Coexistence of Quantized, Time Dependent, Clusters in Globally Coupled Oscillators

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We report on a novel collective state, occurring in globally coupled nonidentical oscillators in the proximity of the point where the transition from the system’s incoherent to coherent phase converts from explosive to continuous. In such a state, the oscillators form quantized clusters, where neither their phases nor their instantaneous frequencies are locked. The oscillators’ instantaneous speeds are different within the clusters, but they form a characteristic cusped pattern and, more importantly, they behave periodically in time so that their average values are the same. Given its intrinsic specular nature with respect to the recently introduced Chimera states, the phase is termed the Bellerophon state. We provide an analytical and numerical description of Bellerophon states, and furnish practical hints on how to seek them in a variety of experimental and natural systems.

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The emergence of coherent phases of interacting oscillators is one of the most important phenomena in nature, and is the foundation for the cooperative functioning of a wealth of different systems. To gather understanding on the mechanisms underlying such organizational behavior, physicists resort to solvable and simplified frameworks, such as the Kuramoto [1] and Kuramoto-like [2–4] models, where a variety of collective states can be described: from full [5,6], to cluster [7,8], to explosive synchronization (ES) [9,4]. Recently, various types of Chimera states (CSs) (the coexistence of coherent and incoherent domains, which occurs, remarkably, for fully identical locally coupled oscillators) [11,10] have been described and observed in experiments [12], including the breathing CS [13], the clustered CS [14], and the multi CS [15].

In this Letter we report on a previously unknown coherent phase that is proper, instead of globally coupled oscillators with widely different frequencies, and which emerges in the proximity of the parameter point where the transition from the system’s incoherent to coherent behavior converts from explosive to continuous. In the novel state, the oscillators form quantized clusters, where neither their phases nor their instantaneous frequencies are locked. Each of the oscillators’ instantaneous speeds is different within the clusters, but the instantaneous frequencies form the same cusped pattern characterizing the average speeds of CS. The oscillators’ instantaneous frequencies behave periodically in time so that their average values are the same. Because of its intrinsic specular nature with respect to CS, the new phase is termed here the Bellerophon state, as Bellerophon was the great hero who, in Greek mythology, confronted the monster Chimera [16].

We start assuming the framework of a Kuramoto-like model of $N$ globally coupled phase oscillators, which reads

\[
\dot{\theta}_i = \omega_i + \frac{\kappa_i}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, \ldots, N, \tag{1}
\]

where the dots denote temporal derivatives and $\theta_i$, $\omega_i$, and $\kappa_i$ are the instantaneous phase, the natural frequency, and the coupling strength of the $i$th oscillator, respectively. The level of synchronization is measured by the order parameter $R = \frac{1}{N} \langle |\sum_{j=1}^{N} e^{i\theta_j}| \rangle_T$, where $\langle \cdot \rangle_T$ and $\langle \cdot \rangle_T$ denote the module and time average, respectively. The set of natural frequencies $\{\omega_i\}$ is drawn from a given frequency distribution (FD) $g(\omega)$. In the following, two distinct cases will be illustrated, which have in common the fact that they sustain both a first- and a second-order-like transition to synchronization; i.e., the coherent phase may occur (for proper parameter choices) abruptly. Namely, case (1) corresponds to setting $\kappa_i = \kappa |\omega_i|$, i.e., to establishing a correlation between the oscillator’s natural frequency and the coupling strength; case (2) considers instead two distinct populations of oscillators (conformists and contrarians) [17], i.e., with $\kappa_i$ only taking two values (either $\kappa_1 < 0$ or $\kappa_2 > 0$).

For the sake of illustration, let us start from case (1), and with a FD that is assumed to be an even function $[g(\omega) = g(-\omega)]$, symmetric, and centered at zero. We take $g(\omega) = \frac{\Delta^2}{2\pi} [1/(\omega - \omega_0)^2 + \Delta^2] + (1/(\omega + \omega_0)^2 + \Delta^2)]$ to be a bimodal Lorentzian distribution, where $\Delta$ is the width parameter (half width at half maximum) of each Lorentzian and $\pm\omega_0$ are their center frequencies. Notice that, depending on $\omega_0/\Delta$, such a FD can be, in fact, either unimodal
As seen in Fig. 2, $\kappa_f$ decreases monotonically as $\omega_0/\Delta$ increases, causing (as $\kappa_b = 2$ always) the hysteresis area to shrink monotonically. When $\omega_0/\Delta = \sqrt{3}$, $\kappa_f = \kappa_b = 2$, and the forward and backward transition points almost coincide [see Fig. 1(c)]. As $\omega_0/\Delta$ gradually exceeds $\sqrt{3}$, the hysteresis area does not immediately disappear [see the inset of Fig. 1(d)]. Actually, at $\omega_0/\Delta = \sqrt{3}$, a Hopf bifurcation occurs during both the forward and backward processes, and both bifurcations are continuous. For the forward direction, the system first undergoes a continuous transition, followed by an ES transition (as $\kappa$ further increases). A similar scenario of transitions characterizes also the backward direction. An initial parameter regime $\omega_0/\Delta > \sqrt{3}$ then exists, where the system undergoes the cascade of one continuous and one explosive transition during both forward and backward continuity. A further increase of $\omega_0/\Delta$ causes the hysteresis area to eventually disappear [Figs. 1(e) and 1(f)], leading to a situation where only continuous transitions occur in the system. It is in this latter regime, i.e., close to a tricritical point in parameter space that novel coherent phases, the Bellerophon states, emerge in the path leading the system from its unsynchronized to its synchronized behavior.

The following step involves characterizing such a novel state, and discussing the differences with other typical coherent states of Kuramoto-type models. For the sake of exemplification, we take the case of $\omega_0/\Delta = 3$ [Fig. 1(f)]. Here, the system exhibits two continuous transitions at $\kappa_1^f = 4/\sqrt{10} \approx 1.26$ and $\kappa_2^f = 2$, respectively. Therefore, three parameter regimes can be identified: $\kappa < \kappa_1^f$ (I), $\kappa_1^f < \kappa < \kappa_2^f$ (II), and $\kappa > \kappa_2^f$ (III). In regime I, the coupling strength is small, and the system features the trivial incoherent state. In regime III, the coupling is so strong that the system goes into the fully synchronized state, in which all oscillators split into two fully synchronized clusters. The Bellerophon phases are steady states...
occurring in the middle regime II, i.e., during the path to full synchronization. In Fig. 3, four typical phases are illustrated, corresponding to the \(\kappa\) values denoted by letters \(A, B, C,\) and \(D\) in Fig. 1(f). They are characterized by three quantities: the instantaneous phases \(\theta_i\), the instantaneous (angular) speed \(\dot{\theta}_i\), and the average speed \(\langle \dot{\theta} \rangle\) (i.e., the oscillators’ effective frequencies), where the bracket stands for a long time average.

In Fig. 3(a), \(\kappa = 1.28\). As \(\kappa\) just exceeds \(\kappa_c^1 = 1.26\), two small symmetric clusters emerge, whose average speeds are equal to each other in magnitude, but opposite in sign. The oscillators in the two clusters rotate with the same average speed (but different instantaneous phases and frequencies). At \(\kappa = 1.60\) [Fig. 3(b)], a multiclustered state emerges. The number of clusters increases in pairs as \(\kappa\) increases, each pair containing oscillators that are symmetric in terms of their natural frequencies. The oscillators inside each cluster have the same average speed [see the staircase structure of Fig. 3(b2)], but different instantaneous frequencies [Fig. 3(b3)]. The clusters coexist with drifting oscillators that are not synchronized. In Fig. 3(c), \(\kappa = 1.80\). This is also a \textit{Bellerophon} state, but different from that of Fig. 3(b). The coherent clusters now occupy almost all the range of natural frequencies, except for a small narrow zone around the central frequency; the increase of \(\kappa\) results in all drifting oscillators being gradually recruited into either one of the clusters. Finally, Fig. 3(d) (\(\kappa = 2.10\)) refers to the fully coherent phase, where two giant clusters are formed. In each cluster, the oscillators with positive or negative frequencies coincide with each other totally: they feature now the same instantaneous speed, so that the whole system behaves like two giant oscillators.

Much better insight is gathered by inspecting the system’s macroscopic and microscopic details. Figure 4(a) reveals that the staircases of coherent clusters at \(\kappa = 1.60\) satisfy in fact a certain rule: they are quantized, and can be expressed as \(\pm (2n - 1)\Omega_1, n = 1, 2, \ldots, \) [20], where \(\Omega_1\) is the lowest frequency, i.e., the principle (or base) system’s frequency. Accordingly, depending on their multiple of \(\Omega_1\), the clusters can be named \(C^1, C^3, C^5, \ldots\), respectively. The key, and also subtle, point here is that, although the average speeds of the oscillators inside each cluster are equal to each other, their instantaneous speeds are generally different and quite heterogeneous. Furthermore, the instantaneous speeds of the oscillators in each cluster are correlated and form the characteristic cusped pattern [Figs. 3(b3) and 3(c3)] analogous to that featured by the average frequencies of the oscillators within the CS. We emphasize that this similarity is between the instantaneous frequencies in the \textit{Bellerophon} state and the average frequencies in CS. Figure 4(b) shows that the instantaneous speeds of the oscillators inside the same cluster evolve periodically, but different oscillators follow different periodic patterns. In other words, the instantaneous speed for each oscillator evolves uniquely. This makes \textit{Bellerophon} states essentially different with respect to other coherent states observed in Kuramoto-like models, such as the partially coherent state [6], the standing wave state [21,22], the traveling wave state [21,23], and the CS [10,11], where the oscillators inside the coherent cluster...
are typically frequency locked. Moreover, even though the instantaneous speed of the clusters’ oscillators varies non-uniformly during one period (particularly for those clusters with large \( n \)), the average speeds during one period for all oscillators in a certain cluster turn out to be the same, i.e., an odd-numbered multiple of \( \Omega_i \) in this case. As the instantaneous speed characterizes the rotations of the oscillators along the unit circle, a very interesting collective motion of the oscillators is observed [Fig. 4(d)]: during one period \( T_i = 1/\Omega_i \), the oscillators in \( C^i \) all perform one loop along the unit circle, and in the meantime, the oscillators in \( C^3 \) and \( C^5 \) rotate three loops and five loops, respectively. In analogy, the oscillators in \( C^{2n-1} \) will perform \( 2n - 1 \) loops. Compared with Fig. 4(b), we further find that during one loop, the instantaneous speeds for all coherent oscillators experience two periods; i.e., each oscillator repeats its motion during the two half periods. In Fig. 4(a), we report the local value of the order parameter (i.e., that contributed by only those oscillators in a certain cluster) in the complex plane, for \( C^3 \), \( C^3 \) and \( C^5 \). Because of the complicated phase relationships among the oscillators in each cluster [see Fig. 4(d)], the resulting value is typically periodic or quasiperiodic, and follows a complicated orbit. Essentially, each cluster can be seen as a giant oscillator, with properties described by the local order parameter. Figure 4(c) reports the order parameters for all oscillators in each cluster (in the complex plane, for \( C^3 \), \( C^3 \) and \( C^5 \)). Because of the complicated phase relationships among the oscillators in each cluster [see Fig. 4(d)], the resulting value is typically periodic or quasiperiodic, and follows a complicated orbit. Essentially, each cluster can be seen as a giant oscillator, with properties described by the local order parameter. Figure 4(c) reports the order parameters for all oscillators in each cluster (in the complex plane, for \( C^3 \), \( C^3 \) and \( C^5 \)).

Let us now start to discuss on the conditions needed for the emergence of Bellerophon states. First of all, the states appear to be robust under the change of the FD, and in particular under relaxing the hypothesis of a symmetric distribution. Indeed, one can consider the same case (1), but under the choice of an asymmetric Lorentzian distribution. Precisely, \( g(\omega) \) is now taken to be \( g(\omega) = \Delta/(\pi[(\omega - \omega_0)^2 + \Delta^2]) \). When keeping \( \Delta = 1 \) as a constant, and changing \( \omega_0 \) (in order to shift the frequency distribution along the positive axis), the typical state that emerges is illustrated in Fig. 5(a).

Furthermore, a correlation between the oscillator’s natural frequency and the coupling strength [inherent to case (1)] seems not to be a necessary condition either. One can indeed consider Eq. (1) under case (2), i.e., in the presence of two populations of oscillators, with \( k_i \) only taking two values (either \( k_1 < 0 \) or \( k_2 > 0 \)). In this case, the natural frequencies \( \{\omega_i\} \) are taken from a symmetric distribution centered at zero, the Lorentzian distribution \( g(\omega) = \Delta/[\pi(\omega^2 + \Delta^2)] \). Now, the oscillators in the ensemble can be generally divided into two groups: those with positive \( k_i \) (which will behave like conformists attempting to follow the global rhythm of the system), and those with negative \( k_i \), which will tend to act as contrarians (always trying to oppose the system’s global trend) [17]). In numerical simulations, one starts from an incoherent state where only the contrarian oscillators interact, and gradually flips a number of contrarians into conformists. In doing so, various strategies can be adopted as rules for the flipping procedure. Three strategies to change contrarians into conformists have been adopted by us. In strategy (i) contrarians are randomly chosen to be flipped into conformists; in strategy (ii) contrarians are ranked according to the absolute value of their natural frequencies \( |\omega_i| \), and then flipped into conformists from the largest \( |\omega_i| \) to the smallest, i.e., the coupling strength of the \( i \)th oscillators will be \( k_i = k_2 \) if \( |\omega_i| > \omega_0 \) and \( k_i = k_1 \) otherwise. Denote the proportion of conformists in the system as \( p \) then \( 1 - p = \int_{-\infty}^{0} g(\omega) d\omega \); strategy (iii) is the opposite of (ii), i.e., \( k_i = k_2 \) if \( |\omega_i| < \omega_0 \) and \( k_i = k_1 \) otherwise, with \( p = \int_{0}^{\infty} g(\omega) d\omega \). In all the three cases, the emerging scenario is qualitatively the same. While the full analytical treatment of the three cases will be presented elsewhere [24], the numerical results of case (iii) are reported in Fig. 5(b) where one can see that, as the proportion of conformists \( p = \int_{0}^{\infty} g(\omega) d\omega \) increases, the system manifests Bellerophon states.

In conclusion, we have provided evidence of a novel, asymptotic, phase of globally coupled oscillators: the Bellerophon state, which differs essentially from all coherent phases described so far in coupled oscillator models. Within the novel state, the oscillators form quantized clusters, where neither their phases nor their instantaneous frequencies are locked. The oscillators’ instantaneous speeds are different, but they behave periodically, and most importantly, their average speed values are the same. Our results support the hypothesis that Bellerophon states are generic, and occur in globally coupled nonidentical oscillators, irrespectively of the symmetric nature of the FD, or of the specific oscillators’ coupling scheme. The required condition for the emergence of these states seems
to be a model system for which continuous and abrupt transitions to synchronization coexist, in such a way that a setting can be chosen in the relative proximity of the parameter point where the switching in the nature of the synchronization transition (from second-order-like to explosive) occurs. The range of parameters over which the new states are observed turns out to be pretty large, and therefore one actually does not even need to be in the immediate vicinity of such a tricritical point. While there is certainly a mathematical challenge for the future, our analytical and numerical description will certainly help physicists seek Bellerophon states in a variety of experimental and natural systems.

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\[ N = 10000 \]


[16] By specular nature we here mean that the conditions under which the system has to be prepared are exactly the opposite of those giving rise to Chimera states. Chimera states emerge, indeed, for locally coupled identical oscillators, while here one needs globally coupled nonidentical oscillators, and actually our Letter shows that the more different the ensemble’s units are, the best is the case for the observation of Bellerophon states.
[18] Numerical values are obtained by a fourth order Runge-Kutta integration method, with an integration time step of 0.005.