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Frequency locking by external force from a dynamical system with strange nonchaotic attractor

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Abstract

Usually, phase synchronization is studied in chaotic systems driven by either periodic force or chaotic force. In the present work, we consider frequency locking in chaotic Rössler oscillator by a special driving force from a dynamical system with a strange nonchaotic attractor. In this case, a transition from generalized marginal synchronization to frequency locking is observed. We investigate the bifurcation of the dynamical system and explain why generalized marginal synchronization can occur in this model.

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1. Introduction

Recently phase synchronization (PS) in chaotic systems has attracted much research attention [1–12]. This interesting synchronization phenomenon was first reported by Rosenblum et al. in 1996 [3,4]. They found that for certain autonomous chaotic flow systems, a suitable phase variable can be defined. Interestingly, it is shown that the phases between two chaotic oscillators can be locked while their amplitudes remain chaotic and uncorrelated. Frequently, PS also refers to the weaker condition when the mean frequencies between two chaotic oscillators are locked. PS between two coupled slightly different chaotic oscillators can be characterized in terms of the Lyapunov exponent (LE) spectra [3]. When PS occurs, the system's null LE becomes negative. Meanwhile, the largest LE of the system remains positive showing that the amplitude is still chaotic. So far, the investigations on PS in chaotic systems are mainly carried out in the following directions. Firstly, the chaotic oscillator entrained by external periodic force is explored [4,5]. Secondly, PS between two coupled chaotic oscillators with different natural frequencies have been intensively studied [3,6–11]. Thirdly, PS in the array of coupled chaotic oscillators has also attracted much attention [12]. In addition to the theoretical works, PS has already been demonstrated experimentally in various fields, such as in electrical circuits [13–15], lasers [16,17], fluids [18], and biological systems [19,20].

In the present work, we investigate the frequency locking in chaotic Rössler oscillator under a special external driving force. For the existing works along this line [4,5], usually the driving force is either periodic or chaotic. Unlike these studies, the driving force considered in this work is from a dynamical system which exhibits a strange nonchaotic attractor (SNA). SNA is a kind of special attractor that is geometrically complicated, but dynamically not chaotic, i.e., the trajectories on it do not show sensitive dependence on initial conditions asymptotically [21]. Therefore, the properties of the dynamics on the SNA are essentially different from that of periodic or chaotic systems. Typically, dynamical system with SNA is driven by quasiperiodical forces, i.e., forces with two incommensurate frequencies. Since signals from the SNA system have two characteristic time scales (see Fig. 1(a) and (b)), is it possible for a phase-coherent

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chaotic system to be frequency-locked to such signals? If so, are there any distinguishing features compared with existing PS in chaotic systems? Motivated by these questions, in this work we study frequency locking in chaotic Rössler system driven by signals from a system with SNA. Interestingly, we observe a transition from generalized marginal synchronization (GMS) [22] to frequency locking with the increase of the coupling strength. Specifically, when the coupling strength is relatively small, the frequency of the driven Rössler oscillator is changed by the driving force, but not locked to it. We observe parameter regimes where the largest conditional Lyapunov exponent (LCLE) in the driven Rössler system is null, indicating that GMS occurs between the coupled systems at this stage. In GMS regime, numerically it is found that two Rössler oscillators under the same driving force exhibit lag synchronization in practical sense. When the coupling strength exceeds certain critical value, it is found that the frequency of the Rössler oscillator becomes locked to one of the driving frequencies. To our knowledge, such a transition from marginal synchronization to frequency locking in chaotic system has not been reported so far. We characterize the bifurcation by the Lyapunov exponent and analyze why GMS can occur in the present model.

This Letter is organized as follows. In Section 2, we briefly introduce the dynamical model. In Section 3, the results of GMS and frequency locking in the Rössler system driven by SNA signals are presented in detail. An analysis to the results can also be found in this section. This Letter ends with the concluding remarks.

2. The dynamical model

In the present work, we aim to study the possible frequency locking between a phase-coherent chaotic oscillator and the SNA system. Usually, SNA appears in dissipative dynamical system driven by several incommensurate frequencies, i.e., in quasiperiodically driven systems. The SNA model used in this work describes the dynamics of quasiperiodically forced damped pendulum [21]. The dynamical equations read

$$\begin{aligned}
\dot{\phi} &= v, \\
\dot{v} &= P\left\{K + V\left[\cos\left(\frac{\omega_1}{\omega_2}w\right) + \cos(w)\right] + \cos(\phi) - v\right\}, \\
\dot{w} &= \omega_2.
\end{aligned} \tag{1}$$

Here, ω_1 and ω_2 are the two incommensurate frequencies of the driving forces. P, K, and V are parameters. Throughout this Letter, the parameters are chosen as: $\omega_1 = (\sqrt{5}-1)/2$ (the inverse golden mean), $\omega_2 = 1$, $P = 0.9715$, $K = 0.8$, and $V = 0.55$. With such parameter values, system (1) exhibits SNA as shown in Fig. 1(c). To visualize the SNA in system (1), usually it is convenient to use the Poincaré surface of section technique. Specifically, we sample the system at time intervals corresponding to the variable $w_n = \omega_2 t_n = 2\pi n$, where $n = 0, 1, \ldots$ (the stroboscopic surface of section). We then examine the dynamical variables ϕ_n (mod 2π) and v_n. Fig. 1(c) shows the (ϕ, v) projection of a trajectory of 10 000 iterations (after 10 000 preiterations) on the stroboscopic surface of section. We use system (1) as the drive system. A time series of the driving signals, i.e., the variable v in Eqs. (1) is shown in Fig. 1(a).

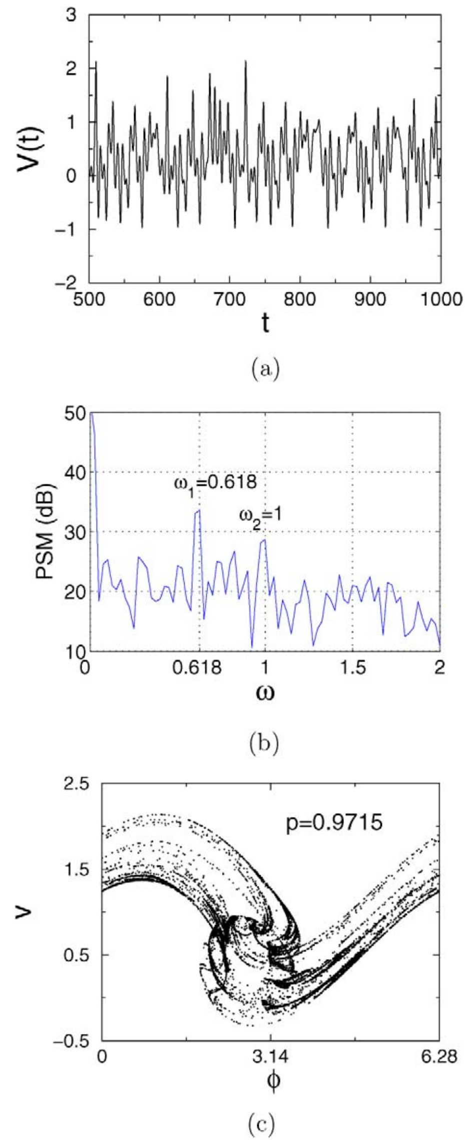

Fig. 1. (a) A time series of $v(t)$ in Eqs. (1). It is used as the driving force in the present work. (b) The power spectrum magnitude (PSM) of (a), where two driving frequencies are visible. (c) The stroboscopic plot of SNA in system (1).

For the response system, we choose the phase-coherent Rössler oscillator. The dynamical equations with the driving are

$$\begin{aligned}
\dot{x} &= -\Omega y - z, \\
\dot{y} &= \Omega x + ay - \epsilon(y - v), \\
\dot{z} &= b + z(x - c),
\end{aligned} \tag{2}$$

where $a = b = 0.2$ and $c = 5.7$ are parameters; ϵ is the coupling strength. Ω characterizes the natural time scale of the Rössler oscillator. We set $\Omega = 1.01$ and numerically found that the natural frequency of the Rössler oscillator (without driving) is about 1.08.

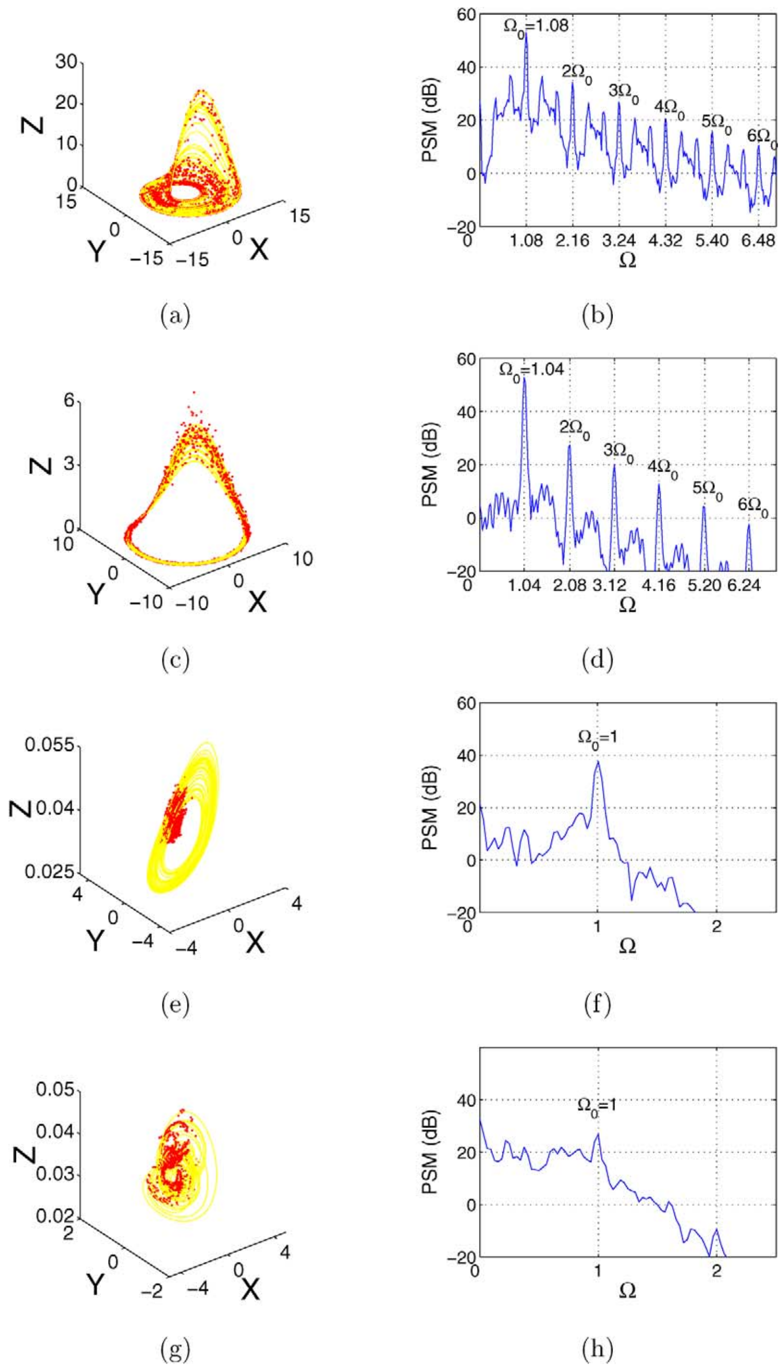

Fig. 2. SNA system (1) drives the Rössler system (2) with $\Omega = 1.01$. (a), (c), (e) and (g): the stroboscopic plots (black dots) and attractors (gray curves) for the response system at $\epsilon = 0$, 0.1, 0.25, and 1.0, respectively. (b), (d), (f) and (h): the PSM of response system corresponding to (a), (c), (e) and (g), respectively. Ω_0 corresponds to the mean frequency of the Rössler system under driving.

3. From GMS to frequency locking

Unlike previous studies, in the present work the Rössler oscillator (2) is driven by signals from the SNA system (1). Since the quasiperiodic driving signals from Eqs. (1) contain two frequencies, i.e., $\omega_1 = (\sqrt{5} - 1)/2$ and $\omega_2 = 1$, it can be expected that under certain condition it is possible for the Rössler oscillator to be phase-locked with one of them. In the present model, the natural frequency of the Rössler oscillator (without forcing) is chosen to be about 1.08. With the driving from system (1), it is found that the frequency of the Rössler oscillator gradually changes and finally becomes locked to one of the driving frequency, i.e., $\omega_2 = 1$. These results are demonstrated in Fig. 2, where the stroboscopic plots of the forced Rössler oscillator as well as the power spectra with different coupling strengths are shown. For comparison, the attractors of the response system have also been shown as background in gray. With the increase of the coupling strength, successive changes occur for the Rössler attractor as shown in Fig. 2. In Fig. 2(a), (c) and (e), the Rössler attractors are phase-coherent. However, when the coupling strength is large enough as in Fig. 2(g), it becomes no longer phase-coherent. Comparing Fig. 2(a) with Fig. 2(c) and (e), it is seen that for small coupling strength such as $\epsilon = 0.1$, the Rössler attractor becomes more phase-coherent under the driving from system (1). Meanwhile, the mean frequency of the Rössler oscillator under driving (denoted as Ω_0 in Fig. 2) has decreased to $\Omega_0 = 1.04$, which corresponds to the basic frequency shown in Fig. 2(d). With further increase of the coupling strength, the Rössler oscillator becomes much more coherent as shown in Fig. 2(e). At this stage, the phase of the Rössler oscillator has already been locked to one of the driving frequency. This can be directly seen from the stroboscopic plot in Fig. 2(e). In Fig. 2(a) and (c), the phases are spread over the whole attractor, while in Fig. 2(e), the phases are concentrated in a small domain on the attractor, indicating phase locking occurs. The phase locking shown in Fig. 2(e) has been further confirmed in Fig. 2(f), where the basic frequency of the driven Rössler oscillator is shown to be locked to the driving frequency, i.e., $\Omega_0 = \omega_2 = 1$ when $\epsilon = 0.25$. Numerically, it is found that the Rössler attractor under driving can remain phase-coherent when $\epsilon < 0.4$. Large driving force will destroy the coherence of the attractor. An example when $\epsilon = 1$ is given in Fig. 2(g) and (h). In this case, the phases in stroboscopic plot are spread over the attractor and the dominant frequency in Fig. 2(f) has almost submerged in the background in Fig. 2(h). Throughout this Letter, we will limit our discussion within the phase-coherent regime of the driven Rössler oscillator, i.e., $\epsilon < 0.4$.

The observation in Fig. 2 suggests that certain bifurcation occurs with the increase of the coupling strength. In order to characterize the bifurcation, we calculate the LE spectra of the response system as well as the frequency difference $\Delta\Omega = \Omega_0 - \omega_2$ between the response system and one of the driving periodic force. The results are shown in Fig. 3(a). With the increase of the coupling strength, it is found that the frequency difference $\Delta\Omega$ gradually decreases. When the coupling strength is greater than a critical value $\epsilon_c \simeq 1.9$, the frequency difference becomes zero, indicating that frequency locking between the Rössler oscillator and one of the periodic force has been established. Numerically, we test many other Ω values in Eqs. (2) and found that as long as they are close to $\omega_2 = 1$, with sufficient large coupling strength the Rössler oscillator will eventually be frequency-locked to the driving frequency $\omega_2 = 1$. In Fig. 3(b), we show the synchronization zone for Ω when the coupling strength is fixed at 0.25. It should be noted that within this synchronization zone, the Rössler oscillator is in the chaotic regime without forcing.

From the LE spectra shown in Fig. 3(a), it can be found that basically there are two qualitatively different regimes. When $\epsilon > \epsilon_c$, the LCLE becomes negative showing that generalized synchronization (GS) between the two systems has been achieved. In fact, the frequency difference $\Delta\Omega$ also vanishes at this point. Therefore, the critical value ϵ_c is a two-fold bifurcation point. Both GS and frequency locking between two systems occur at this point. In other words, in the present model the frequency locking can only be observed after GS has been achieved. As we know, GS and PS both occur in coupled different chaotic systems. Usually they take place at different coupling strength. The relation between these two types of synchronization has been discussed in several works [3,6–8,11]. For coupled system with small parameter mismatch, usually PS is found to occur before GS. Nevertheless, for coupled system with large parameter mismatch or for coupled essentially different system, phase locking or frequency locking might occur after GS, i.e., after the amplitudes of two systems become correlated. In this case, usually relatively large coupling strength is needed to achieve phase locking or frequency locking. From Fig. 3(a), it is also found that the second LCLE λ_2, i.e., the originally null LE becomes negative at much smaller coupling strength than the bifurcation point ϵ_c. This result is consistent with the previous studies on frequency locking in coupled different chaotic systems [8,11], where the occurrence of frequency locking usually cannot be characterized by the originally null Lyapunov exponent. Interestingly, when $\epsilon < \epsilon_c$, there are relatively wide parameter ranges where the LCLE is null. With the null LCLE, the driving and driven system can achieve a special relation known as marginal (projective) synchronization [22–24]. Therefore, in the present model, with the increase of the coupling strength, a transition from marginal synchro-

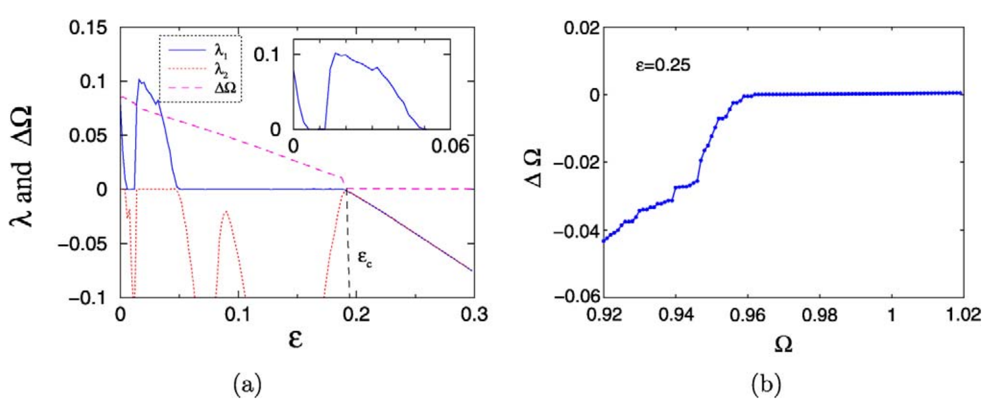

Fig. 3. Characterizing frequency locking in the Rössler system (2) with $\Omega = 1.01$ driven by system (1). (a) The first and the second LCLE (λ_1 and λ_2) and the frequency difference between the response system and one driving periodic force, i.e., $\Delta\Omega = \Omega - \omega_2$ vs. the coupling strength ϵ. (b) $\Delta\Omega$ vs. Ω with the coupling strength $\epsilon = 0.25$.

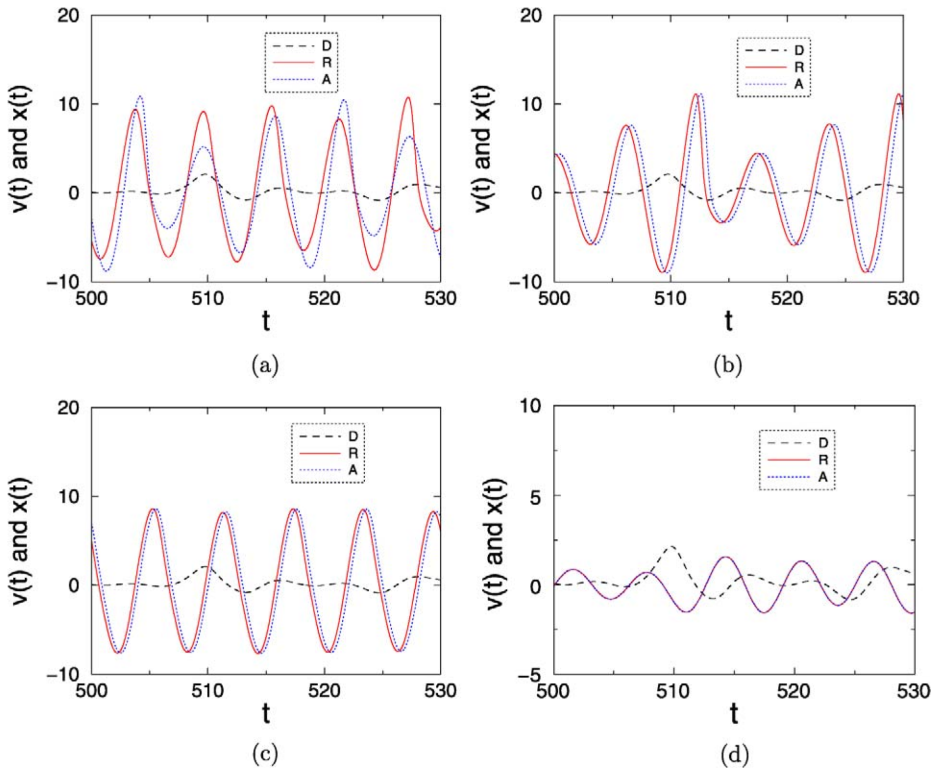

Fig. 4. The evolution of state variables in the drive (D), response (R) and auxiliary (A) systems. (a) $\epsilon = 0$. (b) $\epsilon = 0.01$. (c) $\epsilon = 0.1$. (d) $\epsilon = 0.25$. In (b) and (c), lag synchronization occurs between R and A, indicating a GMS relation between R and D. In (d), complete synchronization exists between R and A, indicating a generalized synchronization relation between R and D.

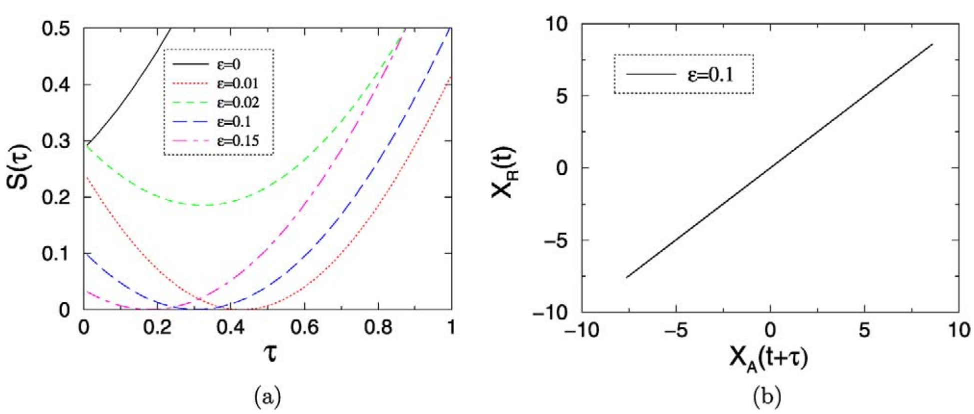

Fig. 5. Characterizing the lag synchronization. (a) Similarity function $S(\tau)$ for different values of coupling strength. For lag synchronization ($\epsilon = 0.01, 0.1, 0.15$), the minima approximately approach zero. (b) $x_R(t)$ vs. $x_A(t+\tau)$ with $\tau = 0.27$ showing lag synchronization between the response and auxiliary systems.

nization (MS) to frequency locking (or GS) has been observed. To our knowledge, the transition has not been reported so far.

Marginal synchronization actually is an interesting synchronization-like dynamical behavior in coupled systems. Since the LCLE of the response system is zero rather than negative, the response system is not a totally passive system under driving. It is only partially entrained by the drive system. The trajectories from different initial conditions in the response system do not converge together, instead they evolve in perfect correlation with the drive dynamics. When the drive and the response systems are identical, the attractor of the response system could be either an amplification of the drive attractor or a shift of the attractor to a different region in phase space. When the drive and the response systems are not identical, generalized marginal synchronization (GMS) can be observed in coupled systems [22]. In this case, if two response systems, usually known as the response and the auxiliary system, are forced by the same driving system, there exists MS relation between them. Fortunately, the response–auxiliary system method [25], which is effective to detect GS, can also be used to detect GMS.

In our work, it is found that the observed GMS exhibits different characteristics. Namely, the attractors in the response and the auxiliary systems do not exhibit invariance under amplification or translation in phase space like in the previous works. However, a careful examination of the evolution of amplitudes in the response and auxiliary system suggests they actually evolve with a time lag, i.e., not only the phases between them are locked, but also their amplitudes coincide with a time lag. These results have been shown in Fig. 4. To verify the lag synchronization between the two driven systems, we calculate the similarity function defined in Ref. [9]. As shown in Fig. 5(a), within the GMS regime, where the LCLE is zero, it is clearly shown that the minima of the similarity function approach zero. This implies that the amplitudes of the two driven systems become approximately the same, but shifted in time with respect to each other. Therefore, for the GMS observed in this work, the response system and the auxiliary system exhibit lag synchronization in practical sense.

In order to explain the approximate lag synchronization between two Rössler oscillators driven by the same signals in the GMS regime, we can introduce the amplitude and phase variable [9]

$$\phi = \arctan(y/x), \qquad A = \sqrt{x^2 + y^2}. \tag{3}$$

Then the amplitude variables of two Rössler system, Eqs. (2), can be rewritten as

$$\dot{A}_{1,2} = (a - \epsilon)A\sin^2\phi_{1,2} - z_{1,2}\cos\phi_{1,2} + \epsilon v \sin\phi_{1,2},$$
$$\dot{z}_{1,2} = b + z_{1,2}(A_{1,2}\cos\phi_{1,2} - c). \tag{4}$$

This is a system of two oscillators driven by periodic forces $\sin\phi_{1,2}$ and $\cos\phi_{1,2}$, as well as the common driving force v from system (1). Since the two driven oscillators are the same and subjected the same driving force, their frequencies are locked. Without losing generality, we assume $\phi_1 = \Omega t$ and $\phi_2 = \Omega t - \Delta\phi$ where $\Delta\phi$ is a constant phase shift. Without this phase shift, the driving forces are the same and one could expect complete synchronization between these two identical oscillators; with the phase shift, an approximate lag synchronization occurs. This can be verified if we introduce the lag variables for the second system $A'_2 = A_2(t+\tau)$, $z'_2 = z_2(t+\tau)$ where $\tau = (\phi_1 - \phi_2)/\Omega$ and reasonably assume the driving amplitude $v(t)$ is a slow variable, i.e., $v(t) \approx v(t+\tau)$ as long as τ is small. In this way, under transformation $t \to t - \tau$, equations of variables (A'_2, z'_2) are the same as that of variables (A_1, z_1). Therefore, we have $A_1 \approx A'_2$ and $z_1 \approx z'_2$ and this implies the approximate lag synchronization between two oscillators described in Eqs. (4). It should be pointed out that strict MS is related to the symmetries of the system's dynamical equation. As shown in Refs. [23], the amplification and the phase space shift of the driving attractor can be attributed to the invariance of the dynamical equation under transformation of coordinate amplification and translation, respectively. However, in the present model, the lag synchronization observed between two driven systems described by Eqs. (4) is in practical sense. Physically, it originates from the approximate invariance under translation of time in Eqs. (4).

4. Concluding remarks

In this Letter, we investigate the frequency locking in chaotic Rössler oscillator driven by a dynamical system with SNA. With the increase of the coupling strength, a transition from GMS to frequency locking is observed. Further analysis reveals that for the GMS found in this model the response system and the auxiliary system turn out to be synchronized with a time lag. This lag synchronization is related to the approximate invariance under translation of time in driven dynamical equations.

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References

[1] A. Pikovsky, M. Rosenblum, J. Kurths, in: Synchronization: A Universal Concept in Nonlinear Science, Cambridge Univ. Press, Cambridge, 2001.
[2] S. Boccaletti, J. Kurths, G. Osipov, D.L. Valladares, C.S. Zhou, Phys. Rep. 366 (2002) 1.
[3] M.G. Rosenblum, A.S. Pikovsky, J. Kurths, Phys. Rev. Lett. 76 (1996) 1804.
[4] A.S. Pikovsky, M.G. Rosenblum, G.V. Osipov, J. Kurths, Physica D 104 (1997) 219.
[5] A. Pikovsky, G. Osipov, M.G. Rosenblum, M. Zaks, J. Kurths, Phys. Rev. Lett. 79 (1997) 47;
E.-H. Park, M.A. Zaks, J. Kurths, Phys. Rev. E 60 (1999) 6627;
T.E. Vadivasova, A.G. Balanov, O.V. Sosnovtseva, D.E. Postnov, E. Mosekilde, Phys. Lett. A 253 (1999) 66;
A. Pikovsky, M.G. Rosenblum, J. Kurths, Int. J. Bifur. Chaos 10 (2000) 2291.
[6] U. Parlitz, L. Junge, W. Lauterborn, L. Kocarev, Phys. Rev. E 54 (1996) 2115.
[7] Z. Zheng, G. Hu, Phys. Rev. E 62 (2000) 7882.
[8] G.V. Osipov, B. Hu, C. Zhou, M.V. Ivanchenko, J. Kurths, Phys. Rev. Lett. 91 (2003) 024101.
[9] M.G. Rosenblum, A.S. Pikovsky, J. Kurths, Phys. Rev. Lett. 78 (1997) 4193.
[10] K.J. Lee, Y. Kwak, T.K. Lim, Phys. Rev. Lett. 81 (1998) 321;
J.Y. Chen, K.W. Wong, J.W. Shuai, Chaos 12 (2002) 100;
M. Zhan, G.W. Wei, C.-H. Lai, Phys. Rev. E 65 (2002) 036202.
[11] S. Guan, C.-H. Lai, G.W. Wei, Phys. Rev. E 72 (2005) 016205.
[12] A.S. Pikovsky, M.G. Rosenblum, J. Kurths, Europhys. Lett. 34 (1996) 165;
G.V. Osipov, A.S. Pikovsky, M.G. Rosenblum, J. Kurths, Phys. Rev. E 55 (1997) 2353;
M. Zhan, Z. Zheng, G. Hu, X. Peng, Phys. Rev. E 62 (2000) 3552;
Z. Zheng, G. Hu, B. Hu, Phys. Rev. E 62 (2000) 7501;
Z. Liu, Y.-C. Lai, F.C. Hoppensteadt, Phys. Rev. E 63 (2001) 055201.
[13] A. Pujol-Peré, O. Calvo, M.A. Matías, J. Kurths, Chaos 13 (2003) 319.
[14] E. Rosa Jr., C.M. Ticos, W.B. Pardo, J.A. Walkenstein, M. Monti, J. Kurths, Phys. Rev. E 68 (2003) 025202(R).
[15] M.S. Baptista, T.P. Silva, J.C. Sartorelli, I.L. Caldas, E. Rosa Jr., Phys. Rev. E 67 (2003) 056212.
[16] K.V. Volodchenko, V.N. Ivanov, S.-H. Gong, M. Choi, Y.-J. Park, C.-M. Kim, Opt. Lett. 26 (2001) 1406.
[17] D.J. DeShazer, R. Breban, E. Ott, R. Roy, Phys. Rev. Lett. 87 (2001) 044101.
[18] D. Maza, A. Vallone, H. Mancini, S. Boccaletti, Phys. Rev. Lett. 85 (2000) 5567.
[19] P. Tass, M.G. Rosenblum, J. Weule, J. Kurths, A. Pikovsky, J. Volkmann, A. Schnitzler, H.-J. Freund, Phys. Rev. Lett. 81 (1998) 3291.
[20] R.C. Elson, A.I. Selverston, R. Huerta, N.F. Rulkov, M.I. Rabinovich, H.D.I. Abarbanel, Phys. Rev. Lett. 81 (1998) 5692.
[21] Y.-C. Lai, U. Feudel, C. Grebogi, Phys. Rev. E 54 (1996) 6070.
[22] A. Krawiecki, A. Sukiennicki, Chaos Solitons Fractals 11 (2000) 1445.

[23] J.M. González-Miranda, Phys. Rev. E 53 (1996) R5;
J.M. González-Miranda, Phys. Rev. E 57 (1998) 7321.
[24] M.A. Matías, J. Güémez, C. Martín, Phys. Lett. A 226 (1997) 264.
[25] H.D.I. Abarbanel, N.F. Rulkov, M.M. Sushchik, Phys. Rev. E 53 (1996) 4528.