Symmetries and conservation laws of one Blaszak–Marciniak four-field lattice equation*

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In this paper, by using the classical Lie symmetry approach, Lie point symmetries and reductions of one Blaszak–Marciniak (BM) four-field lattice equation are obtained. Two kinds of exact solutions of a rational form and an exponential form are given. Moreover, we show that the equation has a sequence of generalized symmetries and conservation laws of polynomial form, which further confirms the integrability of the BM system.

Keywords: Blaszak–Marciniak four-field lattice, symmetries, explicit solution, conservation laws

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1. Introduction

In the past few years, the study of discrete soliton equations has received considerable interest.[1–7] Most discrete soliton equations have various beautiful algebraic and geometric properties.[8–12] One remarkable feature is the fact that they have different kinds of symmetries, such as Lie point symmetries, generalized, high-order or Lie bäcklund symmetries, master symmetries and so on,[13–23] which provide a very effective method to find the exact solutions of nonlinear soliton equations and is a predictor for integrability.[24–36] The other interesting feature of this kind of discrete soliton equations is the conservation law, which is also an important indictor for integrability.[26,27,37]

In this paper, we investigate one Blaszak–Marciniak (BM) four-field lattice equation[4]

$$p_{nt} = s_{n+1} - s_n + p_n(r_{n-2} - r_n), \tag{1a}$$
$$r_{nt} = q_{n+1} - q_n, \tag{1b}$$
$$q_{nt} = p_{n+1} - p_n + q_n(r_{n-1} - r_n), \tag{1c}$$
$$s_{nt} = s_n(r_{n-3} - r_n), \tag{1d}$$

which is derived as an application of r-matrix formalism to the algebra of shift operators;[4] it can degenerate to the well-known Toda lattice equation[1] when $s_n = 0$, $p_n = 0$. In Ref. [4], the bi-Hamiltonian formulation, the Miura-like gauge transformations of this multi-field lattice equation are constructed.

In this paper, we discuss the symmetries and polynomial conservation laws of Eq. (1). Our paper can be organized as follows. In Section 2, we derive the Lie point symmetries of Eq. (1), which admit a three-dimensional nilpotent Lie algebra; in Section 3, equation (1) is degraded into an ordinary differential-difference equation using the similarity reduction technique and two kinds of exact solutions are given; in Section 4, a sequence of generalized symmetries and conservation laws of polynomial form is obtained, which further confirms the integrability of this Blaszak–Marciniak four-field lattice equation.

2. Lie point symmetries

As to Eq. (1), we introduce a one parameter (ε) continuous point transformation

$$n^* = n + \varepsilon\phi_1(n,t,p_n,r_n,q_n,s_n) + \mathrm{O}(\varepsilon^2), \tag{2a}$$
$$t^* = t + \varepsilon\phi_2(n,t,p_n,r_n,q_n,s_n) + \mathrm{O}(\varepsilon^2), \tag{2b}$$
$$p^*_{n^*} = p_n + \varepsilon\phi_3(n,t,p_n,r_n,q_n,s_n) + \mathrm{O}(\varepsilon^2), \tag{2c}$$
$$r^*_{n^*} = r_n + \varepsilon\phi_4(n,t,p_n,r_n,q_n,s_n) + \mathrm{O}(\varepsilon^2), \tag{2d}$$
$$q^*_{n^*} = q_n + \varepsilon\phi_5(n,t,p_n,r_n,q_n,s_n) + \mathrm{O}(\varepsilon^2), \tag{2e}$$
$$s^*_{n^*} = s_n + \varepsilon\phi_6(n,t,p_n,r_n,q_n,s_n) + \mathrm{O}(\varepsilon^2), \tag{2f}$$

where ε is the group parameter, ϕ_i $(i = 1,2,\cdots,6)$ are infinitesimals and the corresponding infinitesimal generator is

$$G = \phi_1\frac{\partial}{\partial n} + \phi_2\frac{\partial}{\partial t} + \phi_3\frac{\partial}{\partial p_n} + \phi_4\frac{\partial}{\partial r_n} + \phi_5\frac{\partial}{\partial q_n} + \phi_6\frac{\partial}{\partial s_n}.$$

We assume the Blaszak–Marciniak four-field lattice Eq. (1) is invariant under this transformation, that is

$$\frac{\mathrm{d}p^*_{n^*}}{\mathrm{d}t^*} = s^*_{n^*+1} - s^*_{n^*} + p^*_{n^*}(r^*_{n^*-2} - r^*_{n^*}), \tag{3a}$$

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$$\frac{\mathrm{d}r^*_{n^*}}{\mathrm{d}t^*} = q^*_{n^*+1} - q^*_{n^*}, \tag{3b}$$

$$\frac{\mathrm{d}q^*_{n^*}}{\mathrm{d}t^*} = p^*_{n^*+1} - p^*_{n^*} + q^*_{n^*}(r^*_{n^*-1} - r^*_{n^*}), \tag{3c}$$

$$\frac{\mathrm{d}s^*_{n^*}}{\mathrm{d}t^*} = s^*_{n^*}(r^*_{n^*-3} - r^*_{n^*}), \tag{3d}$$

then, the invariant equation can be obtained

$$\begin{aligned}
&\phi_{3t} + (\phi_{3p_n} - \phi_{2t})\frac{\mathrm{d}p_n}{\mathrm{d}t} \\
=\ & \phi_6(n+1) - \phi_6(n) + p_n[\phi_4(n-2) - \phi_4(n)] \\
&+ \phi_3(n)(r_{n-2} - r_n), \\
&\phi_{4t} + (\phi_{4r_n} - \phi_{2t})\frac{\mathrm{d}r_n}{\mathrm{d}t} \\
=\ & \phi_5(n+1) - \phi_5(n), \\
&\phi_{5t} + (\phi_{5q_n} - \phi_{2t})\frac{\mathrm{d}q_n}{\mathrm{d}t} \\
=\ & \phi_3(n+1) - \phi_3(n) + q_n[\phi_4(n-1) - \phi_4(n)] \\
&+ \phi_5(n)(r_{n-1} - r_n), \\
&\phi_{6t} + (\phi_{6s_n} - \phi_{2t})\frac{\mathrm{d}s_n}{\mathrm{d}t} \\
=\ & s_n[\phi_4(n-3) - \phi_4(n)] + \phi_6(n)(r_{n-3} - r_n).
\end{aligned}$$

Substituting Eq. (1) into the equation above, we obtain an overdetermined system of equations and by solving it, we have

$$\begin{aligned}
&\phi_1 = \alpha, \quad \phi_2 = -\beta t + \gamma, \quad \phi_3 = 3\beta p_n, \\
&\phi_4 = \beta r_n, \quad \phi_5 = 2\beta q_n, \quad \phi_6 = 4\beta s_n,
\end{aligned}$$

where α, β and γ are arbitrary parameters. Then, the infinitesimal generator becomes

$$\begin{aligned}
G =\ & \alpha\frac{\partial}{\partial n} + (-\beta t + \gamma)\frac{\partial}{\partial t} + 3\beta p_n\frac{\partial}{\partial p_n} \\
&+ \beta r_n\frac{\partial}{\partial r_n} + 2\beta q_n\frac{\partial}{\partial q_n} + 4\beta s_n\frac{\partial}{\partial s_n},
\end{aligned}$$

the generator is

$$\begin{aligned}
G_1 &= \frac{\partial}{\partial n}, \\
G_2 &= -t\frac{\partial}{\partial t} + 3p_n\frac{\partial}{\partial p_n} + r_n\frac{\partial}{\partial r_n} + 2q_n\frac{\partial}{\partial q_n} + 4s_n\frac{\partial}{\partial s_n}, \\
G_3 &= \frac{\partial}{\partial t},
\end{aligned}$$

and the commutator relation holds

$$[G_1, G_2] = 0, \quad [G_1, G_3] = 0, \quad [G_2, G_3] = G_3,$$

indicating that the underlying Lie algebra is nilpotent.

3. Reduction and explicit solutions

In this section, in order to obtain the similarity variable and the similarity transformation according to the set of symmetries (ϕ_1, ϕ_2, ϕ_3, ϕ_4, ϕ_5, ϕ_6), we firstly solve the characteristic equation

$$\frac{\mathrm{d}n}{\phi_1} = \frac{\mathrm{d}t}{\phi_2} = \frac{\mathrm{d}p_n}{\phi_3} = \frac{\mathrm{d}r_n}{\phi_4} = \frac{\mathrm{d}q_n}{\phi_5} = \frac{\mathrm{d}s_n}{\phi_6},$$

then, equation (1) can be reduced into the following two cases.

Case 1 $\beta \neq 0$, we set

$$\begin{aligned}
&\eta = n + \frac{\alpha}{\beta}\log(-\beta t + \gamma), \\
&\nu(\eta) = p_n(-\beta t + \gamma)^3, \quad \omega(\eta) = r_n(-\beta t + \gamma), \\
&\mu(\eta) = q_n(-\beta t + \gamma)^2, \ \zeta(\eta) = s_n(-\beta t + \gamma)^4,
\end{aligned}$$

then, equation (1) can be reduced into

$$\begin{aligned}
&3\beta\nu(\eta) - \alpha\frac{\mathrm{d}\nu(\eta)}{\mathrm{d}\eta} \\
=\ & \zeta(\eta+1) - \zeta(\eta) + \nu(\eta)[\omega(\eta-2) - \omega(\eta)], \\
&\beta\omega(\eta) - \alpha\frac{\mathrm{d}\omega(\eta)}{\mathrm{d}\eta} = \mu(\eta+1) - \mu(\eta), \\
&2\beta\mu(\eta) - \alpha\frac{\mathrm{d}\mu(\eta)}{\mathrm{d}\eta} \\
=\ & \nu(\eta+1) - \nu(\eta) + \mu(\eta)[\omega(\eta-1) - \omega(\eta)], \\
&4\beta\zeta(\eta) - \alpha\frac{\mathrm{d}\zeta(\eta)}{\mathrm{d}\eta} = \zeta(\eta)[\omega(\eta-3) - \omega(\eta)].
\end{aligned}$$

Case 2 $\beta = 0$ and $\gamma \neq 0$, we set

$$\eta = n - \frac{\alpha}{\gamma}t, \ \nu(\eta) = p_n, \ \omega(\eta) = r_n, \ \mu(\eta) = q_n, \ \zeta(\eta) = s_n,$$

and equation (1) can be reduced into

$$\begin{aligned}
&-\frac{\alpha}{\gamma}\frac{\mathrm{d}\nu(\eta)}{\mathrm{d}\eta} = \zeta(\eta+1) - \zeta(\eta) + \nu(\eta)[\omega(\eta-2) - \omega(\eta)], \\
&-\frac{\alpha}{\gamma}\frac{\mathrm{d}\omega(\eta)}{\mathrm{d}\eta} = \mu(\eta+1) - \mu(\eta), \\
&-\frac{\alpha}{\gamma}\frac{\mathrm{d}\mu(\eta)}{\mathrm{d}\eta} = \nu(\eta+1) - \nu(\eta) + \mu(\eta)[\omega(\eta-1) - \omega(\eta)], \\
&-\frac{\alpha}{\gamma}\frac{\mathrm{d}\zeta(\eta)}{\mathrm{d}\eta} = \zeta(\eta)(\omega(\eta-3) - \omega(\eta)).
\end{aligned}$$

Next, two particular solutions for the case $\beta = 0$ are obtained and we plot Figs. 1 and 2 choosing adequate parameters.

(i) Rational solution

$$\begin{aligned}
\nu(\eta) =\ & \frac{(\eta+\eta_0+1)(\eta+\eta_0-2)(a^4c_1^2 + 3c_3c_1^2 - a^4)}{2(\eta+\eta_0)(\eta+\eta_0-1)c_1^2 a} \\
&+ \frac{3c_3}{a(\eta+\eta_0)(\eta+\eta_0-1)},
\end{aligned} \tag{4a}$$

$$\omega(\eta) = \frac{a(\eta+\eta_0-c_2+1)(\eta+\eta_0+c_2)}{c_2(c_2-1)(\eta+\eta_0)(\eta+\eta_0+1)}, \tag{4b}$$

$$\mu(\eta) = -\frac{a^2(\eta+\eta_0-c_1)(\eta+\eta_0+c_1)}{c_1^2(\eta+\eta_0)^2}, \tag{4c}$$

$$\zeta(\eta) = \frac{c_3(\eta+\eta_0+1)(\eta+\eta_0-3)}{(\eta+\eta_0)(\eta+\eta_0-2)}, \tag{4d}$$

where $a = \alpha/\gamma$ and c_1, c_2, η_0 are arbitrary parameters. Then, it holds that

$$p_n = \frac{\gamma(n-\alpha t/\gamma+\eta_0+1)(n-\alpha t/\gamma+\eta_0-2)(\alpha^4 c_1^2/\gamma^4+3c_3c_1^2-\alpha^4/\gamma^4)}{2\alpha(n-\alpha t/\gamma+\eta_0)(n-\alpha t/\gamma+\eta_0-1)c_1^2} + \frac{3c_3\gamma}{\alpha(n-\alpha t/\gamma+\eta_0)(n-\alpha t/\gamma+\eta_0-1)}, \tag{5a}$$

$$r_n = \frac{\alpha(n-\alpha t/\gamma+\eta_0-c_2+1)(n-\alpha t/\gamma+\eta_0+c_2)}{c_2\gamma(c_2-1)(n-\alpha t/\gamma+\eta_0)(n-\alpha t/\gamma+\eta_0+1)}, \tag{5b}$$

$$q_n = -\frac{\alpha^2(n-\alpha t/\gamma+\eta_0-c_1)(n-\alpha t/\gamma+\eta_0+c_1)}{c_1^2\gamma^2(n-\alpha t/\gamma+\eta_0)^2}, \tag{5c}$$

$$s_n = \frac{c_3(n-\alpha t/\gamma+\eta_0+1)(n-\alpha t/\gamma+\eta_0-3)}{(n-\alpha t/\gamma+\eta_0)(n-\alpha t/\gamma+\eta_0-2)}. \tag{5d}$$

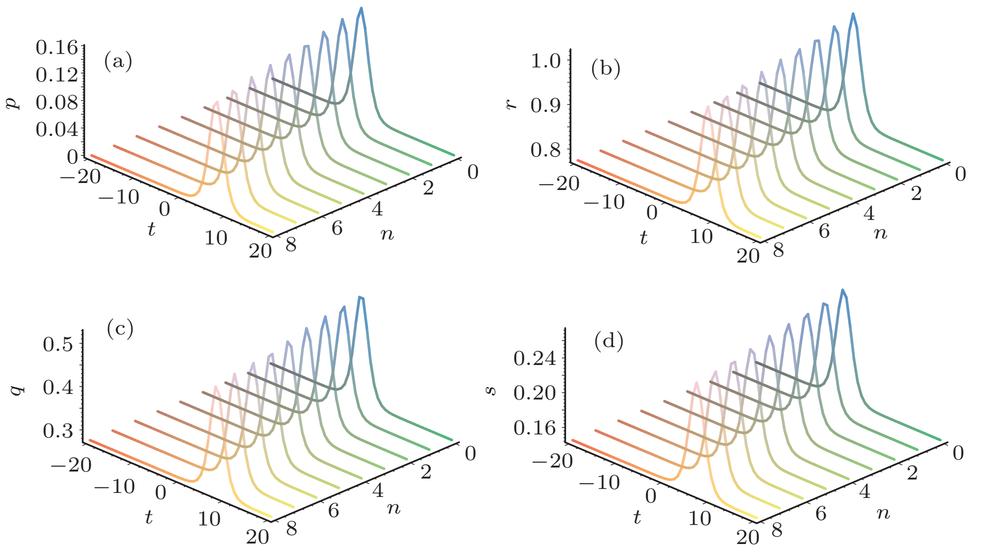

Fig. 1. (color online) Solutions (7) of (a) p_n, (b) r_n, (c) q_n, (d) s_n, with $p=1$, $\alpha=1$, $\gamma=1$, $\eta_0=0$, $c_0=1$, $c_2=2$, $c_5=2$.

(ii) Soliton solution

$$v(\eta) = \frac{a^3 e^p p^3 (c_2-1)(e^{2p}c_2-1)c_0\exp(p\eta+\eta_0)}{(e^p-1)(c_2e^p-1)^2[1+c_0\exp(p\eta+\eta_0)][c_0\exp(p\eta+\eta_0)+e^p]}, \tag{6a}$$

$$\omega(\eta) = \frac{pa(e^p-1)\{1+c_5c_0\exp[p(\eta+1)+\eta_0]\}[c_5+c_0\exp(p\eta+\eta_0)]}{(-1+c_5)(c_5e^p-1)\{1+c_0\exp[p(\eta+1)+\eta_0]\}[1+c_0\exp(p\eta+\eta_0)]}, \tag{6b}$$

$$\mu(\eta) = -\frac{e^p p^2 a^2\{1+c_2c_0\exp[p(\eta+1)+\eta_0]\}\{c_2+c_0\exp[p(\eta-1)+\eta_0]\}}{(c_2e^p-1)^2[1+c_0\exp(p\eta+\eta_0)]^2}, \tag{6c}$$

$$\zeta(\eta) = -\frac{a^4e^{3p}p^4(c_2-1)(e^{2p}c_2-1)\{1+c_0\exp[p(\eta-3)+\eta_0]\}\{1+c_0\exp[p(\eta+1)+\eta_0]\}}{(e^{2p}+e^p+1)(e^p-1)^4(c_2e^p-1)^2\{1+c_0\exp[p(\eta-2)+\eta_0]\}[1+c_0\exp(p\eta+\eta_0)]}, \tag{6d}$$

where p, c_0, c_2, c_5 are arbitrary parameters. Then, it follows that

$$p_n = \frac{\alpha^3e^pp^3(c_2-1)(e^{2p}c_2-1)c_0\exp[p(n-\alpha t/\gamma)+\eta_0]}{\gamma^3(e^p-1)(c_2e^p-1)^2\{1+c_0\exp[p(n-\alpha t/\gamma)+\eta_0]\}\{c_0\exp[p(n-\alpha t/\gamma)+\eta_0]+e^p\}}, \tag{7a}$$

$$r_n = \frac{p\alpha(e^p-1)\{1+c_5c_0\exp[p((n-\alpha t/\gamma)+1)+\eta_0]\}\{c_5+c_0\exp[p(n-\alpha t/\gamma)+\eta_0]\}}{\gamma(-1+c_5)(c_5e^p-1)\{1+c_0\exp[p((n-\alpha t/\gamma)+1)+\eta_0]\}\{1+c_0\exp[p(n-\alpha t/\gamma)+\eta_0]\}}, \tag{7b}$$

$$q_n = -\frac{e^pp^2\alpha^2\{1+c_2c_0\exp[p((n-\alpha t/\gamma)+1)+\eta_0]\}\{c_2+c_0\exp[p((n-\alpha t/\gamma))-1)+\eta_0]\}}{\gamma^2(c_2e^p-1)^2\{1+c_0\exp[p(n-\alpha t/\gamma)+\eta_0]\}^2}, \tag{7c}$$

$$s_n = -\frac{\alpha^4e^{3p}p^4(c_2-1)(e^{2p}c_2-1)\{1+c_0\exp[p((n-\alpha t/\gamma)-3)+\eta_0]\}}{\gamma^4(e^{2p}+e^p+1)(e^p-1)^4(c_2e^p-1)^2\{1+c_0\exp[p((n-\alpha t/\gamma)-2)+\eta_0]\}} \times \frac{\{1+c_0\exp[p((n-\alpha t/\gamma)+1)+\eta_0]\}}{\{1+c_0\exp[p(n-\alpha t/\gamma)+\eta_0]\}}. \tag{7d}$$

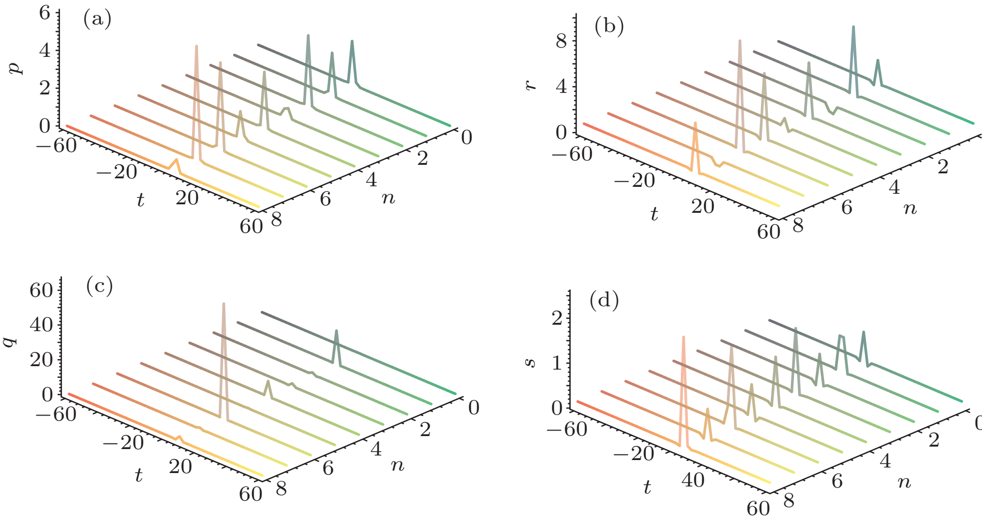

Fig. 2. (color online) Solutions (7) of (a) p_n, (b) r_n, (c) q_n, (d) s_n, with $p=1$, $\alpha=1$, $\gamma=1$, $\eta_0=0$, $c_0=-1$, $c_2=2$, $c_5=2$.

4. Generalized symmetries and conservation laws

In this section, we firstly discuss the generalized symmetries to the Blaszak–Marciniak four-field lattice equation (1) through an algorithmic approach developed by Hereman *et al.*[13–16] It is obvious that equation (1) is invariant under the scaling symmetry

$$(t,p_n,r_n,q_n,s_n)\rightarrow(\lambda t,\lambda^{-3}p_n,\lambda^{-1}r_n,\lambda^{-2}q_n,\lambda^{-4}s_n). \quad (8)$$

Hence, p_n, r_n, q_n, s_n correspond to three, one, two and four derivatives with respect to t, respectively. Next, based on the definition of the "rank" proposed by Göktas and Hereman,[13–16] we firstly derive the generalized symmetries of ranks (3, 2, 4, 5) denoted by $g_1^{(1)}$, $g_2^{(1)}$, $g_3^{(1)}$, $g_4^{(1)}$. Forming all the monomials in p_n, r_n, q_n, s_n of rank 2 gives the following:[13–16]

$$\mathcal{L}_2=\{q_n,r_n^2,r_n\},$$

then, we introduce the necessary t derivatives in each monomial of $\mathcal{L}_2$. Using Eq. (1), we have

$$\mathcal{R}_2=\{r_n^2,q_n,q_{n+1}\},$$

then, $g_2^{(1)}$ can be obtained by considering a linear combination of terms in $\mathcal{R}_2$, that is

$$g_2^{(1)}=b_1r_n^2+b_2q_n+b_3q_{n+1}. \quad (9)$$

In a similar way, $g_1^{(1)}$, $g_3^{(1)}$, $g_4^{(1)}$ can be expressed

$$\begin{aligned}
g_1^{(1)} = {} & a_1r_n^4+a_2q_{n+1}r_n^2+a_3q_{n+1}r_{n+1}^2+a_4q_nr_n^2\\
&+a_5q_nr_nr_{n-1}+a_6q_{n+1}r_nr_{n+1}\\
&+a_7q_nr_{n-1}^2+a_8q_n^2+a_9q_{n+1}q_{n+2}+a_{10}p_nr_n\\
&+a_{11}p_{n+1}r_n+a_{12}p_{n+2}r_n+a_{13}p_nr_{n-1}\\
&+a_{14}p_{n+1}r_{n+1}+a_{15}p_{n+2}r_{n+1}+a_{16}q_{n+1}^2\\
&+a_{17}p_nr_{n-2}+a_{18}p_{n+1}r_{n-1}+a_{19}p_{n+2}r_{n+2}\\
&+a_{20}q_nq_{n-1}+a_{21}q_nq_{n+1}+a_{22}s_n\\
&+a_{23}s_{n+1}+a_{24}s_{n+2}+a_{25}s_{n+3},\\
g_3^{(1)} = {} & c_1r_n^3+c_2q_nr_{n-1}+c_3q_{n+1}r_n+c_4q_{n+1}r_{n+1}\\
&+c_5q_nr_n+c_6p_n+c_7p_{n+1}+c_8p_{n+2},\\
g_4^{(1)} = {} & d_1r_{n+1}s_{n+2}+d_2r_ns_{n+3}+d_3q_nr_{n-1}^3+d_4q_{n+1}^2r_{n+1}\\
&+d_5p_{n+2}r_{n+1}r_{n+2}+d_6q_nq_{n-1}r_{n-2}\\
&+d_7q_nq_{n+1}r_{n+1}+d_8q_nq_{n+1}r_{n-1}+d_9q_{n+1}q_{n+2}r_{n+1}\\
&+d_{10}q_{n+1}q_{n+2}r_{n+2}+d_{11}q_nq_{n+1}r_n\\
&+d_{12}q_{n+1}q_{n+2}r_n+d_{13}p_{n+1}r_nr_{n-1}+d_{14}p_{n+2}r_nr_{n+2}\\
&+d_{15}q_nq_{n-1}r_n+d_{16}q_nr_nr_{n-1}^2\\
&+d_{17}q_{n+1}r_nr_{n+1}^2+d_{18}p_nr_nr_{n-1}+d_{19}p_{n+1}r_nr_{n+1}\\
&+d_{20}p_{n+2}r_nr_{n+1}+d_{21}q_nr_n^2r_{n-1}\\
&+d_{22}q_{n+1}r_n^2r_{n+1}+d_{23}p_nr_nr_{n-2}+d_{24}p_nr_{n-2}r_{n-1}\\
&+d_{25}q_nq_{n-1}r_{n-1}+d_{26}p_{n+1}r_{n-1}r_{n+1}\\
&+d_{27}p_{n-1}q_n+d_{28}q_n^2r_{n-1}+d_{29}r_{n+1}s_{n+3}\\
&+d_{30}p_{n+2}r_{n+2}^2+d_{31}p_{n+1}q_n+d_{32}q_{n+1}r_n^3\\
&+d_{33}r_{n-2}s_n+d_{34}p_{n+1}r_{n-1}^2+d_{35}p_{n+2}q_n\\
&+d_{36}p_{n+2}q_{n+2}+d_{37}r_{n-1}s_{n-2}+d_{38}p_{n+1}r_{n+1}^2\\
&+d_{39}p_nr_{n-1}^2+d_{40}p_{n+2}q_{n+3}+d_{41}p_{n+1}q_{n-1}\\
&+d_{42}r_{n-2}s_{n+1}+d_{43}p_{n+1}q_{n+2}+d_{44}p_nq_{n+1}\\
&+d_{45}p_nq_{n-1}+d_{46}r_{n-1}s_{n+1}+d_{47}r_ns_n+d_{48}q_n^2r_n\\
&+d_{49}r_ns_{n+2}+d_{50}p_{n+2}r_n^2+d_{51}p_{n+1}r_n^2\\
&+d_{52}p_nr_n^2+d_{53}p_{n+2}r_{n+1}^2+d_{54}p_{n+3}q_{n+1}\\
&+d_{55}r_{n-1}s_n+d_{56}r_ns_{n+1}+d_{57}r_{n+1}s_{n+1}+d_{58}r_n^5\\
&+d_{59}r_{n-3}s_n+d_{60}q_{n+1}^2r_n+d_{61}p_nq_n+d_{62}r_{n+2}s_{n+3}\\
&+d_{63}p_nq_{n-2}+d_{64}q_nr_n^3+d_{65}p_{n+2}q_{n+1}
\end{aligned}$$

$$+d_{66}p_nr_{n-2}^2+d_{67}r_{n+3}s_{n+3}+d_{68}r_{n+2}s_{n+2}$$
$$+d_{69}p_{n+1}q_{n+1}+d_{70}q_{n+1}r_{n+1}^3.$$

To determine the coefficients above, we introduce a lattice equation[13,14]

$$\frac{\mathrm{d}p}{\mathrm{d}\tau}=g_1^{(1)},\ \frac{\mathrm{d}r}{\mathrm{d}\tau}=g_2^{(1)},\ \frac{\mathrm{d}q}{\mathrm{d}\tau}=g_3^{(1)},\ \frac{\mathrm{d}s}{\mathrm{d}\tau}=g_4^{(1)},$$

and

$$D_{\rm t}g_1^{(1)}=D_\tau F_1,\ D_{\rm t}g_2^{(1)}=D_\tau F_2,\ D_{\rm t}g_3^{(1)}=D_\tau F_3,\ D_{\rm t}g_4^{(1)}=D_\tau F_4,$$

where $D_{\rm t}$ is the total derivative, F_1, F_2, F_3 and F_4 are the right-hand side terms of Eq. (1). Then, the coefficients can be determined and the symmetries can be expressed as

$$\begin{aligned}
g_1^{(1)}&=s_{n+1}-s_n+p_nr_{n-2}-r_np_n,\\
g_2^{(1)}&=q_{n+1}-q_n,\\
g_3^{(1)}&=p_{n+1}-p_n+q_nr_{n-1}-r_nq_n,\\
g_4^{(1)}&=s_nr_{n-3}-r_ns_n.
\end{aligned}$$

In a similar procedure, we can obtain the generalized symmetries of ranks (5, 4, 3, 6)

$$\begin{aligned}
g_1^{(2)} &= r_{n+1}s_{n+1}+p_nq_{n+1}+r_ns_{n+1}+p_nr_{n-2}^2+p_nq_n\\
&\quad -p_nq_{n-1}-r_{n-2}s_n-r_{n-3}s_n-p_nq_{n-2}-p_nr_n^2,\\
g_2^{(2)} &= -q_nr_{n-1}+q_{n+1}r_n+q_{n+1}r_{n+1}-r_nq_n+p_n-p_{n+2},\\
g_3^{(2)} &= s_n-s_{n+2}-p_nr_{n-1}-q_nr_n^2-p_nr_{n-2}-q_nq_{n-1}\\
&\quad +p_{n+1}r_n+q_nr_{n-1}^2+p_{n+1}r_{n+1}+q_nq_{n+1},\\
g_4^{(2)} &= -r_n^2s_n+s_nq_n+s_nq_{n+1}+s_nr_{n-3}^2-s_nq_{n-2}-s_nq_{n-3}.
\end{aligned}$$

It is straightforward to derive the generalized symmetries with ranks ≥ 7, therefore, we refrain from presenting those here.

After that, we begin to construct conservation laws of Eq. (1). We define ρ_n is the conserved density and J_n is the associated flux, such that

$$\frac{\mathrm{d}\rho_n}{\mathrm{d}t}=J_n-J_{n+1},\tag{10}$$

we firstly consider the conserved density with rank 2. For the terms of $\mathcal{R}_2$, using the criterion applied in Refs. [13] and [14], we set $q_n\equiv q_{n+1}$ and obtain $\mathcal{M}_2=\{r_n^2,q_{n+1}\}$, then the conserved density with rank 2 can be obtained by linearly combining the terms in $\mathcal{M}_2$

$$\rho_n^{(2)}=c_1q_{n+1}+c_2r_n^2.$$

Using Eq. (10), we have

$$\begin{aligned}
\frac{\mathrm{d}\rho_n^{(2)}}{\mathrm{d}t} &= (c_1+2c_2)q_{n+1}r_n+(-c_1-2c_2)q_{n+1}r_{n+1}\\
&\quad +(J_n^{(2)}-J_{n+1}^{(2)}),
\end{aligned}$$

with

$$J_n^{(2)}=-c_1p_{n+1}-2c_2r_nq_n.$$

Considering Eq. (10) and choosing $c_2=1/2$, we have $c_1=-1$. Then, it holds that

$$\begin{aligned}
\rho_n^{(2)}&=-q_{n+1}+\frac{1}{2}r_n^2,\\
J_n^{(2)}&=p_{n+1}-q_nr_n.
\end{aligned}$$

In a similar way, we can obtain a sequence of conservation laws

$$\begin{aligned}
\rho_n^{(3)} &= p_{n+2}+\frac{1}{3}r_n^3-q_{n+1}r_n-q_{n+1}r_{n+1},\\
J_n^{(3)} &= -s_{n+2}+q_nq_{n+1}+p_{n+1}r_n-q_nr_n^2+p_{n+1}r_{n+1},\\
\rho_n^{(4)} &= -s_{n+3}+\frac{1}{2}q_{n+1}^2+\frac{1}{4}r_n^4+p_{n+2}r_{n+2}\\
&\quad +q_{n+1}q_{n+2}-q_{n+1}r_n^2-q_{n+1}r_{n+1}^2+r_np_{n+2}\\
&\quad +r_{n+1}p_{n+2}-r_nq_{n+1}r_{n+1},\\
J_n^{(4)} &= -p_{n+1}q_{n+1}-p_{n+1}q_{n+2}+p_{n+1}r_n^2+p_{n+1}r_{n+1}^2\\
&\quad -q_np_{n+2}-q_nr_n^3-s_{n+2}r_n-s_{n+2}r_{n+1}\\
&\quad -s_{n+2}r_{n+2}+p_{n+1}r_{n+1}r_n+2q_nr_nq_{n+1}+q_nr_{n+1}q_{n+1},\\
\rho_n^{(5)} &= \frac{1}{5}r_n^5-p_{n+2}q_{n+2}-p_{n+2}q_{n+3}+p_{n+2}r_{n+2}^2\\
&\quad -q_{n+1}p_{n+2}-q_{n+1}p_{n+3}-q_{n+1}r_n^3-q_{n+1}r_{n+1}^3\\
&\quad +q_{n+1}^2r_n+q_{n+1}^2r_{n+1}-r_ns_{n+3}+r_n^2p_{n+2}\\
&\quad -r_{n+1}s_{n+3}+r_{n+1}^2p_{n+2}-r_{n+2}s_{n+3}-s_{n+3}r_{n+3}\\
&\quad +q_{n+1}q_{n+2}r_{n+2}+q_{n+1}r_nq_{n+2}+2q_{n+1}r_{n+1}q_{n+2}\\
&\quad +r_np_{n+2}r_{n+2}-r_nq_{n+1}r_{n+1}^2+r_nr_{n+1}p_{n+2}\\
&\quad -r_n^2q_{n+1}r_{n+1}+r_{n+1}p_{n+2}r_{n+2},\\
J_n^{(5)} &= -s_{n+2}r_{n+2}r_n-s_{n+2}r_{n+1}r_n-s_{n+2}r_{n+2}r_{n+1}\\
&\quad -q_nq_{n+2}q_{n+1}-q_nr_{n+2}p_{n+2}+q_nr_{n+1}^2q_{n+1}\\
&\quad -q_np_{n+2}r_{n+1}-p_{n+1}r_{n+2}q_{n+2}-p_{n+1}q_{n+2}r_n\\
&\quad -2p_{n+1}q_{n+2}r_{n+1}+p_{n+1}r_{n+1}^2r_n+p_{n+1}r_{n+1}r_n^2\\
&\quad +3q_nr_n^2q_{n+1}-2q_np_{n+2}r_n-2p_{n+1}r_nq_{n+1}\\
&\quad -2p_{n+1}r_{n+1}q_{n+1}-s_{n+2}r_{n+1}^2+s_{n+2}q_{n+1}\\
&\quad -s_{n+2}r_{n+2}^2-s_{n+2}r_n^2+s_{n+2}q_{n+3}+s_{n+2}q_{n+2}\\
&\quad -q_nq_{n+1}^2+q_ns_{n+3}+p_{n+1}p_{n+3}+p_{n+1}r_n^3\\
&\quad +p_{n+1}r_{n+1}^3+p_{n+1}p_{n+2}-q_nr_n^4+2q_nr_{n+1}q_{n+1}r_n.
\end{aligned}$$

It is straightforward to derive the conservation laws with higher order, so we omit it here.

5. Conclusions

In this paper, we discussed the Lie point symmetries of one Blaszak–Marciniak four-field lattice equation and two interesting solutions of a rational form and a soliton form to the reduced equation are given. A sequence of generalized symmetries and conservation laws are derived, which further confirms the integrability of the BM system. However, it is well known that once the generalized symmetries of a soliton equation are obtained, the master symmetries and hereditary operators may also be constructed.[13–16] Therefore, motivated by the remarkable work of Sahadevan and his collaborators,[13,14] it is still of interest to investigate its master symmetries, hereditary operators, etc.

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