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THE MANY FACES OF SYNCHRONIZATION OF NETWORKS

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Currently, synchronizability of networks is mainly studied in terms of the eigen ratio of the coupling matrix, which is a pure property of network topology. In this work, we clarify that although the eigen ratio is relevant to the *possible range of coupling strength* for achieving synchronization, it cannot fully determine the latter. The magnitude of the eigenvalues also plays a decisive role. We emphasize that synchronizability of networks is inherently related to the local dynamics on networks. It is not appropriate to discuss synchronizability of networks without considering the specific dynamics on them. For three typical types of local dynamics, we discuss the implication of synchronizability of networks.

Keywords: Synchronizability; complex networks; master stability function.

Synchronization of networks inherently involves two factors: the network topology and the local dynamics. The interplay between the network topology and the dynamics is the central issue in the investigation of synchronization of complex networks. Recently there have been many important works contributed in this area which have greatly enhanced our understanding.[1–12] In these studies, however, the synchronizability of a network is represented by a pure quantity of network topology, i.e. the eigen ratio $r = \lambda_N/\lambda_2$, where λ_N and λ_2 are the largest and first nonvanishing eigenvalues of the coupling Laplacian matrix, respectively. What is the exact meaning of synchronizability represented by the eigen ratio? Is it appropriate for us to discuss synchronizability of a network without mentioning any dynamics on it? These questions have not been clearly addressed in existing publications. In this paper, we study the synchronizability of a network for three typical types of master stability functions (MSFs), which are determined by the local dynamics. We clarify the meaning of synchronizability of a network for each situation. We emphasize that in strict sense it is impossible for us to discuss synchronizability of a network without considering the specific dynamics on it.

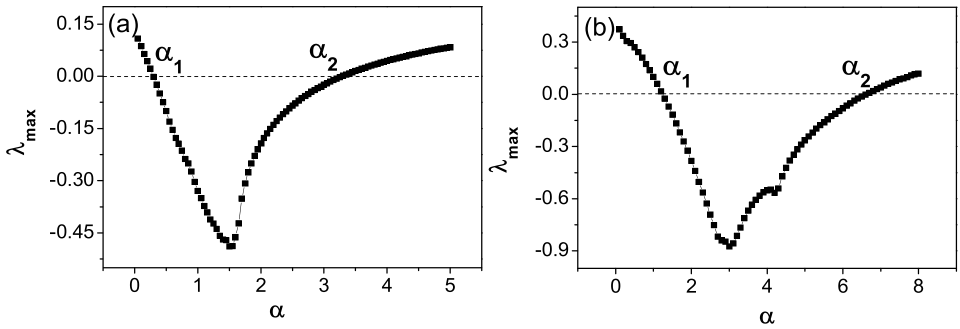

Fig. 1. The first type of master stability functions. (a) The local dynamics is the Rossler system: $\dot{x} = -0.97y - z,\ \dot{y} = 0.97x + 0.25y,\ \dot{z} = 0.4 + z(x - 0.85)$, the coupling is in the equation of x via variable x. (b) The local dynamics is the Lorenz system: $\dot{x} = 10(y-z),\ \dot{y} = 23x-y-xz,\ \dot{z} = xy-z$, the coupling is in the equation of z via variable z.

Let us first recall the theory of master stability function. In 1998,[13] Pecora and Carroll studied the following coupled systems:

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) - \varepsilon \sum_j g_{ij}\mathbf{H}(\mathbf{x}_j)\,, \tag{1}$$

where $\mathbf{x}_i$ denotes the dynamical variable on node i, $\mathbf{F}(\mathbf{x}_i)$ is the local dynamics of node i, ε is the coupling strength, g_{ij} is the element of coupling matrix G representing the network topology, and $\mathbf{H}$ is the coupling function for each node. Under several assumptions, i.e.

(1) the coupled oscillators are all identical,
(2) the coupling functions are all the same, and
(3) the synchronization manifold is an invariant manifold, it is shown that the stability of synchronization manifold of system (1) is determined by the following N variational equations:

$$\dot{\xi}_k = [D\mathbf{F} - \varepsilon\lambda_k D\mathbf{H}]\xi_k\,, \tag{2}$$

where λ_k is the eigenvalue of G, $k = 1, 2, \ldots, N$. When $k = 1$, the variational equation corresponds to the synchronization manifold. All the other $(N-1)$ equations correspond to the transverse modes. If we define a normalized copuling strength $\alpha = \varepsilon\lambda$, we obtain a generic variational equation:

$$\dot{\zeta} = [D\mathbf{F} - \alpha D\mathbf{H}]\zeta\,. \tag{3}$$

The maximum Lyapunov exponents $\lambda_{\max}$ of Eq. (3) is a function of α, which is called a master stability function (MSF). Given a coupling strength ε, if the values of MSF at the following $(N-1)$ points, i.e. $\alpha_k = \varepsilon\lambda_k$, $k = 2, \ldots, N$, are all negative, the synchronization manifold of system (1) is stable.

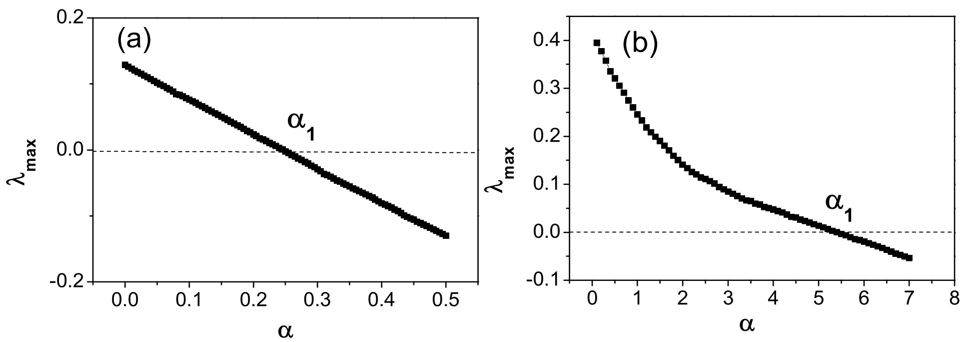

Fig. 2. The second type of master stability functions. (a) The local dynamics is the same as in Fig. 1(a), but the coupling is in the equation of y via variable y. (b) The local dynamics is the same as in Fig. 1(b), but the coupling is in the equation of x via variable x.

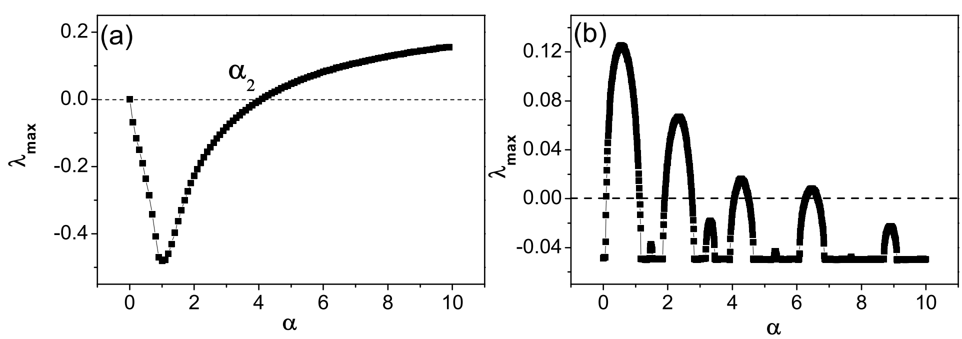

Fig. 3. The third type of master stability functions. (a) The local dynamics is the Rossler system: $\dot{x} = y - z$, $\dot{y} = x + 0.2y$, $\dot{z} = 0.2 + z(x - 2.5)$, the coupling is in the equation of x via variable x. (b) The local dynamics is Duffing system: $\dot{x} = y$, $\dot{y} = -x^3 - 0.1y + 7\sin(t)$, the coupling is in the equation of y via variable x.

According to Eq. (3), MSF is determined by both the local dynamics and the coupling function. In Figs. 1–3, three types of typical MSFs are illustrated, corresponding to different local dynamics and different coupling functions. Without losing generality, in this paper we limit α to be real. Figure 1 is the most typical well-shaped MSF. It has a negative interval between α_1 and α_2, which physically corresponds to the long wave bifurcation point and the short wave bifurcation point, respectively. In previous studies,[1–12] authors explicitly or implicitly assumed that the local dynamics corresponds to this type of MSF. Usually, the coupling matrix G is a symmetric Laplacian. In this case, all eigenvalues of G are real and nonnegative. From small to large, they can be ordered as $\lambda_1 = 0 \leq \lambda_2 \leq \cdots \leq \lambda_N$. The smallest eigenvalue is always zero because all the row sums of L are zero. According to MSF theory, the synchronization manifold is stable if

$$\alpha_1 < \varepsilon\lambda_2 \leq \cdots \leq \varepsilon\lambda_N < \alpha_2 . \tag{4}$$

From Eq. (4), a network is synchronizable if

$$r = \lambda_N/\lambda_2 < \alpha_2/\alpha_1 = \beta\,. \tag{5}$$

In the first paper where the eigen ratio r is defined,[14] it is only pointed out that the eigen ratio condition is related to the synchronization of a network. Later, however, in subsequent studies,[1–12] the eigen ratio r is used as the only measure characterizing the synchronizability of a network. It is believed that the smaller the r, the better the synchronizability of a network, and vice versa. Nevertheless, what is the exact implication of synchronizability related to the eigen ratio? This point is ambiguous from the very beginning. In many works, it is regarded that the compact distribution of eigenvalues, i.e. the small eigen ratio r, implies that the coupled system can be synchronized in a *larger possible range of coupling strength*. However, careful examination reveals this statement to be incorrect. From Eq. (4), we can obtain the result that the coupling strength for achieving synchronization must satisfy $\alpha_1/\lambda_2 < \varepsilon < \alpha_2/\lambda_N$. Then the *possible range of coupling strength* for successful synchronization is

$$\begin{aligned}\Delta\varepsilon &= \frac{\alpha_2}{\lambda_N} - \frac{\alpha_1}{\lambda_2} \\ &= \frac{\alpha_2 - r\alpha_1}{\lambda_N}, \quad \text{or} \quad \frac{\alpha_2/r - \alpha_1}{\lambda_2}\,.\end{aligned} \tag{6}$$

From this formula, it is clear that although $\Delta\varepsilon$ is negatively correlated to the eigen ratio r, the eigen ratio itself cannot fully determine $\Delta\varepsilon$. $\Delta\varepsilon$ requires both the eigen ratio r and the magnitude of one eigenvalue, either λ_1 or λ_N, to determine. For example, we have two networks G^1 and G^2. In this paper, we use superscripts to distinguish networks. If $r^1 < r^2$, this does not necessarily mean $\Delta\varepsilon^1 > \Delta\varepsilon^2$. As long as $\lambda_N^1 > (\alpha_2 - r^1\alpha_1)\lambda_N^2/(\alpha_2 - r^2\alpha_1)$, we can have $\Delta\varepsilon^1 < \Delta\varepsilon^2$ according to Eq. (6). In Ref. 8, it is claimed that a network with $0 = \lambda_1 < \lambda_2 = \cdots = \lambda_N$ has the *widest possible range of coupling strength* to achieve synchronization. However, according to the above analysis, this statement is not accurate in strict sense. If two networks G^1 and G^2 have the same eigen ratios, i.e. $r^1 = r^2$, the network with smaller λ_2 or λ_N will have a larger range of coupling strength for achieving synchronization.

In the above, we have shown that the eigen ratio, which characterizes the network topology, is relevant to, but cannot fully determine the *possible range of coupling strength* for achieving synchronization, let alone represent the full synchronizability of a network. Actually, the *possible range of coupling strength* for achieving synchronization is not the only implication of synchronizability of a network. Synchronizability of a network has many faces. For example, suppose we have two networks G^1 and G^2, and the local dynamics have MSF like the type in Fig. 1. Let us further assume: $\lambda_2^1 \le \cdots \le \lambda_N^1 < \alpha_1$, and $\alpha_2 < \lambda_2^2 \le \cdots \le \lambda_N^2$. To achieve synchronization, for network G_1, we need a coupling strength $\varepsilon^1 > 1$; and for network G_2, we need a coupling strength $\varepsilon^2 < 1$. Since $\varepsilon^2 < \varepsilon^1$, network G^2 is easier to synchronize than network G^1. From the viewpoint of the *minimal coupling strength* to achieve synchronization, synchronizability of G^2 is better than G^1.

However, if eigen ratio $r^1 < r^2$, according to Eq. (6), we have $\Delta\varepsilon^1 > \Delta\varepsilon^2$. Therefore, from the viewpoint of the *possible range of coupling strength* for achieving synchronization, synchronizability of G_1 is better than G_2.

We emphasize that, in principle, synchronizability of a network should be discussed together with the local dynamics. It is the local dynamics that determines the type of MSF which is relevant to the synchronizability of the network. Our previous discussions about synchronizability of a network is only valid when the local dynamics has MSF as shown in Fig. 1. In fact, almost all the existing works studying synchronizability of a network have implicitly assumed this condition. However, MSF shown in Fig. 1 is only one common type of local dynamics. Very often, we have local dynamics whose MSFs are shown in Figs. 2 and 3. For local dynamics of the type in Fig. 2, the coupling strength for achieving synchronization has no upper bound. Thus it does not make sense to relate synchronizability of a network with the *possible range of coupling strength* for achieving synchronization. Instead, the synchronizability of the network can be naturally characterized by the *minimal coupling strength* to achieve synchronization which is only determined by λ_2 i.e. $\varepsilon_m = \alpha_1/\lambda_2$. If the local dynamics is in the limit cycle regime, usually we have MSFs as shown in Fig. 3. For Fig. 3(a), the *minimal coupling strength* does not make sense because an arbitrarily small coupling strength can synchronize the whole network. In this case, the *possible range of coupling strength* for achieving synchronization is only determined by λ_N (since $\alpha_1 = 0$), i.e. $\Delta\varepsilon = \alpha_2/\lambda_N$, which can be used to characterize the synchronizability of the network. For Fig. 3(b), the situation is very complicated due to the multiple synchronous regions of the MSF. In this case, it is almost impossible to define synchronizability of the network in terms of eigenvalues and eigen ratio.

In summary, we have pointed out that the eigen ratio of a network is not a synonym of synchronizability of a network, though it has a close relation to it. The synchronizability of a network has many faces and should be studied together with specific local dynamics. Depending on different local dynamics, the synchronizability of a network has different implications and should be characterized by different criteria.

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