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PHYSICS LETTERS A

Physics Letters A 353 (2006) 30–33

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Understanding synchronization induced by "common noise"

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Received 18 August 2005; received in revised form 23 November 2005; accepted 23 November 2005

Available online 1 December 2005

Communicated by C.R. Doering

Abstract

Noise-induced synchronization refers to the phenomenon where two uncoupled, independent nonlinear oscillators can achieve synchronization through a "common" noisy forcing. Here, "common" means identical. However, "common noise" is a construct which does not exist in practice. Noise by nature is unique and two noise signals cannot be exactly the same. How to justify and understand this central concept in noise-induced synchronization? What is the relation between noise-induced synchronization and the usual chaotic synchronization? Here we argue and demonstrate that noise-induced synchronization is closely related to generalized synchronization as characterized by the emergence of a functional relation between distinct dynamical systems through mutual interaction. We show that the same mechanism applies to the phenomenon of noise-induced (or chaos-induced) phase synchronization.

PACS: 05.45.Xt; 05.45.-a

In the past twenty years, chaotic synchronization, including *complete synchronization, generalized synchronization, phase synchronization and lag synchronization*, has been intensively investigated. Among them, generalized synchronization discovered by Rulkov et al. [1–4] is an interesting phenomenon. It refers to the existence of some functional relation between coupled but nonidentical chaotic oscillators. To detect generalized synchronization, Abarbanel et al. proposed the interesting idea of considering an auxiliary response system and examining the conditional stability of typical trajectories in the driven system [2]. In particular, suppose one wishes to determine whether there is a generalized synchronization between two uni-directionally coupled oscillators, say A and B, the drive and driven system, respectively. One can imagine an auxiliary response system B', which is identical to B and subject to the same driving signal, and asks whether there is synchronization between B and B'. Abarbanel et al. showed that an affirmative answer would imply a generalized synchronization between A and B. Note that the pioneering work on chaotic synchronization by Pecora and Carroll [5] focused on synchronization between identical *subsystems* under a common forcing. The auxiliary response-system approach is equivalent to treating B' as the replica of *subsystem* B in a single dynamical system that comprises A and B. Whether subsystems can be synchronized is determined by the sign of the conditional Lyapunov exponents evaluated for typical trajectories in any of the subsystems under the forcing. Another interesting synchronization phenomenon is the chaotic phase synchronization [12]. It occurs in certain chaotic systems where suitable phases can be defined. In phase synchronization, the phases between two chaotic oscillators can be locked while their amplitudes remain chaotic and uncorrelated. Compared with generalized synchronization, phase synchronization is a weaker form since there is no functional relation between the amplitudes of the two coupled oscillators.

Parallel to the chaotic synchronization mentioned above, the phenomenon of noise-induced synchronization, i.e., synchronization among uncoupled nonlinear oscillators under "com-

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0375-9601/$ – see front matter © 2005 Published by Elsevier B.V.
doi:10.1016/j.physleta.2005.11.067

mon (or identical) noise", has also been intensively studied. The first work along this line was carried out by Maritan and Banavar over a decade ago [6]. This seemingly counter-intuitive phenomenon has since attracted a continuous interest [7–9], partly because it is another powerful demonstration of "noise-induced order" as a result of the interplay between nonlinear dynamics and stochastic processes, in addition to stochastic resonance [10] and coherence resonance [11]. More recently, the phenomenon has been extended [9] to chaotic phase synchronization [12] by Zhou et al. who demonstrated numerically and experimentally that phase coherence between uncoupled chaotic oscillators in a statistical sense can be established by "common noise". They further proposed that the phenomenon is due to the existence of distinct phase-space regions where infinitesimal vectors experience expansion and contraction, respectively, as a result of the common noisy forcing. Very recently, the noise effect on the fully synchronous regime of globally coupled chaotic systems has been investigated [13].

So far, chaotic synchronization and synchronization induced by "common noise" have appeared to be two almost independent research domains. Whether or not these two types of synchronization in chaotic systems can be understood in an unified framework is of particular interest. In this Letter, we argue that noise-induced synchronization can be understood naturally as a manifestation of generalized synchronization. Therefore, these two types synchronization phenomena can be unified conceptually. To state our result, we use the representative setting where two nonlinear oscillators are driven by a common random or chaotic forcing,

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}_1\big[\mathbf{x}, \xi(t)\big], \qquad \frac{d\mathbf{y}}{dt} = \mathbf{f}_2\big[\mathbf{y}, \xi(t)\big], \tag{1}$$

where $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are the dynamical variables of the two oscillators that are governed by vector fields $\mathbf{f}_1$ and $\mathbf{f}_2$, respectively, and $\xi(t)$ denotes the common stochastic or chaotic forcing. To study complete synchronization or phase synchronization, we assume [9] $\mathbf{f}_1 \approx \mathbf{f}_2$. In general, synchronization can be achieved only when there is an interaction (communication) between the oscillators. Since there is no direct coupling between $\mathbf{x}$ and $\mathbf{y}$, the interaction must be provided by the common forcing $\xi(t)$, noisy or chaotic. If the forcing is sufficiently strong, both oscillators tend to follow its evolution to some extent, thereby *indirectly* establishing an interaction between them. Synchronization between $\mathbf{x}$ and $\mathbf{y}$ then implies a generalized synchronization between $\mathbf{x}$ and the random or chaotic forcing ξ (by the auxiliary response-system criterion for generalized synchronization [2]), and vice versa. Quantitatively, the equivalence between noise-induced (or chaos-induced) synchronization and generalized synchronization can be demonstrated by computing the conditional Lyapunov exponents for the driven systems. A negative largest conditional Lyapunov exponent indicates generalized synchronization between the drive and either of the driven systems, which again signals synchronization between the driven systems. A remarkable finding is that the same method applies to noise-induced or chaos-induced phase synchronization, where the driven systems are necessarily nonidentical. In this case, we find that noise-induced or chaos-induced phase synchronization occurs only when the forcing is strong enough so that *both the originally null conditional Lyapunov exponents* become negative.

We first consider noise-induced (or chaos-induced) synchronization between two uncoupled chaotic oscillators. Our driven systems are the following classical Lorenz oscillator:

$$\begin{aligned}\frac{dx_{1,2}}{dt} &= 10(y_{1,2} - x_{1,2}),\\ \frac{dy_{1,2}}{dt} &= r_{1,2}x_{1,2} - y_{1,2} - x_{1,2}z_{1,2} + \xi(t),\\ \frac{dz_{1,2}}{dt} &= x_{1,2}y_{1,2} - (8/3)z_{1,2},\end{aligned} \tag{2}$$

where r_1 and r_2 are the intrinsic parameters of the Lorenz oscillators, and $\xi(t)$ is the common noisy or chaotic forcing. To be able to compare the results from different types of driving signals, we choose

$$\xi(t) = D\frac{\eta(t) - \langle\eta(t)\rangle}{\sigma}, \tag{3}$$

where D is the normalized amplitude of the noisy (or chaotic) forcing, $\eta(t)$ is a Gaussian random process or a chaotic signal, and σ is the standard deviation of $\eta(t)$. To demonstrate the generality of our result, we shall use four different types of forcing: Gaussian random signal, chaotic signals from the Rössler, the Lorenz, and the Mackey–Glass system. Fig. 1 shows for $r_1 = r_2 = 28$, the largest conditional Lyapunov exponent (LCLE) of the Lorenz oscillator Eq. (2) versus the amplitude for four types of noisy or chaotic forcing. The common feature is that there is generalized synchronization in all four cases, insofar as

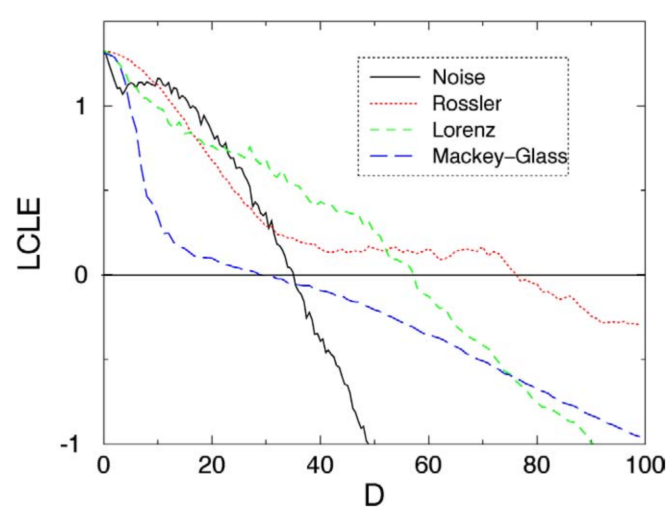

Fig. 1. The LCLE of Lorenz system Eq. (2) for $r = 28$ under chaotic or noisy forcing. Four different types of driving signals are used: (1) Gaussian random signal (solid trace), (2) chaotic Rössler system (dotted trace) given by $\dot{x} = -(y + z)$, $\dot{y} = (x + 0.15y)$, and $\dot{z} = 0.2 + z(x - 8.5)$, (3) chaotic Lorenz system (light-dashed trace) given by $\dot{x} = 10(y - x)$, $\dot{y} = 35x - y - xz$, and $\dot{z} = xy - (8/3)z$, and (4) chaotic Mackey–Glass system (heavy-dashed trace) given by $\dot{x} = -0.1x(t) + 0.2x(t - 100)/[1 + x(t - 100)^{10}]$. In all four cases, generalized synchronization as characterized by a negative LCLE occurs for sufficiently large forcing amplitude.

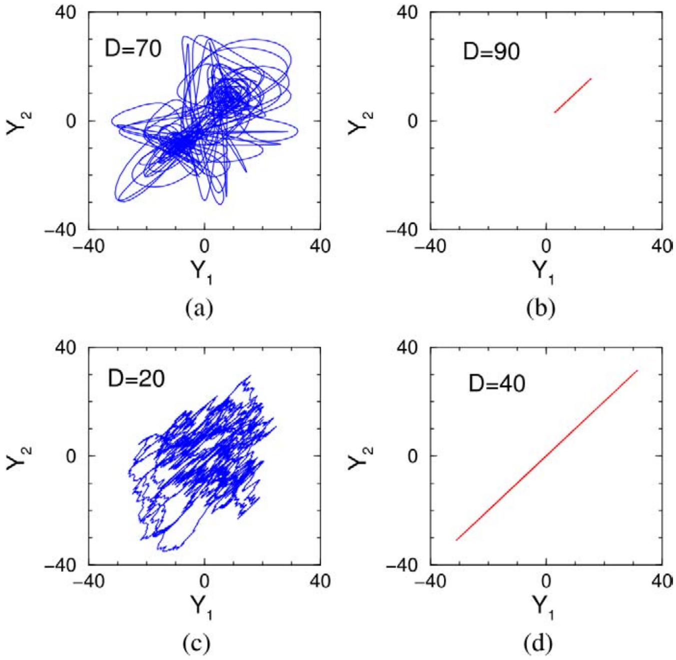

Fig. 2. Evidence of chaos-induced (a, b) and noise-induced (c, d) synchronization between two uncoupled Lorenz chaotic oscillators in Eq. (2). In (a) and (b), the forcing is a chaotic Rössler signal. Chaos-induced synchronization occurs for $D = 90 > D_c \approx 75$. In (c) and (d), the forcing is a Gaussian random signal. We see that noise-induced synchronization occurs for $D = 40 > D_c \approx 35$.

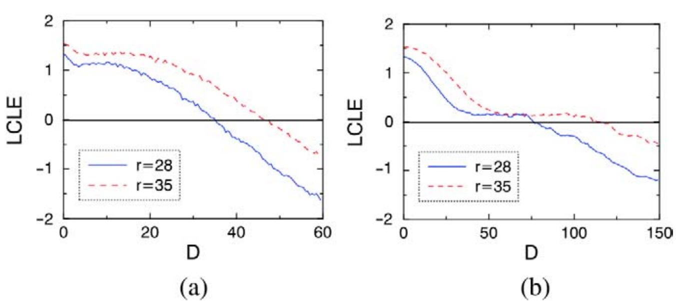

Fig. 3. The LCLEs of two parametrically different Lorenz system Eq. (2) driven by one common signal which is either Gaussian random noise (a) or from chaotic Rössler system (b) as described in the caption in Fig. 1. For each case, an indirect relation has been established between two driven systems when both the LCLEs become negative.

the forcing amplitude is sufficiently large so that the LCLE becomes negative. Evidence for noise-induced or chaos-induced synchronization is shown in Fig. 2, where Figs. 2(a) and (b) show the y-variables of the two Lorenz oscillators under chaotic Rössler forcing of two different amplitudes. From Fig. 1, we see that for this type of forcing, generalized synchronization occurs for $D > D_c \approx 75$ (the dotted trace). Fig. 2(a) shows that chaos-induced synchronization has not occurred for $D = 70 < D_c$, but it does for $D = 90 > D_c$ [Fig. 2(b)]. Figs. 2(c) and (d) show a similar situation for noise-induced synchronization, where the forcing is a Gaussian random signal.

What happens if there is a mismatch between the two oscillators under common noisy or chaotic forcing? In this case, the conditional Lyapunov exponents under the forcing can be calculated for each oscillator. Fig. 3 shows the LCLEs of two

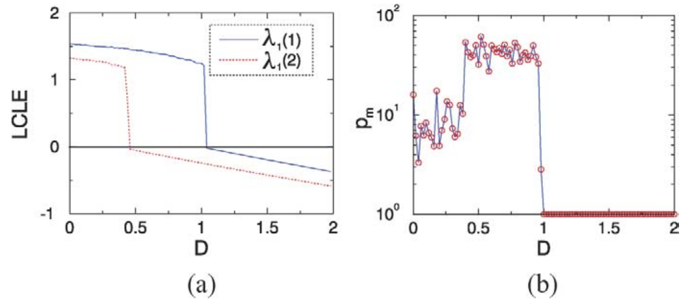

Fig. 4. Direct evidence of generalized synchronization between two uncoupled Lorenz chaotic oscillators in Eq. (2) driven by common chaotic signals. (a) The LCLEs of two driven systems. (b) The MFNN parameter versus the forcing amplitude.

Lorenz systems with $r_1 = 28$ and $r_2 = 35$ driven by a common noisy or chaotic signal. As the forcing amplitude is increased, the LCLEs of the two driven systems become negative one after another, indicating that both driven systems have achieved generalized synchronization with the drive system. This implies a generalized-synchronization relation between the two driven systems. In fact, in Eq. (1), dynamical systems $\mathbf{x}$ and $\mathbf{y}$ can be regarded as two chaotic oscillators bidirectionally coupled through the common forcing $\xi(t)$. Thus the synchronization between two different chaotic systems driven by a common noisy or chaotic signal can be understood as generalized synchronization in mutually coupled chaotic systems [14].

We now provide direct evidence to show that generalized synchronization can be achieved between two mismatched chaotic systems driven by a common chaotic signal. In this example, the drive system is the Rössler system described in the caption of Fig. 1. The two driven systems are two Lorenz systems [Eq. (2)] with $r_1 = 35$ and $r_2 = 28$, respectively. The forcing term $\xi(t)$ in Eq. (2) is replaced by the feedback type $-D(y_{1,2} - y_d)$, where y_d denotes the y variables in the driving Rössler system. In Fig. 4(a), the LCLEs of two driven systems versus the forcing amplitude are plotted. It is seen that both LCLEs become negative when the forcing amplitude is large enough. In this regime, generalized synchronization between the two driven systems occur. In order to show this relation directly, we apply the mutual false nearest neighbor (MFNN) method proposed in Ref. [1]. The MFNN parameter p_m is defined as

$$p_m = \left\langle \frac{|\mathbf{y}_n - \mathbf{y}_{n(NND)}|}{|\mathbf{x}_n - \mathbf{x}_{n(NND)}|} \frac{|\mathbf{x}_n - \mathbf{x}_{n(NNR)}|}{|\mathbf{y}_n - \mathbf{y}_{n(NNR)}|} \right\rangle. \tag{4}$$

Here n is the time index. $\mathbf{x}_n$ is an arbitrary point in the phase space of the drive system and $\mathbf{y}_n$ is its corresponding image in the response system. The nearest phase space neighbor of $\mathbf{x}_n$ has time index $n(NND)$ and the nearest phase space neighbor of $\mathbf{y}_n$ has time index $n(NNR)$. The $\langle\cdot\rangle$ denotes the average of reference point on the attractor. This parameter will be of the order of unity if generalized synchronization exists between two systems. Otherwise, it is much larger than unity. The result of MFNN calculation is shown in Fig. 4(b). We see that for $D > 1$ where both LCLEs are negative, the values of p_m are of the order of unity. This represents direct evidence that generalized synchronization has been established between two different chaotic systems driven by a common chaotic signal.

We finally consider chaos-induced and noise-induced phase synchronization. Noise-induced phase synchronization has been studied in Ref. [9], where it is shown that phase synchronization in a statistical sense can be achieved between two chaotic oscillators in the presence of common noise. However, perfect phase synchronization as characterized by a zero difference between the oscillating frequencies, cannot be achieved because of noise. Here, we show that such phase synchronization can be achieved between two Rössler oscillators with different natural frequencies when they are driven by a common chaotic signal. We use a pair of phase-coherent Rössler oscillators given by

$$\frac{dx_{1,2}}{dt} = -(\omega_{1,2}y_{1,2} + z_{1,2}),$$
$$\frac{dy_{1,2}}{dt} = (\omega_{1,2}x_{1,2} + 0.15y_{1,2}) + \xi(t),$$
$$\frac{dz_{1,2}}{dt} = 0.2 + z_{1,2}(x_{1,2} - 8.5), \tag{5}$$

where $\omega_{1,2}$ are parameters. We choose $\omega_1 = 0.97$ and $\omega_2 = 1.03$ so that the two oscillators are not identical. Fig. 5(a) shows, when the common driving signal $\xi(t)$ comes from another chaotic Rössler oscillator [described by Eq. (5) but with $\omega = 1.0$], the mean frequency difference $\Delta\Omega$ between the two driven oscillators versus the forcing amplitude D. We see that chaos-induced phase synchronization occurs for $D > D_c \approx 0.4$ where $\Delta\Omega$ vanishes. The mean frequency differences between the drive and the two driven oscillators versus D are shown in Fig. 5(b). Fig. 5(c) shows the two sets of LCLEs for the two driven oscillators. We observe that $\Delta\Omega = 0$ apparently requires that both the null conditional Lyapunov exponents from the two oscillators be negative. Fig. 5(d) shows that the driven oscillator remains phase-coherent under the forcing.

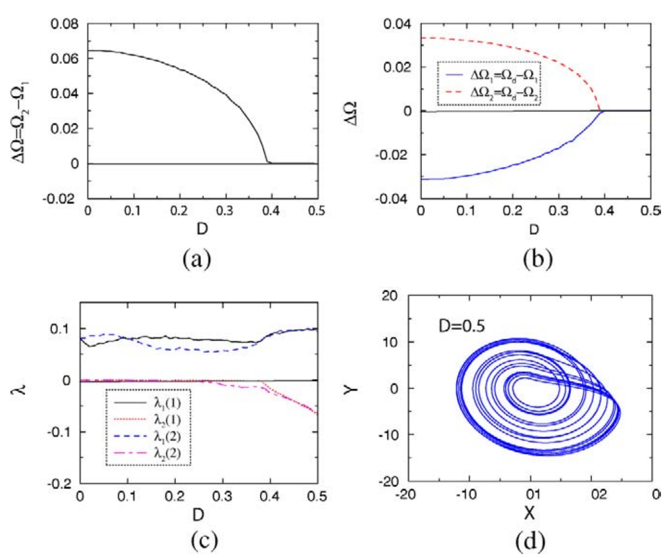

Fig. 5. Chaos-induced phase synchronization between two uncoupled chaotic Rössler oscillators, (a) the mean frequency difference $\Delta\Omega$ between the two driven oscillators versus the forcing amplitude D, (b) the mean frequency differences between the drive and the two driven oscillators versus D, (c) two sets of LCLEs (one from each driven oscillator) versus D, and (d) phase-coherent attractor in the driven oscillator in the synchronized regime.

In summary, we have argued that noise-induced synchronization and the more recent phenomenon of noise-induced phase synchronization can be reasonably understood from the standpoint of generalized synchronization. Both types of synchronization can be predicted based on the signs of the conditional Lyapunov exponents. In particular, noise-induced phase synchronization requires all null conditional exponents be negative. These phenomena are relevant in a variety of contexts and may have potential applications [6,7,9]. Our findings not only unify two seemingly independent domains of synchronization, i.e., the usual chaotic synchronization and noise-induced synchronization, but also can avoid the conceptual difficulty of "common noise" which is frequently used in the study of noise-induced synchronization.

Acknowledgements

Y.C.L. was supported by AFOSR under Grants No. F49620-03-1-0219.

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