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Variation of critical point of aging transition in a networked oscillators system

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In this work, we study the variation of critical point in aging transition in a networked system consisting of both active and inactive oscillators. By theoretical analysis and numerical simulations, we show that the critical point of aging transition actually is determined by the (normalized) cross links between active and inactive subpopulations of oscillators. This reveals how specific configuration of active and inactive oscillators in the network can lead to the variation of transition point. In particular, we investigate how different strategies of targeted inactivation influence the transition point based on the theory. Our results theoretically explain why the low-degree nodes are crucial regarding dynamical robustness in such systems. *© 2014 AIP Publishing LLC.*
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The robustness of a networked dynamical system, i.e., the ability to maintain basic structure and function under attacks or dysfunctions, is vital in practice. It has been shown that the huge blackouts, which inevitably cause tremendous economic loss, are related to the cascading failure of power-grids.[1] Such structural robustness involving network connectedness has been intensively investigated in previous works.[1–3] On the other hand, networked systems typically carry dynamics, for example, the circadian rhythms of mammals, the synchronization of cardiac cells, etc. Of equal importance is the dynamical robustness, i.e., the ability of a networked system to maintain its normal dynamical activity when the topology or the local dynamics are under perturbations.[4–7] In this aspect, the study on aging transition in networked oscillators is helpful for better understanding. Here, we report the variation of critical point in aging transition and offer explanation via theoretical analysis.

I. INTRODUCTION

Complex systems in nature and human society usually comprise a large number of interacting individual elements, such as synchronizing fireflies,[8] neurons in human brain,[9] cardiac pacemaker cells,[10] power grids,[11,12] and Josephson junction arrays,[13] just to name a few. These dynamical systems can be naturally modeled by networked oscillators.[14] One important issue of interest is the collective behaviors in such systems, e.g., synchronization and amplitude death, etc.,[14] which are closely related to the robustness of dynamical systems.[1–4,6,15]

In Ref. 4, Daido and Nakanishi investigated a networked dynamical system which simultaneously consists of active and inactive oscillators. It is found that with the increase of the ratio of inactive oscillators, which we refer to as inactivation, the macroscopic dynamical activity of the system, measured by a global order parameter of amplitude, decreases until it totally vanishes at certain critical point. This phenomenon is termed aging transition. It is shown that in aging transition, the critical point can be used to characterize the dynamical robustness of the networked system.[4,6,15]

In this study, we find an interesting phenomenon, i.e., there always exists variation of the critical point. In fact, for transition phenomena in networked systems, it turns out that the critical point often changes within certain range. In most cases, this is attributed to the small deviations caused by different numerical realizations, including initial conditions, and/or network topologies, etc. However, we found that the critical point of aging transition varies even when the network is totally homogeneous like a regular one, and, in particular, the fluctuation could be large enough to significantly change the robustness property of the system. Therefore, there must be a dynamical mechanism, though ignored in previous studies, which actually underlies this variation of critical point. In this paper, we carried out theoretical analysis and numerical experiments to understand this phenomenon.

Specifically, by applying mean-field approximation and linear stability analysis, we show that the normalized cross links between the subpopulations of active and inactive oscillators play a dominant role for the variation of critical point. Specific inactivation processes lead to different normalized cross links, which, in principle, change the critical point more or less. Based on theoretical analysis, we can explain why the variation in heterogeneous networks is more obvious than that in homogeneous networks for usual random inactivation. In particular, our theory enables us to analyze the dynamical robustness, characterized by the critical point of aging transition, for typical strategies of targeted

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inactivation. We offer an explanation why the low-degree nodes are crucial for dynamical robustness in this system as reported in Ref. 6. These results shed light on how to maintain the stability of networked systems under dynamical perturbations in practice.

The rest of this paper is organized as follows: Sec. II introduces the dynamical model; theoretical analysis and numerical verification are presented in Sec. III; finally, conclusions are given in Sec. IV.

II. MODEL

In this work, we investigate a dynamical model of networked oscillators. It has following features: (1) the dynamical system is described by coupled oscillators; (2) there are two types of oscillators in the system, namely, active and inactive. When isolated, the active oscillators are oscillating (limit cycles) while the inactive oscillators are non-oscillating (fixed points). Note that the inactive oscillators could become oscillating and the active oscillators could become non-oscillating when they are coupled together; (3) the coupling among oscillators forms a network. The general form of this dynamical model can be written as

$$\dot{\mathbf{x}}_j = \mathbf{F}_j(\mathbf{x}_j) + \sigma \sum_{k=1}^{N} c_{jk}(\mathbf{x}_k - \mathbf{x}_j). \tag{1}$$

Here, $j = 1, \cdots, N$ is the index of oscillator (node). $\mathbf{x}$ is the state vector describing the dynamics of oscillators. The first term at the RHS of Eq. (1) describes the local dynamics of an oscillator, and the second term is the interactions among different oscillators via diffusive coupling. σ is the coupling strength. c_{jk} is the element in the adjacent matrix of coupling network, which equals to 1 if nodes j and k are connected, and 0 otherwise. It should be pointed out that similar models have been studied previously. For example, aging transition in such models was investigated in fully coupled network in Ref. 4, and later in regular ring in Refs. 16 and 17. Recently, the study of this model has been extended to various complex topologies.[6,15]

In the present work, we mainly choose the Stuart-Landau (SL) oscillators as the local dynamics, following Refs. 4, 16, and 17. Specifically, the networked SL oscillators can be described by the following coupled ordinary differential equations (ODEs):

$$\dot{z}_j = (\alpha_j + i\Omega_j - |z_j|^2)z_j + \sigma \sum_{k=1}^{N} c_{jk}(z_k - z_j), \tag{2}$$

where z_j and Ω_j are the complex amplitude and the inherent frequency of the jth SL oscillator, respectively. α_j is the control parameter denoting the distance from the Hopf bifurcation point. When $\alpha_j > 0$, the oscillator is a limit cycle with an amplitude $\sqrt{\alpha_j}$. However, it settles down to a fixed point when $\alpha_j < 0$. Thus, the (isolated) oscillator is active when $\alpha_j > 0$ and inactive when $\alpha_j < 0$. In other words, the oscillator will lose its activity as its α value changes from positive to negative. This can be used to model the two distinct dynamical states of oscillators. We define parameter ρ as the proportion of inactive oscillators in the network. Reasonably, the global activity of the networked system can be characterized by the normalized order parameter Q, defined as $Q = \langle Q(t)\rangle = \langle |Z(\rho)|/|Z(0)|\rangle$ with $Z = N^{-1}\sum_{j=1}^{N} z_j$. Here, the bracket means the long time average after transient. As ρ increases to a critical value ρ_c, the networked system will gradually lose its global activity. Q continually decreases until finally $Q = 0$ at ρ_c, i.e., an aging transition occurs as shown in Fig. 1. Because the ratio ρ_c is the largest ratio with which the dynamical system can maintain global activity, it can be reasonably used as a quantitative measure to characterize the dynamical robustness this networked system.[4,6,15–17] The larger the ρ_c, the better the dynamical robustness.

For simplicity, we set $\Omega_1 = \Omega_2 = \cdots = \Omega_N = \Omega$ throughout this paper. For parameter α, we set $\alpha_j = a > 0$ for all active oscillators, and $\alpha = -b < 0$ for all inactive oscillators. This means they are identical within their own subgroups. Of course, α_j can obey other kinds of distributions rather than binary case. In fact, we have examined several typical distributions of α_j such as Gaussian and uniform. It is found that the transition point is actually determined by parameter ρ, i.e., the ratio of the inactive oscillators in the system, regardless of the specific distributions of α_j, as shown in Fig. 1. Without loss of generality, in the following we only consider the situation where α takes binary values.

Besides SL oscillators, we also consider the following networked Rössler oscillators in this paper:[4]

$$\begin{aligned}
\dot{x}_j &= -y_j - z_j + \sigma \sum_{k=1}^{N} c_{jk}(x_k - x_j),\\
\dot{y}_j &= x_j + g_j y_j + \sigma \sum_{k=1}^{N} c_{jk}(y_k - y_j),\\
\dot{z}_j &= d_j + z_j(x_j - e_j) + \sigma \sum_{k=1}^{N} c_{jk}(z_k - z_j),
\end{aligned} \tag{3}$$

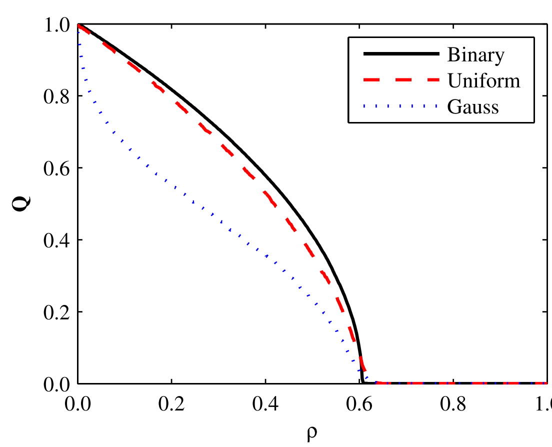

FIG. 1. The aging transitions of three typical α distributions on a small-world network. For binary distribution, $a = 1$, $b = 1$; for uniform distribution and Gaussian distribution, the mean α varies from 3 to -3. The uniformly distributed α varies in a range of length 6, and α of Gaussian distribution has the standard variance 1. System size $N = 500$, the mean degree $\langle K\rangle = 50$, $\sigma = 0.1$, and $\Omega = 3$. The results are averaged over 100 times, and the same results have been obtained in scale-free networks and regular lattices.

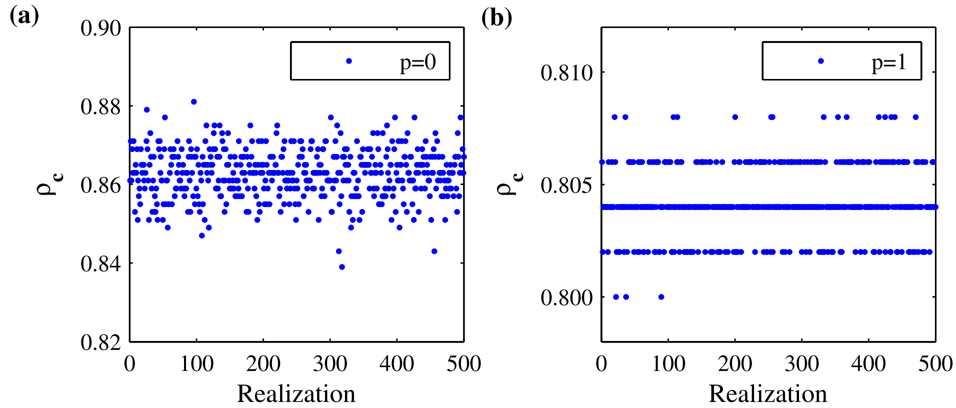

FIG. 2. Fluctuations of the transition points ρ_c in networked system of SL oscillators with random inactivation. To study the influence of network topology on transition point, a simple but effective method is used to continuously change the homogeneity/heterogeneity of a network. The detail of the algorithm is explained in the Appendix. Here, p is the parameter controlling the homogeneity/heterogeneity of the network. As p goes from 0 to 1, the network continuously changes from a heterogeneous scale-free network to a homogeneous network with Poisson degree distribution. $N = 500$, $\langle K \rangle = 50$, $\sigma = 0.1$, $a = 2$, $b = 1$, and $\Omega = 3$.

where x, y, z are the state variables of Rössler oscillator, and g, d, e are parameters. In this system, there are also two types of oscillators by choosing different parameters, i.e., $g = d = 0.2$, $e = 1$ for active oscillators (limit cycles) and $g = d = -0.2$, $e = 2.5$ for inactive oscillators (fixed points). To quantify the global activity, the order parameter can be defined as $Q = \langle Q(t) \rangle = \sqrt{\langle (\mathbf{x}_c - \langle \mathbf{x}_c \rangle)^2 \rangle}$, where $\mathbf{x}_c = \sum_{j=1}^{N} (x_j, y_j, z_j)/N$ is the centroid and the bracket means a long time average after transient. Essentially, the definitions of order parameter Q for SL oscillators and Rössler oscillators are the same, i.e., they represent the amplitude of macroscopic oscillation of the networked system. Throughout this work, numerical integrations are obtained by the fourth order Runge-Kutta method with time step 0.01 with random initial conditions, i.e., uniformly distributed in the range [0, 1]. The order parameter Q is calculated by averaging on the time interval [100, 200].

III. ANALYSIS AND RESULTS

In our study, we find an interesting phenomenon, i.e., even for fixed network topology and coupling strength, the critical point ρ_c varies within certain range, depending on specific inactivation process, i.e., the temporal sequence of converting active oscillators into inactive ones, or more precisely, the configuration of active and inactive oscillators in the network. Typical examples are shown in Fig. 2. We observe variation of critical points ρ_c in both heterogeneous and homogeneous networks. To quantify this feature, in Fig. 3, we plot the variance of ρ_c when the network topology continuously changes from heterogeneous to homogeneous. It is found that for random inactivation this fluctuation is more obvious in heterogeneous networks than in homogeneous networks (with Poisson degree distributions). In the following, we present both theoretical analysis and numerical verification to understand this phenomenon.

A. Linear stability analysis

We start from Eq. (2). When $\rho = 0$, the system contains only identical active oscillators. In this case, it will easily achieve global synchronization as the coupling strength increases. When we randomly choose some active nodes and change them into inactive states, the proportion of inactive nodes ρ becomes greater than 0, and the system contains both active and inactive oscillators. Such a process is called random inactivation. Throughout this paper, it is adopted by default unless otherwise stated.

Usually, the non-trivial aging transition would only occur when the coupling strength is large enough,[4] so before the transition all the active oscillators already well synchronize into a cluster, and so do the inactive ones. We use S_A and S_I to denote the active and inactive subpopulations, respectively. Numerically, the synchronization is found to be approximately complete, and one example has been illustrated in Fig. 4. Following Ref. 4, approximately we can use one single complex variable A to represent the state of all active oscillators, and I to represent the state of all inactive oscillators, respectively. Then the original high-dimensional dynamical system, i.e., Eq. (2), can be essentially reduced as

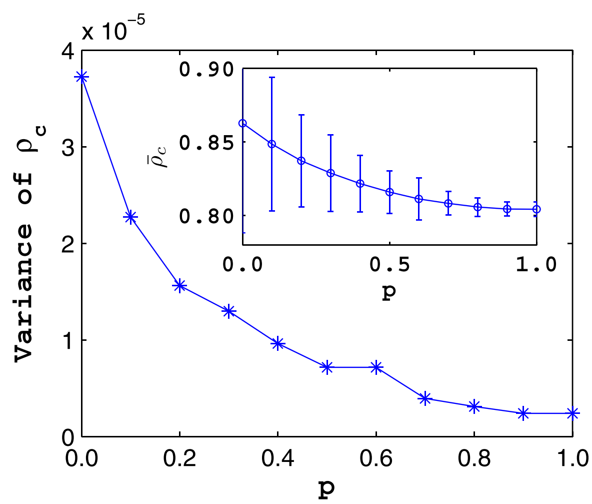

FIG. 3. Variance of ρ_c versus parameter p in networked system of SL oscillators. Inset is the mean value of ρ_c for 500 random inactivation processes, while the network topology is fixed. $N = 500$, $\sigma = 0.1$, $a = 2$, $b = 1$, and $\Omega = 3$.

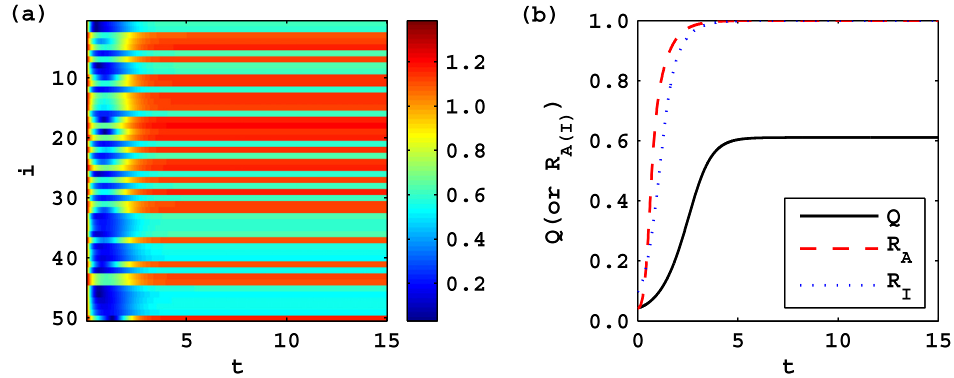

FIG. 4. (a) The evolution of amplitude $|z_i(t)|$ in a homogeneous network ($p=1$) with $N=50$. $\rho=0.5<\rho_c$, $\langle K\rangle=10$, $\sigma=0.1$, $a=2$, $b=1$, and $\Omega=3$. The colour bar denotes the magnitude of amplitude. (b) The global order parameter ($Q(t)$), and the order parameters of active ($R_A(t)$) and inactive ($R_I(t)$) subpopulations in a heterogeneous network ($p=0$). For $R_A(t)$ and $R_I(t)$, the definitions are the same as $Q(t)$ (see the text for the definition of $Q(t)$), but the summation is taken over the subpopulations of active and inactive oscillators, respectively. It is shown that the oscillators inside either subpopulation achieve synchronization ($R_A=R_I=1$), but the whole system does not achieve global synchronization ($Q<1$). Here, $N=500$, $\langle K\rangle=50$, $\sigma=0.1$, $a=2$, $b=1$, and $\Omega=3$. For both (a) and (b), SL oscillators are used.

$$\dot{A}=(a+i\Omega-|A|^2)A+\sigma\mu_jK_j(I-A), \tag{4}$$

$$\dot{I}=(-b+i\Omega-|I|^2)I+\sigma(1-\mu_j)K_j(A-I). \tag{5}$$

Here, K_j means the degree of oscillator j, and μ_j is the proportion of inactive oscillator among the neighbors of oscillator j. Summing the equations of all active and inactive oscillators, we obtain

$$\dot{A}=(a+i\Omega-|A|^2)A+\frac{\sigma(I-A)}{(1-\rho)N}\sum_{j\in S_A}\mu_jK_j, \tag{6}$$

$$\dot{I}=(-b+i\Omega-|I|^2)I+\frac{\sigma(A-I)}{\rho N}\sum_{j\in S_I}(1-\mu_j)K_j. \tag{7}$$

In the above equations, the summing terms at the RHS, i.e., $\sum_{j\in S_A}\mu_jK_j$ and $\sum_{j\in S_I}(1-\mu_j)K_j$, both represent the total number of cross links, denoted by L, i.e., the links between subpopulations of active oscillators and inactive ones. Actually they are the same in different expressions. To avoid the influence of network size, we define the normalized cross links as $\lambda=L/N=\sum_{j\in S_A}\mu_jK_j/N=\sum_{j\in S_I}(1-\mu_j)K_j/N$. Then Eqs. (6) and (7) become

$$\dot{A}=(a+i\Omega-|A|^2)A+\frac{\sigma\lambda(I-A)}{(1-\rho)}, \tag{8}$$

$$\dot{I}=(-b+i\Omega-|I|^2)I+\frac{\sigma\lambda(A-I)}{\rho}. \tag{9}$$

From these equations, we find that the network topology actually affects the global dynamics through parameter λ. Now, we analytically study how parameter λ can lead to variation of the critical point in aging transition, thus affecting the dynamical robustness of such system. With the increase of control parameter ρ, the dynamics of the networked system will gradually lose global activity, i.e., $Q\to 0$ when $\rho\to\rho_c$. At the transition point, the networked system loses its stability and in the mean time the trivial fixed point $z_0=(A,I)=(0,0)$ becomes stable. By a linear stability analysis, we obtain the critical point ρ_c as

$$\rho_c=\frac{ab-\sigma(a+b)\lambda+[ab+\sigma(a+b)\lambda]\sqrt{1-\beta}}{2ab}, \tag{10}$$

with $\beta=4\sigma ab^2\lambda/[ab+\sigma(a+b)\lambda]^2$. For typical dense complex network, $\lambda\gg 1$. Therefore, β is a small quantity. We apply Taylor expansion to $\sqrt{1-\beta}$ in Eq. (10) and keep the linear term. It finally becomes

$$\rho_c=1-\frac{\sigma b}{ab/\lambda+\sigma(a+b)}. \tag{11}$$

B. Analysis to the variation of critical point

Equation (11) shows how ρ_c, i.e., the transition point, depends on λ, i.e., the number of normalized cross links, given that the parameters of the local dynamics and the coupling strength are fixed. Apparently, there is a maximal value $\rho_c^{max}=1$ when $\lambda\to 0$. With the increase of λ, ρ_c will monotonically decrease. When $\lambda\to\infty$, ρ_c approaches the minima $\rho_c^{min}=\frac{a}{a+b}$, as shown in Fig. 5. Physically, this can be understood. Since there are both active and inactive local states in the network, the interaction or influence between the two subpopulations are crucial for the global activity. The normalized cross links λ just characterize this interaction. For example, when it is large, there exists strong interaction between the two subpopulations of active and inactive oscillators. As a result, a small critical value ρ_c can be expected in aging transition.

Now we explain why the critical point ρ_c varies even when the network topology is fixed. Let us analyze the phase diagram on the parameter panel of ρ-λ. As shown in Fig. 5(a), Eq. (10) defines the curve of bifurcation, i.e., the boundary of two areas corresponding to distinct dynamical states of the system. In the upper right area, the networked system is in the quenching state losing global activity; while in the bottom left area, it is in active state oscillating to some extent. The active state loses its stability when the system passes through the curve Eq. (10). For a specific inactivation process, λ is a function of ρ. Actually, $\lambda(\rho)$ is a unimodal

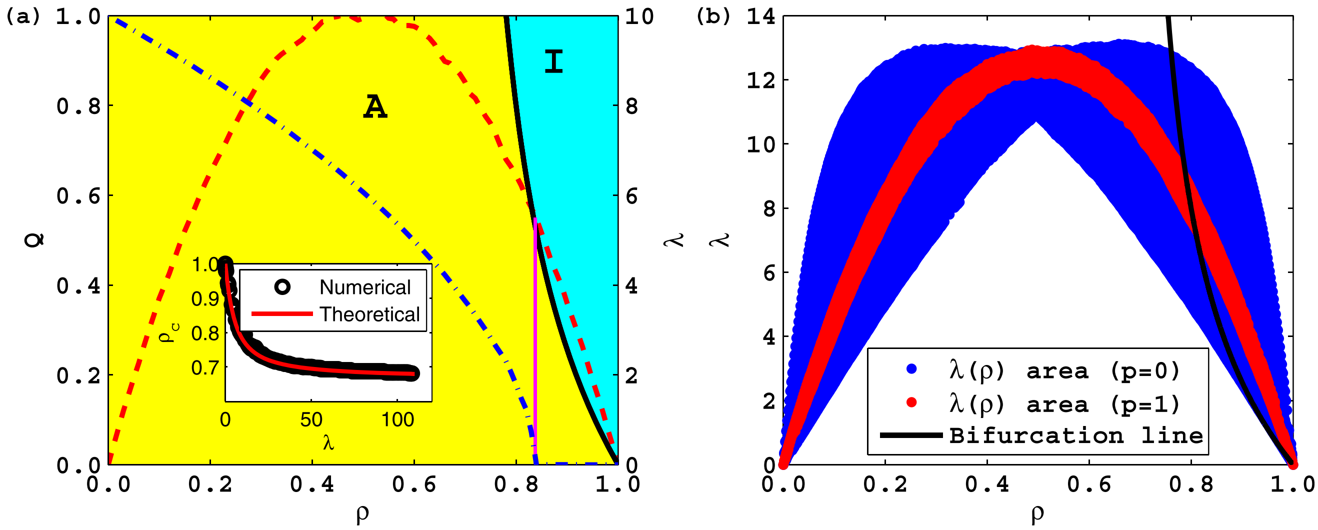

FIG. 5. The ρ-λ parameter panel for networked SL oscillators. (a) The solid black line is the bifurcation curve defined by Eq. (10); the red dashed line is the curve $\lambda(\rho)$ for a specific inactivation process ($p=1$); and the blue dotted dashed line corresponds to the order parameter. Inset compares the theoretical result of ρ_c with that of numerical experiments. (b) The solid black line is the same as in (a); the blue area shows the variation area of $\lambda(\rho)$ curve in heterogeneous network ($p=0$), while the red area corresponds to that in homogeneous network ($p=1$). Other parameters are the same in both (a) and (b): $N=500$, $\langle K\rangle=50$, $\sigma=0.1$, $a=2$, $b=1$, and $\Omega=3$.

function satisfying $\lambda(0)=\lambda(1)=0$, as shown in Fig. 5(a). The curve $\lambda(\rho)$ intersects the bifurcation curve Eq. (10), and the crosspoint determines the critical point ρ_c. The key point here is that $\lambda(\rho)$ is actually a multi-valued function of ρ. Given ρ, even when the network topology is fixed, λ could be different depending on the specific configuration of active and inactive oscillators in the network. Therefore, for different inactivation processes with the same ρ, we will get different curves $\lambda(\rho)$, which intersect with the bifurcation curve defined by Eq. (10) at different points. This is the reason why we always observe the variation of critical point in numerical simulations.

Naturally, one may want to know which inactivation process will give the maximal ρ_c or the minimal ρ_c, i.e., the variation scope of ρ_c, given the network topology. Interestingly, this problem can be mapped into the ground state problem of anti-ferromagnetic Ising model or the MAX-CUT problem in combinatorial optimization.[18–20] It has been proven that the solution of these problems in general networks is a NP-Complete problem, i.e., we cannot find an effective algorithm to determinate it in polynomial time.[18] Since analytical method is not available to get the variation scope of ρ_c, we turn to numerical study. To this end, we numerically plot the possible curves $\lambda(\rho)$ on the parameter panel of ρ-λ. The two crosspoints that define the largest range on the axis of ρ roughly gives the fluctuation range of ρ_c. As shown in Fig. 5(b), it is found that $\lambda(\rho)$ for a heterogeneous network has a much larger variation area compared to a homogeneous network. Therefore, more obvious variation of ρ_c is observed in the former case. Physically, this result can be heuristically understood. We know that the variation of ρ_c is caused by the multi-valued $\lambda(\rho)$, which depends on specific inactivation processes. Compared with homogeneous networks, there are huge differences among the node degrees in the heterogeneous network, so it is not difficult to imagine that λ would vary more in the latter case when random inactivation process is applied.

One may think that the variation is induced by the non-uniform degree distributions of networks. It may vanish when the network is exactly homogeneous, i.e., every node has the same degree. To verify whether it is correct or not, let us examine an example in the following. Consider a regular ring network where all oscillators have the same degree K, as schematically shown in Fig. 6(a). Then let us do the inactivation in two different ways. Due to the simplicity of the topology, the normalized cross links $\lambda(\rho)$ in these cases can be easily estimated. (1) Indexed inactivation: Flip the oscillator from active to inactive in sequence, i.e., $1\rightarrow 2 \rightarrow 3 \rightarrow \cdots$ as in the schematic example. Apart from the short periods of starting and ending stages, $\lambda_1 = K(K+2)/4N$. (2) Random inactivation as we have discussed above: $\lambda_2 \approx \rho(1-\rho)K$. The above analysis is illustrated in Fig. 6(b). It is seen that these two $\lambda(\rho)$ curves in Fig. 6(b) intersect with the bifurcation line at different points, giving different ρ_c for each case. The transition points at the same ρ but with different inactivation processes vary even on regular ring network. Based on our theory, we understand that this is due to the difference of normalized cross links induced by different configurations of oscillators in the network. The above analysis has been verified by our numerical simulations as shown in Fig. 6(a). Here, we point out that our theoretical predictions are only qualitatively consistent with the numerical results because synchronization in case 1 is approximate.

The above example emphasizes the importance of the configuration of oscillators in networks. As one application of the theory, we now consider the case of targeted inactivation rather than random activation in network. This situation is closely related to the dynamical robustness of networked systems. In Ref. 6, it has been reported that under targeted inactivation the dynamical robustness of the system,

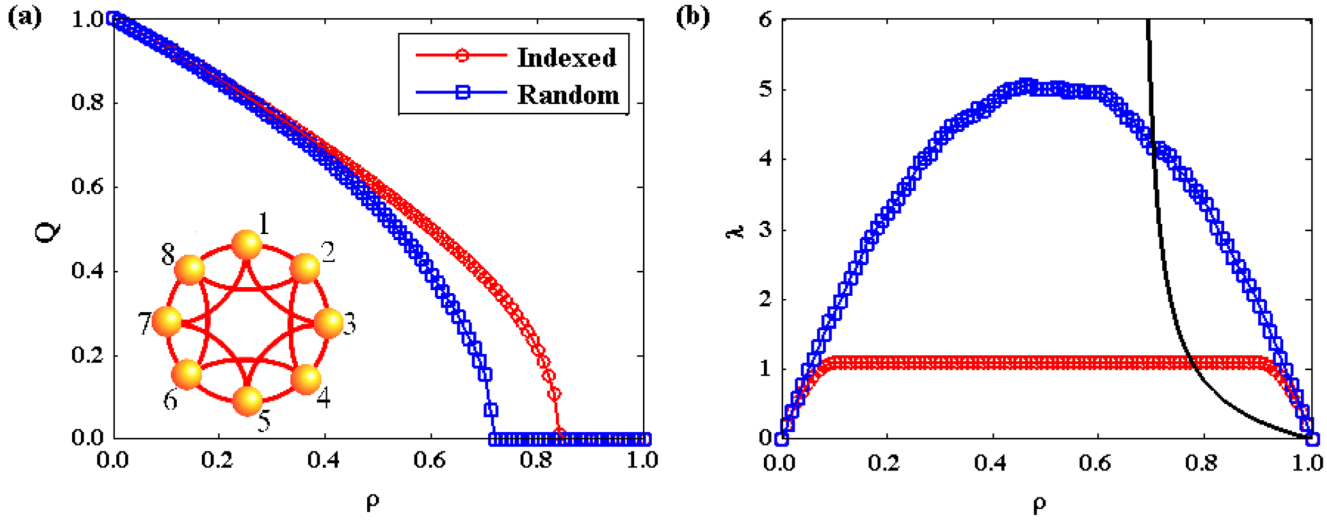

FIG. 6. (a) Fluctuation of transition points in strictly regular ring network under two different inactivation strategies. (b) $\lambda(\rho)$ curves corresponding to (a) intersect with the bifurcation line (solid black line). The legend in (b) is the same as in (a). Networked SL oscillators are used with parameters $\sigma = 1.0$, $K = 20$, $N = 100$, $a = 2$, $b = 1$, and $\Omega = 3$.

characterized by the critical point in aging transition, depends more on the low-degree nodes rather than the hubs. This important finding reveals the crucial role of the low-degree nodes in the context of dynamical robustness. Based on our analytical treatment, we can provide an explanation here. In our study, we apply three typical strategies of inactivation: (1) Inactivation goes from the node with the maximal degree to the one with the minimal degree; (2) Inactivation takes the inverse order of (1); (3) Random inactivation. For all inactivation strategies, we increase ρ from 0 to 1. Physically, this means that all the active oscillators in the network will be gradually changed into inactive ones. Numerical results for both networked SL oscillators and networked Rössler oscillators are shown in Fig. 7. In both situations, ρ_c under strategy 1 is always greater than that under strategy 2, regardless of the heterogeneity/homogeneity of the network topology. Because larger ρ_c implies good dynamical robustness, this result, counterintuitive somehow, shows that the networked system is more robust when the targeted inactivation starts from the hubs rather than the low-degree nodes.

To understand the result, we plot the curves $\lambda(\rho)$ on the λ-ρ parameter plane. As shown in Fig. 8, for $\rho = 0$ all oscillators are active, while for $\rho = 1$ all oscillators are inactive. In both cases, $\lambda = 0$. Apart from these two points, all other λ should be greater than 0. For random inactivation, i.e., strategy 3, it can be expected that λ has the maxima approximately at $\rho = 0.5$. Thus, in this case the curve $\lambda(\rho)$ is unimodal and approximately symmetric with respect to 0.5. For strategies 1 and 2, the corresponding $\lambda(\rho)$ curves are still unimodal, but not symmetric with respect to 0.5. Interestingly, inactivating all initially active oscillators (ρ goes from 0 to 1) with strategy 1 is just the inverse process of activating all initially inactive oscillators with inverse strategy 2 (ρ goes from 1 to 0), so actually the two $\lambda(\rho)$ curves with strategies 1 and 2 are basically the same if ρ goes from 0 to 1 for the former and from 1 to 0 for the latter. Considering these three inactivation strategies, it is not difficult to figure out that with the increase of ρ, curve 1 increases much faster than curve 2 (and curve 3 is in between) because in strategy 1 the inactivation starts from the hubs, and it rapidly leads to large λ. Similarly, the curves decrease oppositely when ρ approaches 1. Since the bifurcation curve defined by Eq. (10) intersects with the $\lambda(\rho)$ curves at the decreasing stage when ρ is close to 1, we have $\rho_c^1 > \rho_c^3 > \rho_c^2$, as shown in Fig. 8. The above analysis successfully explains why the global dynamics is the most vulnerable when inactivation starts from the low-degree nodes. It should be pointed out that this situation is quite different from the case of structural robustness, where usually the hubs play an important role.

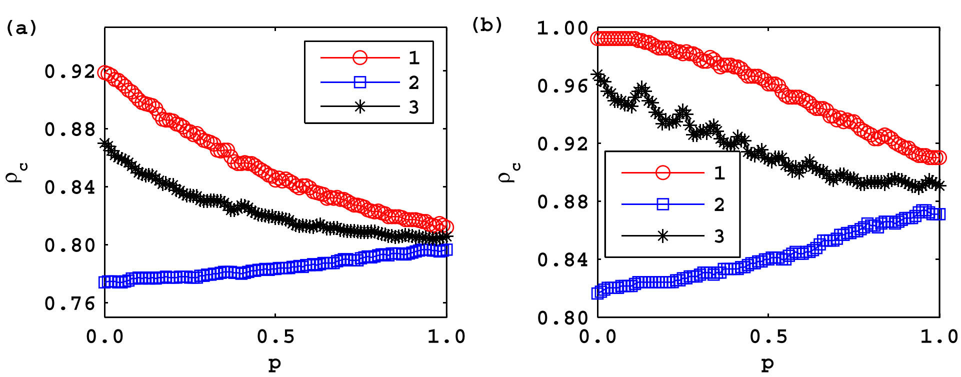

FIG. 7. Dynamical robustness characterized by ρ_c under three typical strategies of inactivation. (a) Networked system of SL oscillators. $\sigma = 0.1$, $a = 2$, $b = 1$, and $\Omega = 3$. (b) Networked system of Rössler oscillators. $g = d = 0.2$, $e = 1$ for active oscillators and $g = d = -0.2$, $e = 2.5$ for inactive ones. $\sigma = 0.002$. Other parameters are the same for (a) and (b): $N = 500$, $\langle K \rangle = 50$.

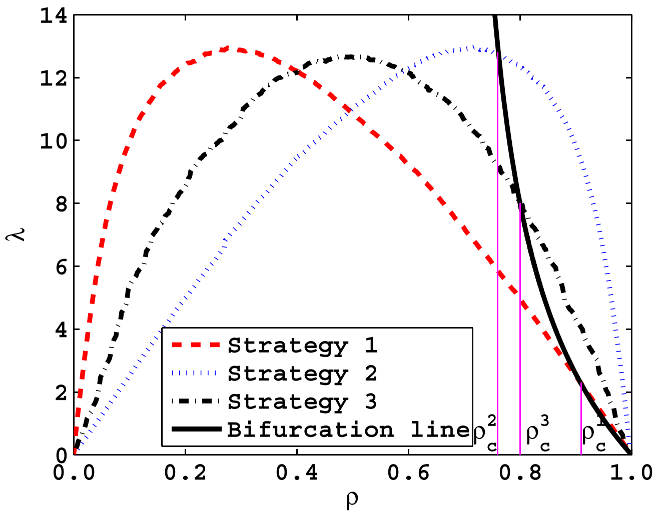

FIG. 8. Identifying the critical points under three typical strategies of inactivation in networked system of SL oscillators. The meanings of curves are the same as in Fig. 5. $p = 1$, and other parameters are the same as in Fig. 7.

C. Random inactivation

In the above analysis, the only limitation for the network topology is that it should be dense enough so that the normalized cross links $\lambda \gg 1$, and there is no requirement for the strategy of inactivation. Thus, the above result holds for general inactivation processes. In fact, in our study, we find for random inactivation, Eqs. (8) and (9) can take another forms, and the theoretical treatment can be simplified.

We notice that in principle for different oscillator j, K_j, and μ_j are not necessarily the same. However, they must satisfy the constraint $\sum_j K_j/N = \langle K\rangle$ and $\sum_j \mu_j/N = \rho$ over all the nodes when the network is fixed. Let $\mu_j = \rho + \xi_j$, where ξ_j is the deviation of μ_j from its mean value. Normally, a node with larger degree K_j would have smaller deviation. For random inactivation in a dense network, the active and inactive oscillators are mixed evenly, it is reasonable to expect that $|\xi_j| \propto 1/K_j$ and μ_j distributes symmetrically around its mean value. And the distributions of μ_j in active and inactive subpopulations will be approximately the same as in the whole system. Thus, we have $\sum_{j\in S_A} \xi_j K_j = \sum_{j\in S_I} \xi_j K_j = 0$. Substitute it and $\mu_j = \rho + \xi_j$ into the expression of λ, we get $\lambda = L/N = \sum_{j\in S_A}(\rho + \xi_j)K_j/N = \frac{\rho}{N}\sum_{j\in S_A} K_j$, and $\lambda = L/N = \sum_{j\in S_I}[1-(\rho+\xi_j)]K_j/N = \frac{(1-\rho)}{N}\sum_{j\in S_I} K_j$. Substitute these relations into Eqs. (8) and (9), we get

$$\dot{A} = (a + i\Omega - |A|^2)A + \sigma\rho K_A(I - A), \tag{12}$$

$$\dot{I} = (-b + i\Omega - |I|^2)I + \sigma(1-\rho)K_I(A - I), \tag{13}$$

where $K_A = \frac{1}{(1-\rho)N}\sum_{j\in S_A} K_j$ and $K_I = \frac{1}{\rho N}\sum_{j\in S_I} K_j$ are the mean degrees of active and inactive subpopulations, respectively. Similarly, by applying linear stability analysis, we can analytically obtain the critical point ρ_c as

$$\rho_c = \frac{ab + \sigma a K_I}{\sigma(aK_I + bK_A)}. \tag{14}$$

From this equation, we can immediately find that the critical point ρ_c are determined by the mean degrees of active and inactive subpopulations in the case of random inactivation. The point is, even though the mean degree of the whole network is fixed, there is still some degree of freedom for K_A and K_I as long as they satisfy the following constraint:

$$\rho K_I + (1-\rho)K_A = \langle K\rangle. \tag{15}$$

On the parameter panel of K_A-K_I, only part area can satisfy this condition, as shown in Fig. 9. In particular, parameters K_A and K_I not only are related to the network topology but also to the specific strategy of inactivation. Usually, each different realization results in different K_A and K_I, causing the variation of ρ_c observed above. We can see in Fig. 9 that our theoretical result is qualitatively consistent with the numerical verifications in both networked SL system and Rössler system. Extensive numerical results have shown that Eq. (14) is valid as long as the mean degree is large enough, e.g., $\langle K\rangle \geq 40$.

Furthermore, a trivial solution always exists for Eq. (15), i.e., $K_A = K_I = \langle K\rangle$. In this circumstance, Eq. (15) degenerates as

$$\rho_c = \frac{a(b + \sigma\langle K\rangle)}{(a+b)\sigma\langle K\rangle}. \tag{16}$$

In a strict sense, this result only holds for very homogeneous network. In this case, the critical point ρ_c only involves the mean degree $\langle K\rangle$, which means that for random

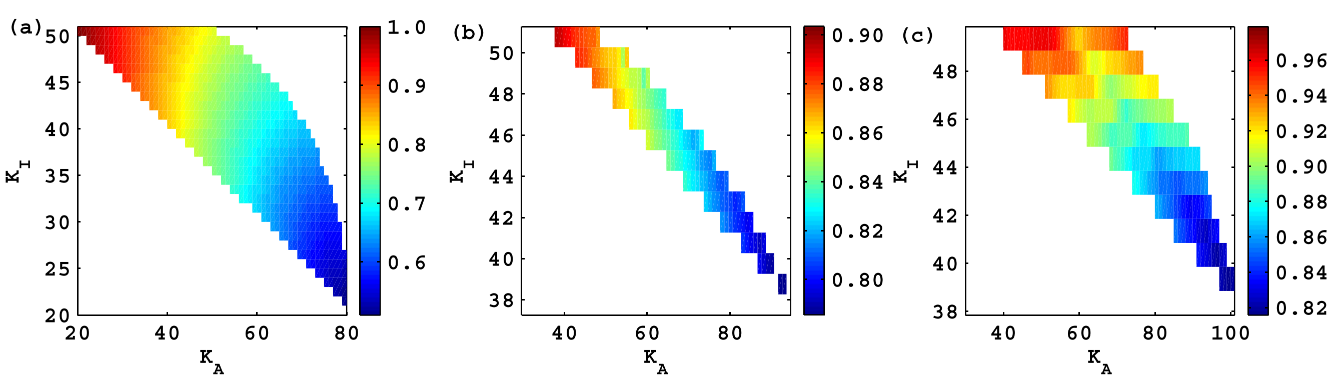

FIG. 9. ρ_c as a function on the panel of K_A-K_I. The white area means that K_A and K_I cannot satisfy the constraint Eq. (15). (a) Theoretical result (SL oscillators) with fixed coupling strength and mean degree. (b) Numerical results for networked SL oscillators. (c) Numerical results for networked Rössler oscillators. $p = 0$, $N = 500$, $\langle K\rangle = 50$. Other parameters are the same as in Fig. 7.

inactivation in homogeneous networks the variation of critical point is almost neglectable. However, as we have shown before, the variation of critical point induced by different inactivation strategies still exits even in a regular ring. Interestingly, Eq. (16) coincides with one situation investigated in Ref. 6, where Eq. (16) is derived by a different approach. Moreover, Eqs. (14) and (16) can recover Eq. (4) in Ref. 4 when the topology is globally connected; and under the strong coupling limit, Eq. (16) can reproduce the results in Refs. 5 and 21. Therefore, all these studies provide insights from different perspectives into the variation of critical point in aging transition.

IV. CONCLUSION

In this work, we investigated the variation of critical point in aging transition. For a networked system with both active and inactive oscillators, we found that the critical point of aging transition varies even thought the coupling strength and the network topology are fixed. By analytical treatment and numerical experiments, we successfully explained why this variation occurs and how it relates to the normalized cross links determined by the specific configuration of active and inactive oscillators in the network. We further studied the dynamical robustness under three strategies of targeted inactivation. The result revealed that the global dynamics in this system is the most vulnerable when inactivation starts from the low-degree nodes, rather than the hubs. The present work provided helpful understanding of transition phenomena in networked systems. It might also shed light on designing effective strategies to enhance/destroy the dynamical robustness in real circumstances.

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APPENDIX: METHOD TO TUNE THE NETWORK TOPOLOGY

Previously, there are several methods to change network topology from homogeneous to heterogeneous.[22,23] For example, the degree distribution of network can be tuned between exponential and power-law. However, in the present work, we expect a much wide tuning range that the degree distribution of homogeneous network could be Poisson form or even approximate Delta function. So we propose a simple but effective method for this purpose. The main idea is to gradually rewire edges from an initially heterogeneous network. Here, are the main steps: (1) Generate a heterogeneous network using the strategy of preferential attachment, i.e., newly added edges have more chance to connect to nodes with large degree. Typically, a network with power law degree distribution can be obtained using this strategy which is called BA (Barabási-Albert) network;[24] (2) Choose an arbitrary edge and compare the degrees of its both ends; (3) Disconnect the edge from the node with higher degree and randomly rewire it to a node in the network. We define the rewiring probability p as the number of edges rewired normalized by the total number of edges in the network. When p varies from 0 to 1, a heterogeneous scale-free network gradually converts into a homogeneous one. It should be emphasized that by homogeneous here we mean its degree satisfies Poisson distribution, rather than strictly regular network. Numerically, this has been verified. Therefore, p is the parameter that can control the extent of heterogeneity/homogeneity in a network. In our simulations, we have also started from other scale-free networks with power law exponents between 2 and 3 (the BA network has a power law exponent of 3), but the results are qualitatively the same.

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