,[15] and anticipate synchronization,[16] etc. For example, in CS, the dynamics of two coupled systems totally coincide with each other; in GS, certain functional relation exists between the dynamics of two coupled systems which are usually nonidentical. Moreover, PS is a weaker synchronization form in which the phases of two oscillators can be locked, while their amplitudes remain uncorrelated and chaotic. Recently, the study of synchronization has been extended to the area of complex networks.[17–31] For example, synchronization on small-world networks,[17–20] scale-free networks,[21–23] modular networks,[24–27] weighted networks,[28] and gradient networks[29] has been investigated. These studies aim to explore the interplay between network topology and dynamics on network. They are important for us to understand the real situations in complex systems comprising interacting elements in both nature and human society.

So far, most works on synchronization in complex networks study the situations of PS and CS. Specifically, PS in complex networks is mainly investigated through the generalized Kuramoto models.[20,27,30–32] In this model, the node dynamics is very simple that is governed by an ideal phase oscillators, $\dot{\phi}=\omega$, where ω is the frequency. The heterogeneity in the node dynamics can be modeled by assigning different, usually random, frequencies to different phase oscillators. The generalized Kuramoto models in complex networks have the advantage that they can still be treated analytically in many aspects.[27,30] On the other hand, CS on complex networks is often investigated through the approach of master stability function.[19,33] To apply this approach, the node dynamics in complex networks must be assumed to be identical, and then the theory of master stability function provides a general mathematical framework to relate the synchronizability of a network to the spectral properties of the corresponding coupling matrix.[19,22,25,28,34] Recently, a connection graph based stability method has been proposed, which is able to give the upper bounds of minimum coupling strength for achieving global synchronization of coupled oscillators on complex networks.[35] Apart from CS, this method has an important advantage in dealing with approximation synchronization on networks and even synchronization on networks with time-varying coupling.[35]

One interesting question in studying network synchronization is that can different synchronization forms, or collective behaviors, in coupled low-dimensional dynamical systems still be observed in complex networks? We notice that GS phenomenon has not been carefully investigated on networks in previous works. In fact, GS is an important and very useful concept in the analysis of coherence in biological[10] and physical[11] systems consisting of multiple interacting components. For example, in Ref. 11, by applying the GS detecting method, it has been successfully demonstrated that GS relation exists in He–Ne lasers and liquid crystal spatial light modulators. These important experiments thus provide direct evidence showing that different spatiotemporal dynamics could have strong coherence between them. In this paper we present a detailed numerical study on GS phenomenon in various complex networks. Our particular interest is paid to the occurrence and development of GS on networks. For typical complex networks, including scale-free networks, small-world networks, random networks, and modular networks, interestingly, we observe that GS generally occurs, regardless of whether the node dynamics are identical or nonidentical. For networked identical oscillators, we find that usually there is a GS regime before the final global CS. We further carried out extensive numerical experiments to demonstrate how the development of GS on networks can be affected by several factors, such as the network topology, the local dynamics, as well as the specific coupling strategy.

In processing this paper, we noticed a recent work which reported GS phenomenon in scale-free networks.[36] Compared with the results in Ref. 36, the present work is different in the following aspects. First, in our work GS is extensively investigated in various network topologies, including scale-free networks, random networks, small-world networks, and modular networks, while in Ref. 36, GS is mainly studied in a very special network, i.e., the scale-free network with tree-like structure. Second, the present work characterizes a typical path for coupled identical oscillators on complex networks, i.e., from nonsynchronization to global CS via GS, while in Ref. 36, global CS has not been achieved. Third, the present work investigates the development of GS for coupled nonidentical oscillators, either parametrically different or physically different, while Ref. 36 does not consider this general setting where the occurrence of GS is naturally justified. Finally, in the present work, we discuss the effect of different coupling strategies on the development of GS on network. Especially, we analyze and explain the different observations regarding the GS evolution on networks in our work and in Ref. 36. Therefore, in many aspects the current study deepens and widens the work in Ref. 36 and can offer more thorough and comprehensive insight for understanding GS phenomenon on complex networks.

This paper is organized as follows. In Sec. II, the methods and measures to characterize GS and CS on networks are described. In Sec. III, GS of coupled identical oscillators is studied on typical complex networks. In Sec. IV, GS of coupled nonidentical oscillators is considered on networks. It is shown that the development of GS on networks can be affected by both network topology and the local dynamics. In Sec. V, the effect of different coupling strategies on the development of GS on networks is analyzed. Especially, we show that GS can be observed in coupled system with hybrid oscillators even when their local dynamics are physically different. Finally, a section with discussions and summary ends this paper.

II. APPROACHES CHARACTERIZING GS AND CS ON NETWORKS

The auxiliary-system approach has been extensively used to detect GS in two coupled chaotic systems.[4] Here, we can extend it to detect GS on complex networks. The key observation is that for any given node in a network, the coupling from other nodes can be regarded as a kind of "driving." In particular, we consider the following linearly coupled identical oscillators on a network:

$$\dot{\mathbf{x}}_i = \mathbf{F}_i(\mathbf{x}_i) - \varepsilon \sum_j a_{ij} H(\mathbf{x}_i - \mathbf{x}_j) \tag{1}$$

for $i=1,\ldots,N$, where $\mathbf{x}_i$ denotes the dynamical variables of node i, $\mathbf{F}_i(\mathbf{x}_i)$ is the local vector field governing the evolution of $\mathbf{x}_i$ in the absence of interactions with other nodes, a_{ij} is the element of the network adjacency matrix $\mathbf{A}$ ($a_{ij}=1$ if there is a link between node i and node j, $a_{ij}=0$ otherwise, and $a_{ii}=0$), H is the output matrix, and ε is the coupling strength. To apply the auxiliary-system approach, we consider a replica for each oscillator in the original network,

$$\dot{\mathbf{x}}_i' = \mathbf{F}_i(\mathbf{x}_i') - \varepsilon \sum_j a_{ij} H(\mathbf{x}_i' - \mathbf{x}_j) \tag{2}$$

for $i=1,\ldots,N$. Note that the driving variable $\mathbf{x}_j$ is identical for both Eqs. (1) and (2). If, for initial conditions $\mathbf{x}_i(0) \neq \mathbf{x}_i'(0)$, we have $|\mathbf{x}_i(t)-\mathbf{x}_i'(t)| \to 0$ as $t\to\infty$, node i then is entrained in the sense that its dynamics is no longer sensitive to the initial conditions. In other words, there is GS relation between $\mathbf{x}_i$ and $\mathbf{x}_j$ for $j=1,\ldots,N$. Numerically, we can examine the following local distance of GS between a node and its auxiliary counterpart:

$$d(\varepsilon,i) = \frac{1}{t_2 - t_1} \sum_{t_1}^{t_2} |\mathbf{x}_i(t) - \mathbf{x}_i'(t)|, \tag{3}$$

where t_1 is chosen to be larger than the typical transient time of the local dynamics $\mathbf{F}_i(\mathbf{x}_i)$. For oscillators on complex networks, GS may be gradually developed with the increase in the coupling strength. To characterize the development of GS on the whole networks, we can define the distance of global GS as $l_g(\varepsilon)=\langle d(\varepsilon,i)\rangle$. Here, $\langle\cdot\rangle$ denotes the spatial average over all nodes. If $l_g=0$, global GS has been achieved between any two pairs of oscillators on the whole network.

For coupled identical oscillators on complex networks, CS is generally expected to take place. To characterize CS, we can define $l_c(\varepsilon)$ as the distance of global CS, which measures the distance between the dynamics of all oscillators and their average, i.e.,

$$l_c(\varepsilon) = \frac{1}{t_2 - t_1} \sum_{t_1}^{t_2} \langle |\mathbf{x} - \langle \mathbf{x} \rangle| \rangle, \tag{4}$$

where the meaning of t_1 and t_2 is the same as that in Eq. (3). If $l_c=0$, global CS has been achieved on the whole network.

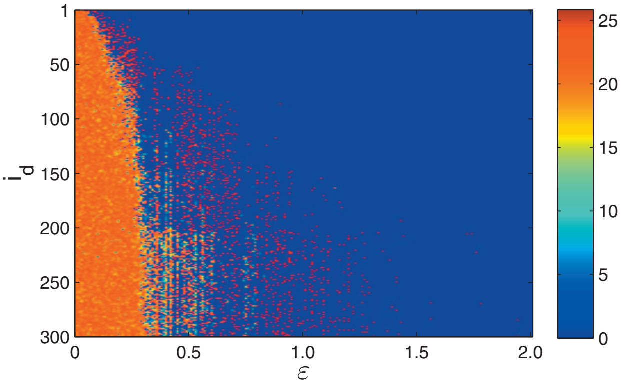

FIG. 1. (Color) Color map of $d(\varepsilon,i_d)$ in the two-dimensional parameter space (ε,i), characterizing the development of GS for 300 identical chaotic Lorenz oscillators on a scale-free network. The network is the Barabási−Albert (BA) model with $m_0=m=4$ (Ref. 21). We see that with the increase in coupling strength, GS can be achieved gradually from the hubs and then spread to the rest of the network.

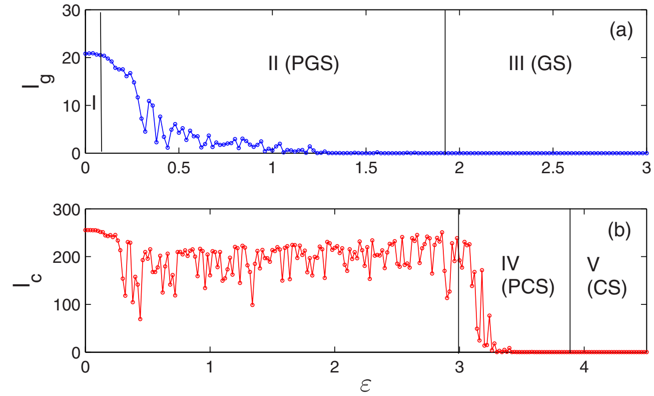

FIG. 2. (Color online) Characterizing the path toward global CS on scale-free network. (a) The distance of global GS vs the coupling strength. (b) The distance of global CS vs the coupling strength. Numerically, if $l_g(l_c)<0.001$, then global GS (CS) is regarded to have been achieved. The notations of the regimes in this figure are used throughout this paper.

For individual node dynamics, most of the time in this paper we choose the chaotic Lorenz oscillator,

$$\mathbf{F}_i(\mathbf{x}_i)=[10(y_i-x_i),r_ix_i-y_i-x_iz_i,x_iy_i-(8/3)z_i]^T, \qquad (5)$$

where $\mathbf{x}_i\equiv(x_i,y_i,z_i)$ are the state variables of the Lorenz oscillator. Note that in studying GS, the local dynamics could be different from each other. This can be modeled by setting different r_i values for Lorenz oscillators in the network. Without losing generality, the coupling between two nodes in Eq. (1) is through the x variable, i.e., the output matrix is $H=[1,0,0;0,0,0;0,0,0]$. For convenience, in the present work, we use two kinds of node indices: i_d and i_r. In the first index, we order the degrees of the network so that $i_d=1$ denotes the node with the largest degree, $i_d=2$ is for the node with the second largest degree, and so on. In the second index, we rank the Lorenz oscillators in the network according to their r_i values, i.e., $i_r=1$ denotes the node with the largest parameter r, $i_r=2$ is for the node with the second largest parameter r, and so on. These two indices of nodes are used throughout the paper. Besides the Lorenz oscillator, other node dynamics such as the Rossler oscillator and the logistic map have also been used in our study.

III. COUPLED IDENTICAL OSCILLATORS: THE OCCURRENCE AND DEVELOPMENT OF GS

Previously, in the study of synchronization of two coupled chaotic oscillators, it is found that CS generally takes place when the coupled oscillators are identical, while GS usually occurs when the coupled oscillators are nonidentical. However, this generally accepted view turns out to be not true when synchronization is studied on complex networks. In this section, we report the existence of GS for coupled identical chaotic oscillators on various complex networks, including scale-free networks, random networks, small-world networks, and modular networks.

As an example, we first study the occurrence and development of GS for 300 chaotic Lorenz oscillators on a scale-free network. In this case, $r_i=28$ in Eq. (5) for all nodes, i.e., all oscillators are identical. Intuitively, with the increase in coupling, we can expect that the coupled chaotic oscillators will finally achieve global CS. The interesting finding here is that before the system achieves the global CS state, there exists another synchronization regime, namely, the GS regime, which usually occurs with much smaller coupling strength compared with CS. To illustrate the GS development on scale-free network, we plot the color map of the distance matrix $d(\varepsilon,i_d)$ in Fig. 1. From Fig. 1, we can see that when the coupling strength is small, $d(\varepsilon,i_d)$ is greater than 0 for all nodes, showing that the system is in the nonsynchronous state. In addition, when the coupling strength is large enough, $d(\varepsilon,i_d)$ is 0 for all nodes, showing that all oscillators have been entrained and the coupled system is in the global GS state according to the auxiliary-system approach criterion. Between these two regimes, it is the transient regime of partial GS (PGS), where part of the oscillators has been entrained but the others were not.

On the other hand, for coupled identical oscillators, CS generally occurs as long as the connections of the network are dense enough, and this is the case in the above example. As the coupling strength is further increased after GS, the coupled system will finally go to the global CS state. Usually, there is a transient stage before the system achieves global CS. In this stage, part of the nodes has achieved CS in practical (approximation) sense with each other. These nodes form a synchronous cluster, while the other nodes do not synchronize with them. We call this stage the partial CS (PCS) regime.

To characterize the development of GS for coupled Lorenz oscillators on scale-free network, in Fig. 2(a) we plot the distance of global GS l_g versus the coupling strength. From this figure, three regimes of coupling can be clearly identified, and global GS on this specific network is found to be achieved when $\varepsilon>1.8$. Similarly, in Fig. 2(b) we plot the distance of global CS l_c versus the coupling strength and find that the global CS on the network can be achieved when $\varepsilon>3.8$. Combining these results together, we can get an overall picture on the path toward global CS in the scale-free network. Specifically, five dynamical regimes can be identified as follows. I: $\varepsilon\leq 0.1$, the nonsynchronization regime; II:

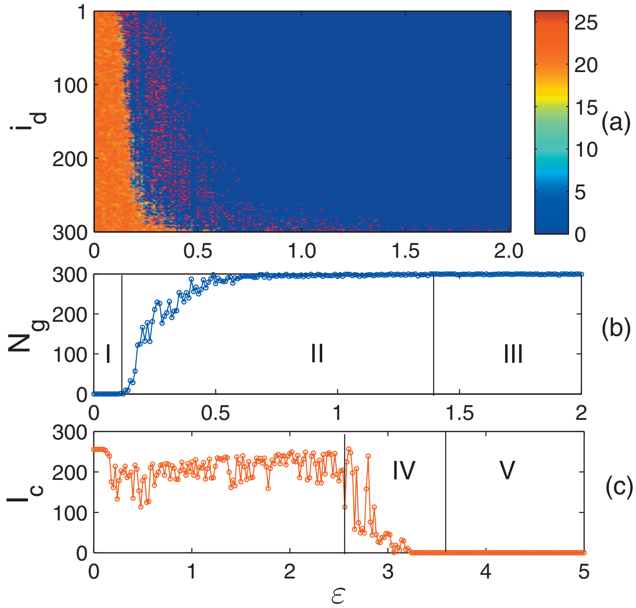

FIG. 3. (Color) Characterizing the development of GS and the path toward global CS of 300 coupled identical Lorenz oscillators on a random network with average degree of 10. (a) The color map of $d(\varepsilon, i_d)$ in the two-dimensional parameter space (ε, i). (b) The number of nodes which has achieved GS vs the coupling strength. (c) The distance of global CS vs the coupling strength.

$0.1<\varepsilon\leq 1.8$, the PGS regime; III: $1.8<\varepsilon<3.0$, the global GS regime; IV: $3.0<\varepsilon<3.8$, the PCS regime; and V: $\varepsilon>3.8$, the global CS regime. These regimes represent a typical path from nonsynchronization state to global CS state via GS for networked identical oscillators. Since GS is a weaker synchronization form, achieving GS on network usually requires smaller coupling strength than achieving CS.

In the above we have observed GS for coupled identical chaotic oscillators on a scale-free network. How about other network topologies? In our study, we have also considered the following networks. (1) A random network consisting of 300 nodes. The average degree is 10. (2) A small-world network, which is obtained by rewinding 20 links in a regular network consisting of 100 nodes and each node has ten nearest neighbor connections.[17] (3) A modular network consisting of 100 nodes which is evenly divided into five modules.[27] Inside each module node is fully connected. Any two nodes in different modules have probability $p=0.01$ to connect each other. (4) As a special case of complex networks, a regular network consisting of 100 nodes and each node has ten nearest neighbor connections. For all these typical complex networks studied, GS has been observed. Here, we show one more example in the case of random network. In Fig. 3, the development of GS on a random network is characterized. Comparing Fig. 3(a) with Fig. 1, we can see that the development of GS at the beginning stage on the random network is different from the situation on the scale-free network. Mainly, most nodes in random network are entrained at approximately the same coupling strength. This is due to the fact that random network has approximately homogeneous degrees and no hubs exist as in scale-free networks. Although the development of GS on networks depends on the different network topologies, qualitatively, the path toward global CS is the same as shown in Figs. 2 and 3. In fact, we have similar observations for other types of complex networks. We emphasize that this path to the global CS for networked identical chaotic oscillators is a typical one, not the complete one. One may observe different paths toward the global CS, depending on the local dynamics on networks. For example, in our study we have also observed the path to the global CS as: nonsynchronization $\rightarrow$PS$\rightarrow$GS$\rightarrow$CS.

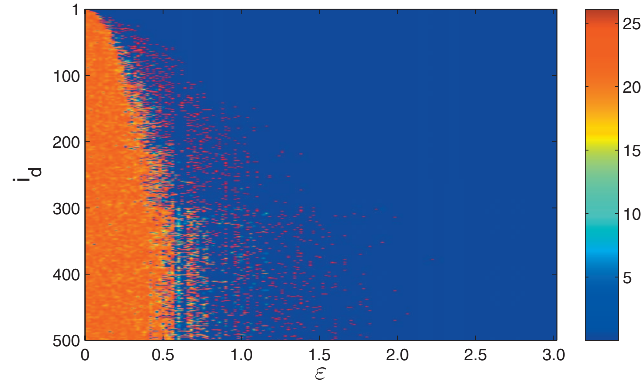

FIG. 4. (Color) Color map of $d(\varepsilon, i_d)$ in the two-dimensional parameter space (ε, i), characterizing the development of GS for 500 nonidentical Lorenz chaotic oscillators on a scale-free network. The network is the BA model with $m_0=m=3$ (Ref. 21). The development of GS in the network is similar to the situation shown in Fig. 1.

IV. COUPLED PARAMETRICALLY DIFFERENT OSCILLATORS: TOPOLOGY VERSUS LOCAL DYNAMICS

In realistically physical or biological situations, the node dynamics are usually different from each other. For example, in neuron networks any two pairs of neurons cannot be exactly the same. This raises an important question: is it possible for nonidentical oscillators on network to achieve GS, or certain extent of coherence? If so, how do network topologies and local dynamics affect the development of GS? To address these questions, in this section we will consider coupled nonidentical oscillators in complex networks. To model different oscillators on network, we randomly set the parameter r_i in Lorenz system in the interval [28.0, 30.0]. By applying auxiliary-system approach, we find that GS generally occurs in such coupled systems on networks. As an example, Fig. 4 illustrates the development of GS for coupled Lorenz oscillators with parameter mismatches on a scale-free network. It is shown that with the increase in coupling strength, nodes in the network are entrained first from the hubs and gradually spread to other nodes. When the coupling strength is large enough, all nodes in the network are entrained, showing that global GS has been achieved. In spite of the different local dynamics, the above example shows a similar development of GS on the scale-free network as in the case of coupled identical oscillators. This implies that here the network topology plays a dominant role in the development of GS on network.

Previously, many works have revealed how network topology affects the network synchronizability under the setting of coupled identical oscillators. For networked noniden-

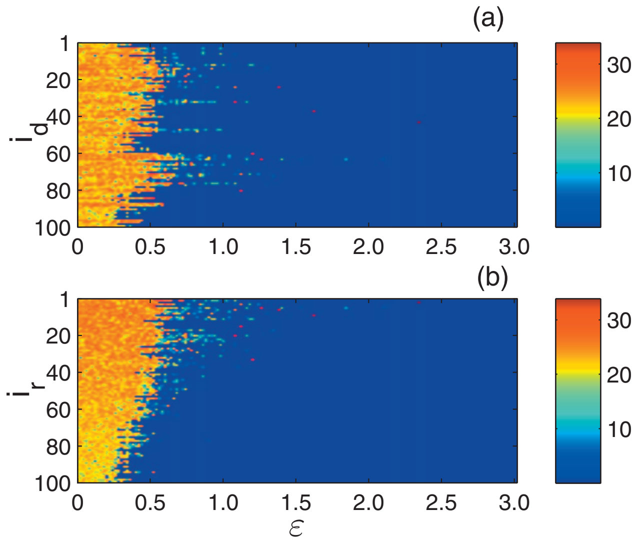

FIG. 5. (Color) Color maps of $d(\varepsilon,i_d)$ and $d(\varepsilon,i_r)$ in the two-dimensional parameter space (ε,i) for nonidentical Lorenz oscillators on a regular network. The network has 100 nodes, and each node has six nearest neighboring connections.

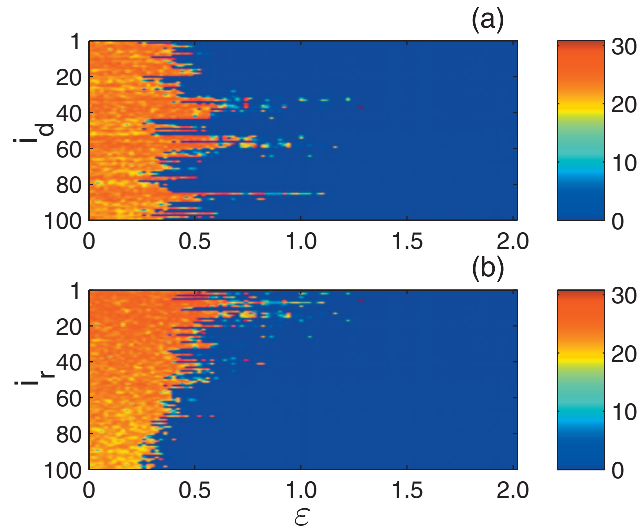

FIG. 6. (Color) Color maps of $d(\varepsilon,i_d)$ and $d(\varepsilon,i_r)$ in the two-dimensional parameter space (ε,i) for nonidentical Lorenz oscillators on a small-world network. The network is obtained by rewiring a small part of connections on a regular network with 100 nodes (Ref. 17).

tical oscillators studied in this work, apparently there are two factors affecting the development of GS on complex networks, i.e., network topology and heterogeneity in the local chaotic dynamics. In Secs. II and III, we have shown that for scale-free networks, GS typically starts from the hubs and then spreads to others nodes with relatively smaller degrees, regardless of whether the oscillators are identical or nonidentical. In these cases, heterogeneity in the degree distribution appears to be the dominant factor governing the development of GS. An interesting issue then is that for networks with homogeneous degree distributions, such as regular, small-world, or certain modular networks, how does heterogeneity in the local dynamics affect GS? In the following, we study this question through numerical simulations.

We first consider a regular network of $N=100$ nodes. Each node in the network has $k_d=6$ connections to its nearest neighbors. As a special case, this is an exactly homogeneous network with regular degree sequence. The local dynamics in this example and in the following two examples are those of Lorenz chaotic oscillators with different parameter r_i randomly distributed in the interval [28, 38]. Figure 5 shows the color maps of the local synchronization distances $d(\varepsilon,i_d)$ and $d(\varepsilon,i_r)$, where i_d and i_r are the node indices arranged according to the decreasing node degree and decreasing values of parameter r_i, respectively. For the Lorenz oscillator under the parameter setting in our study, we find that the larger the value of parameter r, the larger the largest Lyapunov exponent of the chaotic attractor. Thus, in Fig. 5, the index i_r actually corresponds to the decreasing value of the largest Lyapunov exponent of the local dynamics. Specifically, larger value of i_r corresponds to a small value of r_i so that the corresponding local dynamics is less chaotic in the sense that its largest Lyapunov exponent has a relatively smaller value. Comparing Fig. 5(a) with Fig. 5(b), we find that when transient behaviors are disregarded, nodes whose dynamics are less chaotic require smaller value of the coupling strength to be entrained. Entrainment of more chaotic nodes requires stronger coupling. Thus the development of GS on regular network is determined by the local dynamics.

We next consider the occurrence and development of GS on small-world networks. A representative example is shown in Fig. 6. We see from Fig. 6(a) that there is no apparent synchronization sequence of nodes according to the degree index i_d. However, as can be seen from Fig. 6(b), GS starts from nodes with smaller values of the largest Lyapunov exponent. This demonstrates that for a small-world network, heterogeneity in the local dynamics plays a dominant role in the development of GS, which is similar to the situation in regular networks.

Lastly, we investigate GS in a type of modular network. A modular network is characterized by a number of sparsely connected subnetworks, each with dense internal connections. For such a network, synchronization within each individual cluster can usually be achieved readily due to the dense internal connections, so the occurrence of global synchronization is of particular interest.[27] An example of the development of GS on modular network is shown in Fig. 7. We see that global GS can be achieved despite the sparse intercluster connections. An interesting phenomenon is that the development of GS does not seem to strongly depend on i_d or i_r. Nevertheless, comparatively, it can still be found that nodes with less chaotic local dynamics are easier to be entrained with smaller coupling strength.

To summarize, we have observed that the development of GS on complex networks can be affected by both the network topology and the local chaotic dynamics when they both have heterogeneity. For heterogeneous networks, such as scale-free networks, the network topology plays a leading role; while for approximately homogeneous networks, such as small-world networks and modular networks, the local dynamics is the dominant factor determining the organization and spreading of GS on networks.

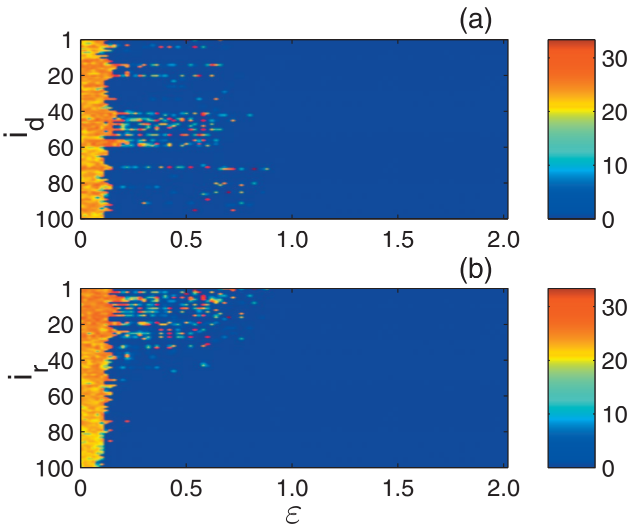

FIG. 7. (Color) Color maps of $d(\varepsilon, i_d)$ and $d(\varepsilon, i_r)$ in the two-dimensional parameter space (ε, i) for nonidentical Lorenz oscillators on a clustered network. The network has 100 nodes which are evenly divided into five modules. Nodes within each module are fully connected. Any two nodes in different modules have probability $\rho=0.005$ to connect each other.

V. EFFECT OF DIFFERENT COUPLING STRATEGIES

A. The linear coupling and the normalized coupling

In the above numerical experiments, we have seen that the hub nodes behave as "seeds" to develop GS in scale-free networks. Nevertheless, in a recent works,[36] it is reported that only for a kind of special scale-free network with tree-like structure, GS is observed to develop from the hubs and then gradually spread to other nodes in the network; while for usual scale-free networks, the heterogeneity of network seems to have little effect on the development of GS in the network. In attempting to find the reason that leads to different observations between the present work and Ref. 36, we notice that a different coupling strategy is used in Ref. 36. Take a coupled one-dimensional map network as an example, the coupled system in Ref. 36 is

$$x_i^{n+1} = (1-\varepsilon) f(x_i^n) + \frac{\varepsilon}{k_i} \sum_j a_{ij} f(x_j^n), \tag{6}$$

where x_i^n is the state variable of node i at time step n, $f(x)$ is the local map, and k_i is the degree of node i. This coupling scheme with each node in the network driven by the local mean field is essentially the same as the following normalized coupling strategy:

$$\dot{\mathbf{x}}_i = \mathbf{F}_i(\mathbf{x}_i) - \frac{\varepsilon}{k_i} \sum_j a_{ij} H(\mathbf{x}_i - \mathbf{x}_j), \tag{7}$$

where notations are the same as those in Eqs. (1) and (6). In this coupling scheme, the coupling strength for each node i is normalized by its degree. Therefore, the effective coupling strength each node received from its network neighbors is of the same order of magnitude, regardless of the fact if it is a hub with very large degree or a node with very small degree. Obviously, due to the normalization, the effect of network topology on the development of GS has been suppressed. As a consequence, there should be no distinct difference among the coupling strength when nodes in the network achieve GS.

The above heuristical idea can be further understood through the local stability analysis for nodes in the network. The conditional stability of coupled system (1) is equivalent to the stability of CS manifold between system (1) and its corresponding auxiliary system (2).[4] Letting $\Delta\mathbf{x}_i = \mathbf{x}_i' - \mathbf{x}_i$ and subtracting Eq. (1) from Eq. (2), we obtain

$$\begin{aligned}\Delta\mathbf{x}_i &= \mathrm{F}_i(\mathbf{x}_i') - \mathrm{F}_i(\mathbf{x}_i) - \varepsilon \sum_j a_{ij} H \Delta\mathbf{x}_i \\ &\approx [\mathbf{D}\mathrm{F}(\mathbf{x}_i) - \varepsilon k_i H] \cdot \Delta\mathbf{x}_i.\end{aligned} \tag{8}$$

We see that for node i, the effective coupling strength is proportional to its degree k_i. For a fixed value of ε, the coupling between a hub node and its counterpart in the auxiliary system can be significantly larger than the coupling for nodes with smaller degree, leading to "earlier" synchronization between the hub nodes in the original and in the auxiliary systems. Therefore, for usual linear coupled oscillator system, the general observation is that, in a complex network with heterogeneous degree distribution, the set of hub nodes provides a skeleton around which synchronization is developed. The above analysis is also suitable for coupled system (7). In this case, there will be no factor k_i as in Eq. (8). This implies that the effect of network topology on the development of GS in network is almost eliminated.

In the following, we further present two examples where the oscillators in a scale-free network are coupled with the normalized coupling strategy. The first example is coupled Lorenz oscillators described by Eq. (7), and the second example is the following coupled Logistic maps:

$$x_i^{n+1} = f(x_i^n) - \frac{\varepsilon}{k_i} \sum_j a_{ij} [f(x_i^n) - f(x_j^n)]. \tag{9}$$

For both cases, the local dynamics are identical, i.e., $r_i=28$ for all Lorenz oscillators and $f(x)=4x(1-x)$ for all Logistic maps. Figure 8 illustrates the development of GS for the above two systems on a scale-free network. The effect of different coupling strategies on the development of GS can be verified by comparing Fig. 8 with Fig. 1. For system (1), GS first occurs on the hubs and then gradually spread to other nodes; while for systems (7) and (9), GS almost simultaneously takes place on the hubs and the other nodes. Actually, there is no significant difference among all nodes on networks.

B. Coupled hybrid oscillators

In Sec. IV, we have shown that the development of GS on networks can be affected by two heterogeneous factors: the network topology and the local dynamics. In the following, we further investigate the situations when the parameter mismatches of local dynamics are very large, or the local dynamics are physically different.

By adopting the normalized coupling strategy, we can conveniently investigate this issue. Here we present two examples showing how GS is developed for coupled hybrid oscillator system in networks. By hybrid we mean that the oscillators in the network can be classified into different

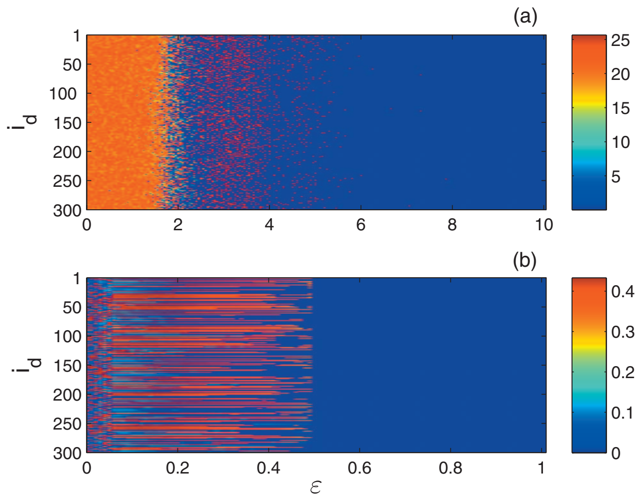

FIG. 8. (Color) Color maps of $d(\varepsilon, i_d)$ in the two-dimensional parameter space (ε, i), characterizing the development of GS for 300 identical Lorenz chaotic oscillators (a) and 300 identical Logistic maps (b) on a scale-free network. The network is the same as in Fig. 1. Compared with Fig. 1, the effect caused by the normalized coupling strategy is obvious.

types which are parametrically different or physically different. In the first example, we consider a system coupled with hybrid Lorenz oscillators. To be specific, we arbitrarily select 5% Lorenz oscillators in the network and make their parameter r_i be randomly distributed in the interval [30, 40]. For the rest oscillators, they are the same with $r_i = 28$, which are significantly smaller than that of the 5% oscillators. In Fig. 9, the development of GS for such a hybrid system is illustrated on a scale-free network. It is seen that most oscillators in the network are entrained at $\varepsilon \approx 2$. However, a small number of oscillators require significantly larger coupling strength to be entrained. A careful examination of the locations of these nodes reveals that they just correspond to the 5% oscillators with larger r values. In the second example, the coupled hybrid system consists of two kinds of oscillators which have different dynamical equations. Similar to the first example, 5% nodes are randomly selected to be the identical Rossler oscillators, while the rest 95% percent nodes are the identical Lorenz oscillators with $r_i = 28$. The dynamical equations of the Rossler oscillator read as

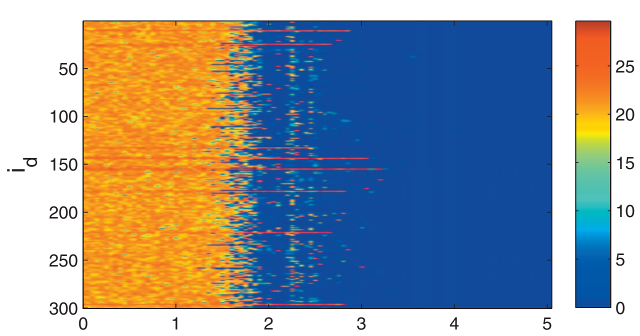

FIG. 9. (Color) Color map of $d(\varepsilon, i_d)$ in the two-dimensional parameter space (ε, i), characterizing 300 hybrid Lorenz chaotic oscillators on a scale-free network. The network is the BA model with $m_0 = m = 3$ (Ref. 21).

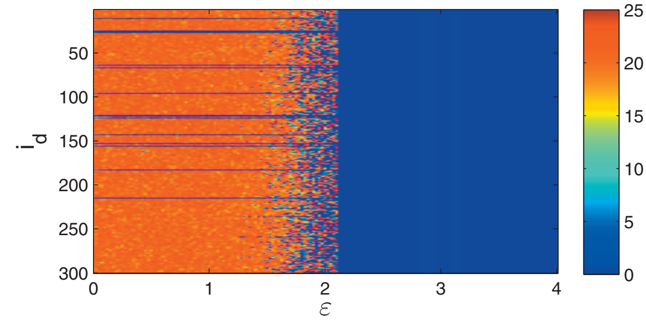

FIG. 10. (Color) Color map of $d(\varepsilon, i_d)$ in the two-dimensional parameter space (ε, i), characterizing 300 hybrid Lorenz and Rossler oscillators on a scale-free network. The network is the BA model with $m_0 = m = 3$ (Ref. 21).

$$\mathbf{F}(\mathbf{x}_i) = [-(y_i + z_i), x_i + 0.2y_i, 0.2 + z_i(x_i - 5.7)]^T, \qquad (10)$$

where $\mathbf{x}_i \equiv (x_i, y_i, z_i)$ are the state variables of the Rossler oscillator. In Fig. 10, we plot the local distance of GS versus the coupling strength for all nodes in a scale-free network. It is clearly shown GS can be achieved for such a coupled hybrid system. The development of GS in network has two distinct stages: the Rossler oscillators are much easier to be entrained compared with the Lorenz oscillators. The latter needs much larger coupling strength to achieve GS. Note that the largest Lyapunov exponent of the Lorenz oscillator is proportional to its r value, and the Rossler oscillator has a much smaller Lyapunov exponent than that of the Lorenz oscillator; we can conclude from the above two examples that GS usually develops from the nodes in the network where the local dynamics are less chaotic in the sense that the largest Lyapunov exponents of the local dynamics have relatively smaller values. Although our numerical simulations are carried out on a scale-free network, we believe this conclusion can still hold for other network topologies when the normalized coupling strategy is applied.

VI. DISCUSSIONS AND SUMMARY

The concept of GS was first developed in studying synchronization of two coupled nonidentical oscillators with drive-response configuration.[3] Since the dynamics of the drive and the response are different, strict CS is impossible to occur in such systems. However, there may be certain functional relation between the dynamics of two oscillators, which is defined as the GS. Extensive studies have shown that GS relations are usually very complicated and odd and, in principle, these relations are difficult to be identified analytically. In Ref. 5, a more rigorous definition of differentiable GS is discussed. To detect the occurrence of GS in coupled systems, Abarbanel *et al.*[4] proposed an effective method known as auxiliary-system approach, which has been successfully applied both numerically and experimentally.[11] Although GS was initially investigated in directionally coupled systems, it is natural to extend this concept to the mutually coupled systems and networks as long as there exist certain functional relations among the node dynamics. For example, in Ref. 10, a mutual prediction method was proposed to detect dynamical interdependence and GS in a neural ensemble; also in Ref. 6, a mutual false nearest neighbors method was used to characterize GS in bidirectionally coupled dynamical systems. Later, in Ref. 7, auxiliary method was extended to mutually coupled nonidentical chaotic systems, and in Ref. 9, the noise effect on GS was addressed. Recently, in Ref. 36 GS was reported in a treelike scale-free network. In fact, in physical systems, the mutually coupled systems are as common as directionally coupled systems. It turns out that the GS concept is also very helpful in investigating the coherence and dependence of dynamics in mutually coupled systems and networks. We emphasize that apart from GS, there are other important synchronization regimes, such as practical synchronization, pulse synchronization, phase (frequency) locking, which are very useful in describing the collective behaviors in coupled dynamical systems.

In summary, we have investigated the occurrence and development of GS on various complex networks, including scale-free networks, small-world networks, random networks, and modular networks. It is shown that GS generally takes place in such networks for both coupled identical oscillators and nonidentical oscillators. For coupled identical oscillators, there exists a typical path toward global CS, i.e., nonsynchronization $\rightarrow$GS$\rightarrow$CS. We find that the development of GS on complex networks depends on both network topology and local dynamics when the coupled oscillators are nonidentical. Moreover, the specific coupling strategy also plays an important role during the evolution of synchronization. Under the linear dissipative coupling scheme, for heterogeneous networks, GS generally starts from a small number of hub nodes and then spreads to the rest nodes in the network; while for homogeneous networks, GS usually starts from the nodes whose local dynamics are less chaotic in the sense that the largest Lyapunov exponents have relatively smaller values. We further show that the effect of network topology on the development of GS can be suppressed if the coupling strengths of nodes in the network are normalized. Under such coupling scheme, the development of GS is essentially determined by the chaotic extent of local dynamics. We also demonstrate that GS can occur in coupled systems with hybrid oscillators on complex networks, and the development of GS has distinct stages due to physically different local dynamics. We emphasize that while there are extensive works on synchronization in complex networks, prior to this work the general existence of GS on typical networks and the factors affecting the development of GS have not been carefully investigated. Our work reveals that on complex networks, coupled oscillators may present fundamentally different synchronization regimes which deserve further study.

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