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Characterizing generalized synchronization in complex networks

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Abstract. Recently, it was shown that generalized synchronization (GS) can generally occur in systems of networked oscillators. In this paper, we further characterize the states of GS by both theoretical analysis and numerical experiments. We show that the entrainment of local dynamics in a network can be characterized by the conditional Lyapunov exponent of the local oscillator. Meanwhile, different types of states of GS can be identified by analyzing the Lyapunov exponent spectra of the coupled system. Most importantly, we further provide direct evidence demonstrating that node dynamics in a network in a chaotic GS state can indeed achieve functional relations, although they may not directly connect to each other in typical complex networks.

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1. Introduction

Frequently we encounter two dynamical systems with different natures coupled and interacting with each other. Since the coupled dynamics are different, generally there will be no manifold of complete synchronization (CS) except for some very special cases [1]. Usually, the collective coherence in such coupled non-identical dynamical systems could occur in the form of generalized synchronization (GS) or other forms [2]–[4]. Specifically, GS refers to the phenomenon where there exists a certain functional relation between the dynamics of two coupled systems that are usually non-identical [5]–[8]. Initially, the GS concept was defined in two unidirectionally coupled dynamical systems, i.e. the drive-response configuration. Later, this concept was extended to mutually coupled dynamical systems [9]. In addition, the phenomenon of GS was also observed and investigated in spatially extended systems. For example, it was experimentally demonstrated that the GS relation exists in coupled laser systems [10] and in liquid crystal spatial light modulators [11]. Very recently, the GS phenomenon has been observed in microwave electronic systems in experiments [12]. So far, GS has been extensively studied in the domain of chaotic synchronization [1], [5]–[15], [17]–[33].

It has been shown that the functional relation between two different dynamical systems in GS is so complicated that it is generally difficult to identify it analytically [13]–[16]. However, the signature of GS can still be characterized by various approaches. For example, typically the occurrence of GS can be characterized by the largest conditional Lyapunov exponent of the driven system [34, 35]. In [6], the auxiliary system approach was proposed, which has been successfully used to detect the GS relation in two coupled dynamical systems, both numerically and experimentally. Remarkably, if the underlying functional relation satisfies certain conditions [8], the existence of GS between two dynamics can even be identified by certain direct methods, such as nearest neighbors [5] and other methods along this line [17]–[19].

Recently, the GS phenomenon has been studied in the field of complex networks. In [36], GS was reported in a special scale-free network with tree-like structure. In [37], it was shown that oscillators in networks can be entrained by the local field of dynamics from their network neighbors, and GS can generally take place in coupled oscillator systems in typical complex networks, such as small-world, scale-free, random networks, as well as clustered networks. Furthermore, it was revealed that the development of GS in networks is the result of the interaction between the local dynamics and the network topology if the oscillators are

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different. The subtle point here is that, according to the auxiliary system approach criterion, the entrainment of an oscillator in a network only implies that there is a GS relation between the dynamics of the oscillator and the local driving field. In typical complex networks, usually most of the node pairs are not connected, especially in sparse networks. In such situations, can the dynamics of two oscillators, which may not directly connect to each other in a network, really achieve GS in the strict sense, i.e. achieve the functional relation? Motivated by this important and interesting question, in this work, we carried out both theoretical analysis and numerical simulations to characterize GS states in systems of networked oscillators. Specifically, by applying Lyapunov exponent spectra analysis, we have shown that both the entrainment of local dynamics and the global bifurcations of the GS states can be characterized. It is found that there exist two types of GS states for networked chaotic oscillators, namely the simple GS (SGS) state and the chaotic GS (CGS) state. In particular, by applying the mutual false nearest-neighbor method [5], we are able to provide direct evidence demonstrating that node dynamics in a network in the CGS state can indeed achieve functional relations, no matter whether they are directly connected or not.

This paper is organized as follows. In section 2, we briefly describe the dynamical models. In section 3, we characterize the entrainment of local oscillators by the conditional Lyapunov exponent. In sections 4 and 5, we analyze and characterize two distinct GS states, namely the SGS and the CGS. In section 6, based on the mutual nearest-neighbor method, we further carry out an analysis to directly characterize the relations among the dynamics of networked oscillators. Finally, a summary is given.

2. Dynamical model

The dynamical models in this study are the same as in [37]. For self-consistency, we here briefly describe them in the following.

In this work, we investigate a large number of chaotic oscillators that are linearly coupled in a network as

$$\dot{\mathbf{x}}_i = \mathbf{F}_i(\mathbf{x}_i) - \varepsilon \sum_j a_{ij} H(\mathbf{x}_i - \mathbf{x}_j), \tag{1}$$

or

$$\dot{\mathbf{x}}_i = \mathbf{F}_i(\mathbf{x}_i) - \frac{\varepsilon}{k_i} \sum_j a_{ij} H(\mathbf{x}_i - \mathbf{x}_j),$$
(2)

for i = 1, ..., N, where \mathbf{x}_i denotes the dynamical variables of node i, $\mathbf{F}_i(\mathbf{x}_i)$ is the local vector field governing the evolution of \mathbf{x}_i in the absence of interactions with other nodes, a_{ij} is the element of the network adjacency matrix \mathbf{A} ($a_{ij} = 1$ if there is a link between node i and node j, $a_{ij} = 0$ otherwise, and $a_{ii} = 0$), H is the output matrix, ε is the coupling strength and k_i is the degree of node i, i.e. the number of neighbors connected to node i in the network. We emphasize that generally the local dynamics described by $\mathbf{F}_i(\mathbf{x}_i)$ could be different.

To characterize the occurrence of GS in networks, the auxiliary system approach can still be used [6]. The key point is to construct a replica for each oscillator in the original network,

$$\dot{\mathbf{x}}_{i}^{\prime} = \mathbf{F}_{i}(\mathbf{x}_{i}^{\prime}) - \varepsilon \sum_{j} a_{ij} H(\mathbf{x}_{i}^{\prime} - \mathbf{x}_{j}), \qquad (3)$$

for i = 1, ..., N. Note that the driving variable \mathbf{x}_j is identical for both equations (1) and (3). If, for initial conditions $\mathbf{x}_i(0) \neq \mathbf{x}'_i(0)$, we have $|\mathbf{x}_i(t) - \mathbf{x}'_i(t)| \rightarrow 0$ as $t \rightarrow \infty$, node *i* is then entrained in the sense that its dynamics are no longer sensitive to initial conditions. If all nodes are entrained, we say that the networked oscillator system achieves the GS state.

For the local node dynamics, we typically choose the chaotic Lorenz oscillator,

$$\mathbf{F}_{i}(\mathbf{x}_{i}) = [10(y_{i} - x_{i}), r_{i}x_{i} - y_{i} - x_{i}z_{i}, x_{i}y_{i} - (8/3)z_{i}]^{\mathrm{T}},$$
(4)

where $\mathbf{x}_i \equiv (x_i, y_i, z_i)$ are the state variables of the Lorenz oscillator. Parameters r_i can be chosen to uniformly distribute in a certain interval, modeling different oscillators in the network. In this work, the coupling between two nodes in equation (1) is through the *x* variable, i.e. the output matrix is H = [1, 0, 0; 0, 0, 0; 0, 0, 0]. Besides the Lorenz oscillator, other node dynamics, such as the logistic map and the Rossler oscillator, have also been used in our study.

For the network topology, we choose the Barabási–Albert (BA) model that has been extensively used to represent the scale-free networks [38]. In such a model, the network grows from the initial m_0 nodes. Then, at each time step, m new nodes are added to the network. A new node randomly picks an existing node to connect to, but with some bias. More specifically, a node's probability of being selected is directly proportional to its degree. This is the rule that is called preferential attachment. In this way, a scale-free network with degree distribution satisfying power law can be generated.

3. The conditional Lyapunov exponents

By applying the auxiliary system approach, the occurrence and development of GS in networks can be revealed. For each node dynamics \mathbf{x}_i in the network, an imaginary auxiliary oscillator \mathbf{x}'_i can be constructed, as shown in equation (3). An important observation is that node \mathbf{x}_i and its auxiliary counterpart have exactly the same 'driving' forces, i.e. they are in the same local 'mean field'. If this local mean field is large enough, the dynamics on node *i* can be entrained in the sense that its dynamics are no longer sensitive to the initial conditions. Theoretically, the entrainment of local node dynamics can be characterized by the local largest conditional Lyapunov exponent λ_c [34, 35]. Let us first recall the concept of the Lyapunov exponent and its calculation. As we know, the most important dynamical property of chaotic system is its sensitivity to initial conditions, i.e. from any two nearby starting points, the trajectories will separate exponentially in phase space. This property of chaotic dynamics can be characterized by the Lyapunov exponent, which can be understood as the averaged expanding rates of an infinitesimal tangent vector along a chaotic trajectory in phase space. Specifically, suppose we have chaotic dynamics described by $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$. Given a trajectory $\{\mathbf{x}_n\}_{n=0}^{N-1}$ in phase space by time discretization, the following matrix product determines this exponent,

$$\mathbf{Q} \equiv \prod_{n=0}^{N-1} \mathbf{D} \mathbf{F}(\mathbf{x}_n), \tag{5}$$

where $\mathbf{DF}(\mathbf{x}_n) = \partial \mathbf{F} / \partial \mathbf{x}|_{\mathbf{x}_n}$ is the Jacobian matrix of the chaotic system evaluated at the trajectory point (\mathbf{x}_n). The largest Lyapunov exponent is given by

$$\lambda_1 = \lim_{N \to \infty} \frac{1}{N} \ln |\mathbf{Q}\mathbf{u}|,\tag{6}$$



Figure 1. Color maps of $d(\varepsilon, i_d)$ (a) and the largest conditional Lyapunov exponents $\lambda_c(\varepsilon, i_d)$ (b) in the two-dimensional (2D) parameter space (ε, i_d) , characterizing the development of GS for 100 identical chaotic Lorenz oscillators on a scale-free network described by equation (1). $r_i = 28$ for all Lorenz oscillators. i_d is the node index ranked according to the descending degree of nodes throughout this paper. The network is the BA model with $m_0 = m = 3$.

where **u** is a randomly chosen unit vector at the initial point \mathbf{x}_0 in the system. Wolf *et al* [39] provided a method that is frequently used to compute Lyapunov exponents. In the drive-response configuration of two coupled chaotic systems, we can treat the response system as a separate dynamical system and calculate its Lyapunov exponents as usual for that system alone regardless of the driving signals. These exponents will, of course, depend on the drivings, and for this reason they are called the conditional Lyapunov exponents. Similarly, the conditional Lyapunov exponents can be computed for nodes on a network. Given node *i* and coupling strength ε , we treat the local dynamics of node *i* as an independent system driven by the local mean field from its network neighbors and then calculate the conditional Lyapunov exponents accordingly. In figures 1 and 2, we provide two examples where the color maps of $\lambda_c(\varepsilon, i_d)$ matrices are plotted to illustrate how the development of GS in a network can be characterized by the largest conditional Lyapunov exponents of node dynamics. For comparison, we also characterize the occurrence of GS by directly measuring the 'distance' between a node *i* and its auxiliary oscillator, i.e.

$$d(\varepsilon, i) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |\mathbf{x}_i(t) - \mathbf{x}'_i(t)| \, \mathrm{d}t,$$
(7)

where $[t_1, t_2]$ is a time interval, and t_1 should be chosen to be larger than the typical transient time of the local dynamics. This definition is suitable for continuous chaotic flow, such as the Lorenz system and the Rossler system. For maps like the Logistic map, the integral can be replaced by summation. From figures 1 and 2, we can see that the results of these two methods



Figure 2. The same as figure 1 except that the dynamical model is equation (2).

are totally consistent. This implies that, for mutually coupled oscillators on networks, the largest conditional Lyapunov exponent is still a useful concept to characterize the local dynamical behavior.

4. The simple generalized synchronization (GS) states

In the above section, we have shown that the entrainment of local oscillators in a network can be characterized by the largest conditional Lyapunov exponents. If they are negative for all network nodes, we say that the networked oscillator system has achieved the GS state. Interestingly, by carefully examining the GS states at different coupling strengths, we reveal that the GS states can be different in terms of the final state of node dynamics. In this section, we extend the concept of GS to characterize one typical entrainment state for networked identical oscillators, namely the SGS state. By SGS we mean that all oscillators in the network are in trivial asymptotic states, either fixed points or limit cycles. However, the important point here is that all the trivial local attractors are different in phase space. For networked identical oscillators and linear diffusive coupling, as described by equation (1), it is easy to verify that there exists the CS manifold, i.e. all node dynamics coincide with each other, either on a trivial attractor or on a chaotic attractor. However, for the networked identical oscillator system (1), our work reveals that the system may go to such an asymptotic state that the dynamics of any two nodes *i* and *j* have the trivial functional relation $\mathbf{x}_i = \mathbf{x}_j + \mathbf{d}_{ij}$, where \mathbf{x}_i and \mathbf{x}_j could be fixed points or limited cycles, and $\mathbf{d}_{ij} \neq 0$.

The existence of the SGS state can be understood by the following heuristical analysis. Consider networked identical oscillators described by equation (1). Suppose that, in the asymptotic state, the local dynamics settle down on fixed points, and these fixed points have



Figure 3. The largest Lyapunov exponent λ_1 versus the coupling strength ε , characterizing the bifurcations from the spatiotemporal chaos to the SGS state, and finally to the CS state for networked identical oscillators with the increase in coupling strength. The dynamical model is the same as described in figure 1. $m_0 = m = 3$ for (a) and $m_0 = m = 6$ for (b).

the relation $\mathbf{x}_i = \mathbf{x}_j + \mathbf{d}_{ij}$. Then equation (1) becomes

$$0 = \mathbf{F}(\mathbf{x}_i) - \varepsilon \sum_j a_{ij} H(\mathbf{x}_i - \mathbf{x}_j),$$
(8)

i.e.

$$\sum_{j} a_{ij} H \mathbf{d}_{ij} = \mathbf{C}_i, \tag{9}$$

where $\mathbf{C}_i = \mathbf{F}(\mathbf{x}_i)/\varepsilon$ is a constant (vector) for each *i*. Equation (9) is a linear algebraic equation with respect to the unknowns \mathbf{d}_{ij} for oscillator *i*. Therefore, for all *N* oscillators, we can have a set of linear algebraic equations where the number of unknowns \mathbf{d}_{ij} equals $\sum k_i/2$ and the number of equations is *N*. For typical complex networks, the total degree is much larger than the node number. Therefore, equation (9) could generally have non-zero solutions, i.e. $\mathbf{d}_{ij} \neq 0$.

Our numerical simulations have verified the above analysis. In the previous section, GS is characterized by the largest conditional Lyapunov exponents of oscillators or distances between oscillators and their auxiliary systems. For networked identical oscillators, these methods cannot distinguish GS and CS because CS can also be regarded as a special kind of GS with identical functional relation. In fact, for a system of networked chaotic oscillators, initially it is in the spatiotemporal chaos regime if the coupling strength is small. With the increase in coupling strength, they may bifurcate into the GS regime. Given that the network connectivity is not too sparse, finally the networked system will enter into the CS regime if the coupling strength is large enough. These global bifurcations of networked dynamics can be characterized by computing the largest Lyapunov exponent for the networked system.

In figures 3 and 4, we provide examples to support our arguments. In figure 3, the bifurcations with the increase in coupling strength for a system of networked identical oscillators



Figure 4. Two SGS states corresponding to figure 3(a) at $\varepsilon = 2$ (a and b) and $\varepsilon = 5.25$ (c and d). The left panels are the snapshots of the system at t = 1000. The circles and stars denote the oscillators in equation (1) and their auxiliary counterpart, respectively. They coincide with each other. The right panels are the evolution of the variable *x* at nodes 1, 34 and 100.

have been characterized by the largest Lyapunov exponent of the system. Typically, with the increase in coupling strength, the system will experience the spatiotemporal chaos, the SGS and the CS regimes, respectively. The spatiotemporal chaos regime is characterized by the positive largest Lyapunov exponent. Then, in the SGS regime, the oscillators are all entrained to become trivial attractors, and the largest Lyapunov exponent of the system is less than or equal to zero. Finally, when the coupling strength is further increased, the largest Lyapunov exponent of the system becomes positive again, showing that the coupled system has settled down on the chaotic CS manifold. Comparing figure 3(b) with 3(a), it was found that, if the network connectivity is increased, normally the SGS regime will shrink. For fully connected networks, we cannot observe the SGS state.

In figure 4, we show two typical SGS states in phase space for 100 identical chaotic Lorenz oscillators in a scale-free network. Figures 4(a) and (b) plot the SGS states where the asymptotic states for all nodes in the network are fixed points. In this state, the largest Lyapunov exponent of the system is negative. Figures 4(c) and (d) plot the SGS states where the asymptotic states for all nodes in the network are limited cycles. In such a state, the largest Lyapunov exponent of the system is equal to zero. Here, we emphasize that the most important point for the SGS state is that all the trivial local attractors are not identical. Mathematically, they have simple functional relations that can be regarded as the trivial case of normal GS relation.

It should be noted that the SGS state studied above has both relations with and differences from the phenomenon of amplitude (or oscillation) death in coupled dynamical systems [40]. The latter refers to the fact that the oscillation in coupled dynamical systems, either limit cycles or chaotic oscillators, can be quenched so that the system settles down on fixed points in a certain parameter regime. Usually, strong coupling, large parameter mismatch, time-delay coupling or dynamic coupling can lead to this behavior. Physically, this is because the interaction among

oscillators can induce new trivial fixed points or just stabilize the unstable fixed points already existing in the uncoupled system. In the present work, the concept of SGS has a certain overlap with amplitude death, but is broader, because the SGS is defined as the trivial asymptotic states in a networked chaotic oscillator system, including both fixed points and limit cycles. In fact, it is found that these two states usually occur alternately in a certain parameter regime. As we mentioned above, although CS manifold exists for the networked system with identical oscillators, the system finally goes to a trivial state in which all nodes are different from each other. This suggests that this state, although trivial, is induced by the interaction among oscillators in the network. Thus it is reasonable to think that there are certain relations among the trivial node dynamics in this state. For this reason, we extend the concept of GS to this trivial state. We emphasize that the SGS between two fixed points is merely a pro forma extension of the GS concept. In the strict sense, GS should refer to the functional relation between two oscillatory dynamics.

5. The chaotic GS states

For networked identical oscillators, we have observed transitions from non-synchronization to the SGS state and then to the global CS. One natural question is: how about the system of networked non-identical oscillators? Actually, GS is inherently defined in such coupled nonidentical systems, whether they are high-dimensional or low-dimensional. In our study, as expected, we have also observed both the SGS and the CGS states for coupled nonidentical oscillators in networks. In the following, we provide two examples to illustrate them. Figures 5–7 characterize the GS state occurring in a system of 100 non-identical Lorenz oscillators in a scale-free network. The system is described by equation (1). To make the oscillators different, we specially choose parameter r_i from the uniform distribution in the interval [28, 38]. In this parameter regime, the individual Lorenz oscillator has the chaotic attractor. We first characterize the occurrence of GS in this system by computing the largest conditional Lyapunov exponents on the parameter plane (ε , i_d). The results are shown in figure 5(a). It is found that approximately when $\varepsilon > 0.5$, GS can be achieved in this system. In figures 5(b) and (c), we further characterize the GS states by the Lyapunov exponent spectra of the coupled system. From figure 5(b), it is found that the largest Lyapunov exponent of the system can exhibit different behavior within the GS regime, showing that different GS states exist. By carefully examining the evolution of local dynamics, we have confirmed that both the SGS and the CGS states exist in such a system. As shown in figure 5(b), the SGS state is characterized by the non-positive largest Lyapunov exponent, and the CGS state is characterized by the positive largest Lyapunov exponent. We have also computed the Lyapunov exponent spectra, as shown in figure 5(c). It is seen that in the spatiotemporal chaos regime, the largest Lyapunov exponent is positive, while in the SGS regime it is equal to or less than zero. Interestingly, in the CGS regime, only the largest Lyapunov exponent is positive; the second largest Lyapunov exponent is equal to zero, and all other Lyapunov exponents are negative. This suggests that, although the original networked system is high-dimensional, in the CGS state it behaves just like a 3D chaotic attractor although all oscillators are parametrically different!

We have carried out detailed computation of the largest Lyapunov exponent in the SGS state. The results show that this regime alternately consists of many small intervals of negative largest Lyapunov exponent and many small intervals of zero largest Lyapunov exponent, as



Figure 5. Characterizing the GS bifurcations for 100 non-identical chaotic Lorenz oscillators on a scale-free network. The dynamical equation is (1), and $r_i \in [28, 38]$. The network is the BA model with $m_0 = m = 6$. (a) Color map of the largest conditional Lyapunov exponents $\lambda_c(\varepsilon, i_d)$ in the two-dimensional parameter space (ε, i_d) . (b, c) The largest Lyapunov exponent (b), and spectra of the largest ten Lyapunov exponents (c) versus the coupling strength ε .

shown by points A and B in figure 5(b). Similar to the case of coupled identical oscillators, in the SGS state with negative largest Lyapunov exponent, the local dynamics of the coupled system settle on different fixed points, whereas in the SGS state with zero largest Lyapunov exponent, the local dynamics go to different limit cycles. In figures 6 and 7, we provide two such examples to directly illustrate the SGS and CGS states. Figure 6 corresponds to point B shown in figure 5(b), where the largest Lyapunov exponent is zero and the asymptotic attractors of the local dynamics are limit cycles. Figure 7 corresponds to point C shown in figure 5(b), where the largest Lyapunov exponent is positive and all node dynamics go to different chaotic attractors.

In our study, we have also observed the CGS state in coupled identical dynamical systems in networks. Here we show one typical example. The networked system is described by equation (2), and the local dynamics are chosen as the chaotic logistic maps, i.e. f(x) =4x(1-x) for all nodes. In figure 8(a), we plot the color map of the distance matrix $d(\varepsilon, i_d)$ on the parameter plane (ε , i_d). Due to the normalized coupling in equation (2), all nodes in the network are entrained almost simultaneously despite the heterogeneous structure of the



Figure 6. Characterizing the local dynamics in the SGS state corresponding to point B in figure 5(b), $\varepsilon = 2.226$. The first row: the limit cycles for nodes 1, 34 and 100 in the (x, z) plane. The middle row: the variable z versus the variable z' of the auxiliary oscillator, showing the local entrainment of oscillators. The bottom row: SGS relation among nodes 1, 34 and 100.

scale-free topology. Since the local dynamics in the coupled system are identical, generally both GS and CS could occur. To distinguish them, in figures 8(b) and (c), we plot, respectively, the distance of global GS as $l_g(\varepsilon) = \langle d(\varepsilon, i) \rangle$, where $\langle \cdot \rangle$ denotes the spatial average over all nodes, and the distance of global CS, $l_c(\varepsilon)$, which is basically the asymptotic time average of the instantaneous standard deviations of the dynamical variables, i.e.

$$l_{c}(\varepsilon) = \frac{1}{t_{2} - t_{1}} \sum_{t_{1}}^{t_{2}} \langle |\mathbf{x}_{i}(t) - \langle \mathbf{x}_{i}(t) \rangle | \rangle.$$
(10)

Here, the meaning of t_1 and t_2 is the same as that in equation (7), which are chosen after discarding a number of transients. Apparently, if $l_c = 0$, global CS has been achieved in the whole network. It is shown that, for this networked system, it has a GS regime before it finally goes to the CS state. To our surprise, unlike the networked identical oscillators we studied in section 3, this GS state can be further characterized as CGS by computing the Lyapunov exponent spectra of the coupled system. In figure 9, we plot the full Lyapunov exponent spectra for 300 networked logistic maps. Immediately, we have the following observations. Firstly, the system has three distinct dynamical regimes, i.e. the spatiotemporal chaos, the CGS and the CS, which can be characterized by the largest Lyapunov exponent of the system. In particular, the GS state in this example has the positive largest Lyapunov exponent, showing that this state is physically different from the SGS states studied in coupled identical dynamical systems in previous sections. Secondly, we have found that the second largest Lyapunov exponent plays a unique role in characterizing the bifurcations of the system. As shown in figure 9, the



Figure 7. Characterizing the local dynamics in the CGS state corresponding to point C in figure 5(b), $\varepsilon = 5$. The figure caption is the same as in figure 6.

bifurcation from the spatiotemporal chaos state to the GS state is manifested by the separation of the largest Lyapunov exponent and the second largest Lyapunov exponent of the system. Thirdly, at the very initial stage, all the Lyapunov exponents are positive, showing that the system is in spatiotemporal chaos with very high dimension. With the increase in coupling strength, the oscillators in the network have been gradually entrained and the system enters the CGS stage. At the beginning of the CGS stage, the system still has a large number of positive Lyapunov exponents, showing that the dynamics of the networked system are in a certain high-dimensional phase space. In this sense, there is no distinct difference between the spatiotemporal chaos regime and the CGS regime. At the CGS stage, with further increase in coupling strength, the number of positive Lyapunov exponents gradually decreases until all the Lyapunov exponents except the largest one become negative, when the system finally goes to the CS state. Interestingly, during this stage, the largest Lyapunov exponent of the networked system keeps increasing until it reaches a plateau when the system goes to the CS regime. Therefore, the Lyapunov exponent spectra in the CGS regime present a vivid scenario of the transition of highdimensional dynamical systems: the dimension of the coupled system continuously collapses while simultaneously the system's dynamics become more and more chaotic!

6. Identifying functional relations among node dynamics

In previous sections, we have shown that coupled oscillators in networks can be entrained to form the GS state. So far, the GS state only means the entrainment of local node dynamics by its network neighbors, as described in equations (1) and (2). This entrainment of local dynamics can be characterized by the largest conditional Lyapunov exponent of oscillators. In our study, we find a subtle conceptual problem when GS in networks is considered. As we know, GS in



Figure 8. Characterizing the global bifurcations for 300 identical chaotic logistic maps on a scale-free network. The dynamical equations are described by equation (2), and the network is the BA model with $m_0 = m = 10$.



Figure 9. The full Lyapunov exponent spectra corresponding to the dynamical model described in figure 8. The red line and the green line denote the largest and the second largest Lyapunov exponent, respectively.

the strict sense requires a functional relation between two dynamics. However, according to the auxiliary system approach criterion, the entrainment of an oscillator in a network only implies that there is a GS relation between the dynamics of the oscillator and the local driving field. Whether there is indeed a functional relation between the dynamics of any pair of nodes in the network remains unknown. Especially in typical complex networks, usually most of the node pairs are not directly connected. In such situations, do the dynamics of two oscillators in a



Figure 10. Direct characterization of GS relations among node dynamics on networks. The network is the BA model with $m_0 = m = 6$. Network size N = 100. The parameter mismatch among Lorenz oscillators is $\Delta r = 1$. (a) The largest Lyapunov exponent versus the coupling strength. (b) The matrix of the MFNN parameter for $\varepsilon = 0.2$ showing the spatiotemporal chaos (STC) state. (b) The matrix of the MFNN parameter for $\varepsilon = 5$ showing the CGS state.

network (they may not be directly connected) really achieve GS in the strict sense or not? This is a very important and interesting question concerning network synchronization.

In this work, we are able to answer the above question by providing direct evidence that there indeed exist functional relations among node dynamics when the networked oscillator system is entrained. For this purpose, we investigate a system of 100 coupled Lorenz oscillators in a scale-free network. The dynamical equations of such a system are described by equation (1), where the local dynamics are chosen to be different by setting r in [28, 29] uniformly. In figure 10(a), the largest Lyapunov exponent of the networked system versus the coupling strength is plotted. It is shown that the networked system successively experiences different dynamical regimes as the coupling strength is increased, namely the spatiotemporal chaos regime, the SGS regime and the CGS regime. Here, we want to demonstrate that, in the CGS regime where all the oscillators are entrained, GS in the strict sense occurs between any two pairs of node dynamics in the network.

In order to show this relation directly, we apply the mutual false nearest neighbor (MFNN) method proposed in [5]. The MFNN parameter P is defined as

$$P(i, j) = \left\langle \frac{|\mathbf{x}_n^j - \mathbf{x}_{n(NNI)}^j|}{|\mathbf{x}_n^i - \mathbf{x}_{n(NNI)}^i|} \frac{|\mathbf{x}_n^i - \mathbf{x}_{n(NNJ)}^i|}{|\mathbf{x}_n^j - \mathbf{x}_{n(NNJ)}^j|} \right\rangle.$$
(11)

Here, *n* is the time index. \mathbf{x}_n^i is an arbitrary point in the phase space of oscillator *i* and \mathbf{x}_n^j is its corresponding counterpart for oscillator *j*. The nearest phase space neighbor of \mathbf{x}_n^i has

time index n(NNI) and the nearest phase space neighbor of \mathbf{x}_n^j has time index n(NNJ). The $\langle \cdot \rangle$ denotes the average of the reference point on the attractor. This parameter will be of the order of unity if a certain functional relation exists between oscillators *i* and *j*. Otherwise, it is approximately the square of the size of the local chaotic attractor, which is much larger than unity for the Lorenz oscillator. In figures 10(b) and (c), we plot the color maps of the MFNN parameter P(i, j) for two typical spatiotemporal chaos states and the CGS state, respectively.

In figure 10(b), the coupling strength is small ($\varepsilon = 0.2$). At this stage, none of the oscillators is entrained, as shown in figure 10(a). From the color map of the MFNN parameter P(i, j), we can see that all the elements are of the order of 10^2-10^4 (269–23912). Apparently, there is no functional relation among node dynamics at this phase. In figure 10(c), the coupling strength is large enough to entrain all the oscillators in the network. Remarkably, in this case, all elements in the matrix P(i, j) are of the order of 1, showing that certain functional relations have been achieved among the dynamics of nodes in the network. Combining the results in figure 10, we can conclude that, for networked non-identical oscillators in the CGS state, i.e. when all the oscillators have been entrained, the node dynamics can indeed achieve GS in the strict sense, although they may not be directly connected to each other in the network.

7. Summary

In this work, we carry out both theoretical analysis and numerical experiments to characterize the GS occurring on a scale-free network. It is shown that the entrainment of oscillators in networks can be characterized by the conditional Lyapunov exponent of the local dynamics. By analyzing the Lyapunov exponent spectra, we are able to characterize the global bifurcations of the networked system. We reveal that there are two different types of GS states existing in the GS regime, i.e. the SGS and the CGS. In the former state, the system has non-positive largest Lyapunov exponent, and the oscillators in the network go to different trivial attractors, such as fixed points or limited cycles. In the latter, the system has the positive largest Lyapunov exponent, and the oscillators in a network in the CGS state can indeed achieve functional relations although they may not be connected directly on typical complex networks. Our work will be helpful to understand the collective behavior and coherence in networked dynamical systems.

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