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Research paper

The special class of second integrals of the KdV equation

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ABSTRACT

The special class of second integrals, distinguished for its infinite dimension, emerges naturally when generalizing the second integrals of ordinary differential equations to partial differential equations. The conserved quantities of the KdV equation are a special class of second integrals. We proved its uniqueness under the assumption on the cofactor operator. The special class is so peculiar that to find it is almost an algorithm. Thus we managed to generalize the special class of the conserved quantities of the KdV equation to a new 2-parameter special class. Among the special classes, the special class of nonlocal second integrals plays an extra role. As an example, the special class that generates the multi-soliton solutions of the KdV equation is presented.

1. Introduction

For convenience, consider an ordinary differential equation (ODE)

$$\dot{X} = F(X), \tag{1}$$

where $X = (x_1, x_2, \cdots, x_n)^T$, $F(X) = (f_1(X), \cdots, f_n(X))^T$, $x_i = x_i(t)$, and the dot means the derivative with respect to t. The second integral[1] $J = J(X)$ of (1) is a function that satisfies

$$\dot{J} = \alpha J, \tag{2}$$

where $\alpha = \alpha(X)$ is called the cofactor and X is confined as any solution of (1). The first integrals are extremely important second integrals with $\alpha = 0$. The definition of the second integral is generalized if J is a vector and α is a matrix. The third integral, see Goriely [1] for the definition, is the case that α is triangular 2×2 matrix. In literature, the second integrals have many names, such as the Darboux polynomials, eigenpolynomials, algebraic invariant manifolds, special integrals, stationary solutions, etc. Darboux [2], Poincaré [3], Painlevé [4] are among those pioneers who studied the second integrals of ODEs. To date, researches on the second integrals of the ODEs are still hot, see e.g. [5] and the references therein.

Now let us generalize the concept of second integral from ODE to partial differential equation (PDE). For simplicity we assume that the independent variables are $(x,\ t)$ and that u is a vector $u = (u_1(x,t), u_2(x,t), \cdots, u_n(x,t))$. We also assume that the PDE is of evolution type

$$\frac{\partial u}{\partial t} = F[u], \tag{3}$$

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[1] For simplicity we only consider time-independent second integrals.

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where $[u]$ denotes u and its derivatives with respect to x.

Definition. The second integral of PDE (3) is a function $J = J[u]$ that satisfies $\dot{J} = \alpha J$ when u is confined as any solution of (3) and α is a linear differential operator matrix. We will call α the cofactor operator.

Remark 1. In fact, the above definition of the second integral for PDE (3) is practical and is only appropriate for local ones. In general, J may be nonlocal, for example, it is possible that J depends on $\int udx$. In the following, unless it is declared as nonlocal, we assume $J = J[u]$ for PDEs.

In the following we assume that α is a scalar (for ODEs) or a linear differential operator (for PDEs).

Example 1. [6] Let $J = q^3r + q_x^2 - qq_{xx}$ and the PDE for (q, r) be the third order AKNS system

$$\begin{aligned} q_t &= -q_{xxx} + 6qrq_x, \\ r_t &= -r_{xxx} + 6qrr_x. \end{aligned} \tag{4}$$

We can verify $\frac{d}{dt}J = \left(-\partial^3 + 3q^{-1}q_x\partial^2 + 3q^{-1}(2q^2r - q_{xx})\partial\right)J$, where $\partial = \frac{d}{dx}$. So J is a second integral of (4).

In contrast to the case of the ODEs, the second integrals are much less used to study PDEs. One of the early applications of second integral to PDE is Lax's work on the KdV equation in 1976 [7]. In his pioneer work, he proved that the functional minimizing $F_{N+1}[u]$ subject to the constraint $F_j[u] = A_j$ $(j = 0, \cdots, N)$ forms $N-$dimensional tori which are invariant under the KdV flow, where $F_j (j = 0, \cdots, N)$ are the well-known conserved quantities. These invariant tori (the second integrals of the KdV equation) are latter known to describe the algebraic-geometric solutions. The KdV equation is one of such cases that the second integrals are generated by their infinite symmetries. Thus it is surprising that Bagderina found a new infinite sequence of invariant manifolds for the Sawada–Kotera equation in addition to the known two sequences of its symmetries and conservation laws [8].

The second integral is important for many reasons. Firstly, $J = 0$ is an invariant space of (1), i.e., the solution $X = X(t)$ will satisfy $J(X) = 0$ if at some time $t = t_0$ we have $X(t_0) = X_0$ and $J(X_0) = 0$. In dynamics analysis, the invariant space $J = 0$ is a barrier for the motion. Also $J = 0$ is often used to solve the particular solutions for an ODE or PDE. Secondly, the second integrals have deep relations with the first integrals. The Darboux theorem for the second integrals of an ODE establishes a link between the first integrals and the second integrals.

Theorem 1. *[1] Let F in (1) be a polynomial vector field of degree d and the flow admits q irreducible Darboux polynomials* $(J_1, J_2, \cdots, J_q)$. *Then, if* $q > \binom{n+d-1}{n}$, *the system admits a first integral of the form*

$$I = \prod_{i=1}^{q} J_i^{\lambda_i},$$

where $\lambda_i \in \mathbb{C}$. *Moreover,* $q > \binom{n+d-1}{n} + n$ *if and only if the system admits a rational first integral (i.e.,* $\lambda_i \in \mathbb{Z}$*).*

The first half of Theorem 1 was proved by Darboux [9]. The latter half was showed by Jouanolou in his book [10], see [11,12] for simplified proofs. Theorem 1 means that without additional constraints we have to know many enough second integrals to construct a first integral. But a well-known result is that if Q_1 and Q_2 are two second integrals with the same cofactor, then $\frac{Q_1}{Q_2}$ is a first integral [1]. The conclusion is still true even without the polynomial conditions for Q_1, Q_2 and F. Thus it is reasonable to classify the second integrals by their cofactors. An *irreducible second integral* with cofactor α will be said to belong to α-class. For ODEs there is no second integral that belongs to both α and β class if $\alpha \neq \beta$. For example, if there is a Q that belongs both to α and β, then $\alpha = \frac{\dot{Q}}{Q} = \beta$. So this classification is well-defined for ODEs. The 0-class is crucial since it contains the first integrals. Furthermore, we can define addition and scalar multiplication in the second integrals of α-class to form vector space. Thus the dimension of the α-class makes sense as well. The classification can be generalized to PDEs.

Remark 2. If the α-class is defined for PDE as the same as ODE, it will be possible that the second integral Q belongs to both α-class and β-class, $\alpha \neq \beta$. For example, if $\dot{Q} = Q_{xx}Q = \left(Q\partial^2\right)Q$, we can say that Q belongs to both Q_{xx}-class and $Q\partial^2$-class. To overcome the non-unique problem, we impose constraints on the form of α. Suppose the order of Q is q and α is of form $\alpha = \sum_{j=0}^{N} \alpha_j \partial^j$. We demand that the order of α_j is no more than $j + q$. Then the cofactor operator α is unique for Q.

Moreover, for PDEs we can also define the *special class*.

Definition. The α-class is called *special* if its dimension is infinity.

Remark 3. The special class does not exist in any ODE system, or else the ODE system will have infinitely many first integrals.

Remark 4. $J = 0$ for a special class generates not a single solution but a hierarchy of solutions for the PDE since J can have arbitrary number of parameters.

This paper investigates the special class of second integrals of the KdV equation [13]

$$u_t = u_{xxx} + 6uu_x. \tag{5}$$

Though the KdV equation has been studied so vastly and deeply by so many classical methods, such as the inverse scattering, the Darboux transformation, Hirota direct method, algebraic-geometric method, etc. see e.g. [14–19] and the references therein, quite surprisingly, some aspects of the second integrals of (5) found below are completely new and unexpected.

The main idea of this paper comes from the work of Li et al. [20] where they examined some special forms of Lax pairs of several integrable systems. They found that the special forms of Lax pairs spontaneously lead to constraints for the integrable system. A surprising induction of them is that all examined closed-form hierarchy solutions, including the algebro-geometric solutions, come from those constraints. The induction means that all examined closed-form hierarchy solutions come from the special class of second integrals by $J = 0$ or $J = C$ depending on the form of the cofactor operator. So they concluded that the special classes play an important role for integrable PDEs. However, they did not investigate the 'completeness' for those special classes, i.e., they did not answer how many special classes there are for a given PDE.

Our first result is the following theorem.

Theorem 2. *For the KdV Eq. (5), if the cofactor differential operator α_0 for the special class of second integrals depends only on u, i.e., no derivatives of u appear in α_0, then α_0 is unique and $\alpha_0 = \partial^3 + 6u\partial$.*

Maybe Theorem 2 is natural for those who are familiar with the conserved quantities ρ_{2j} of the KdV equation because $\frac{d}{dt}\rho_{2j} = (6u\partial + \partial^3)\rho_{2j}$ and ρ_{2j} are differential polynomials of u. One may also immediately recall that the symmetries σ_{2j+1} of the KdV equation satisfy $\frac{d}{dt}\sigma_{2j+1} = (\partial^3 + 6u\partial + 6u_x)\sigma_{2j+1}$. Considering that no other such kind of quantities is known for the KdV equation, he would guess that the cofactor of such kind is unique.

Conjecture 1. *For the KdV Eq. (5), if the cofactor differential operator α_1 for the special class of second integrals depends only on u and u_x, then α_1 is unique and $\alpha_1 = \partial^3 + 6u\partial + 6u_x$.*

The conjecture is still unproven. However, it has been verified for the lower order second integrals. Considering Theorem 2 and Conjecture 1, our next result will be completely out of expect.

Theorem 3. *Let*

$$d_0 = 3 + c_1(u + c_2)^2. \tag{6}$$

For any $c_1, c_2 \in \mathbb{C}$, there is a cofactor differential operator

$$\alpha_2 = \partial^3 + \left(6u + 36c_1\frac{u_x^2}{d_0^2} + 6c_1\frac{(u + c_2)u_{xx} - u_x^2}{d_0}\right)\partial + \left(-72c_1^2\frac{(u + c_2)u_x^3}{d_0^3} + 36c_1\frac{u_xu_{xx}}{d_0^2}\right) \tag{7}$$

for the special class of second integrals of the KdV equation.

Notice that the cofactor in Theorem 3 with $c_1 = 0$ reduces to the one in Theorem 2. The cofactor in Theorem 2 is associated with the Lax pair $\hat{L}_1 = \partial^2 + 4u - 2\partial^{-1}u_x$, $\hat{P}_1 = \partial^3 + 6u\partial$; while the cofactor in Conjecture 1 is related to the Lax pair $L_2 = \partial^2 + 4u + 2u_x\partial^{-1}$, $\hat{P}_2 = \partial^3 + 6u\partial + 6u_x$. Therefore, one may expect the cofactor in Theorem 3 is also related to some unusual Lax pair. Since the cofactor in Theorem 3 is a generalization of the cofactor in Theorem 2 and the second integrals in Theorem 2 generate the algebro-geometric solutions, one may wonder whether the second integrals in Theorem 3 generate new hierarchy of solutions that generalize the algebro-geometric solutions of the KdV equation. Note that the second integral equation $\frac{d}{dt}J = \alpha J$ with α defined in Theorem 3 breaks the scalar symmetry $\partial \to k\partial$, $\frac{d}{dt} \to k^3\frac{d}{dt}$, $u \to k^2u$ for fixed c_1 and c_2. But if we demand $c_1 \to k^{-4}c_1$, $c_2 \to k^2c_2$ in addition to the scalar symmetry, then the form of the second integral equation is invariant. So the special classes defined by Theorem 3 with different (c_1, c_2) should be studied as a whole.

The following notations for the derivatives with respect to x, t, u or u_x etc. have been adopted: if $f = f(u)$, then $f' = \frac{df}{du}$; $\dot{f} = f_t = \frac{df}{dt}$; $f' = f_x = \frac{df}{dx}$; $f_u = \frac{\partial f}{\partial u}$; $f_{u_x} = \frac{\partial f}{\partial u_x}$.

The paper is organized as follows. Section 2 gives some examples to illustrate the basic facets of the second integral for a PDE. Section 3 and 4 are devoted to the proofs of Theorems 2 and 3, respectively. Section 5 studies the $J = 0$ solutions of α_2-class by some examples. Section 6 deals with a special class of nonlocal second integrals. Section 7 is the discussions.

2. Simple examples

In this section, we will illustrate the basic facets of second integrals by examining the simplest second integrals of the KdV Eq. (5).

Example 2. u is a second integral of the KdV equation (5), since $\frac{d}{dt}u = \left(\partial^3 + 6u\partial\right)u$.

Example 3. u^2 is not a second integral of the KdV Eq. (5).

$$\frac{d}{dt}u^2 = \left(\partial^3 - 3\frac{u_x}{u}\partial^2 + 3\left(2u + \frac{u_x^2}{u^2}\right)\partial\right)u^2.$$

But the cofactor operator $\partial^3 - 3\frac{u_x}{u}\partial^2 + 3\left(2u+\frac{u_x^2}{u^2}\right)\partial$ is singular when it is confined on $u^2 = 0$.

Example 3 shows a difference between the second integral of a PDE and the second integral of an ODE: for an ODE J^2 must be a second integral if J is. Note this is a very special example: in most cases the second integrals can be transformed freely from one form to another form, see Examples 5 and 6.

Example 4. $u^2 + c$ is a second integral of the KdV Eq. (5) if $c \neq 0$.

$$\frac{d}{dt}(u^2 + c) = \left(\partial^3 - 3\frac{u_x}{u}\partial^2 + 3\left(2u + \frac{u_x^2}{u^2}\right)\partial\right)(u^2 + c).$$

Example 5. Find f_1 such that $J = u_x + f_1(u)$ is a second integral of the KdV Eq. (5).

$$\begin{aligned}\frac{d}{dt}J = &\left(\partial^3 + (6u - 3f_1''u_x)\partial + \left(-f_1'''u_x^2 + (6 + 3f_1'f_1'' + f_1f_1''')(u_x - f_1)\right)\right)J \\ &+ f_1^2\left(6 + 3f_1' + f_1'' + f_1f_1'''\right).\end{aligned} \tag{8}$$

Therefore, $f_1 = 0$ or $f_1(u) = \pm\sqrt{-2u^3 + c_1u^2 + c_2u + c_3}$ make $J = u_x + f_1(u)$ a second integral of the KdV equation. Note $f_1(u) = \pm\sqrt{-2u^3 + c_1u^2 + c_2u + c_3}$ is the solution of $6 + 3f_1' + f_1'' + f_1f_1''' = 0$.

Example 6.

$$J = u_x^2 + 2u^3 - c_1u^2 - c_2u - c_3 \tag{9}$$

is a second integral of the KdV Eq. (5).

$$\frac{d}{dt}J = \left(\partial^3 - 3\frac{u_{xx}}{u_x}\partial^2 + 3\left(2u + \frac{u_{xx}^2}{u_x^2}\right)\partial\right)J.$$

In fact the second integral $u_x \pm \sqrt{-2u^3 + c_1u^2 + c_2u + c_3}$ in Example 5 and the second integral $u_x^2 + 2u^3 - c_1u^2 - c_2u - c_3$ in Example 6 can be regarded as equivalent since they can be converted to each other. Both $J = 0$ in Example 5 or in Example 6 generate the traveling wave solution of the KdV equation.

Example 7. If the second integral of the KdV Eq. (5) is of form $J = u_{xx} + f(u, u_x)$, then f will satisfy

$$\begin{aligned}&f^3 f_{u_xu_xu_x} + 3u_x^2(2f_{u_x} + f_u f_{uu_x}) + 3f^2(f_{u_x}f_{u_xu_x} - f_{uu_x} - u_x f_{uu_xu_x}) \\ &-3u_x f(6 + f_{u_xu_x}f_u + f_{u_x}f_{uu_x} - f_{uu}) + 3u_x^2 f f_{uuu_x} - u_x^3 f_{uuu} = 0.\end{aligned} \tag{10}$$

Since (10) is a complicated PDE, there is no hope to solve it completely. The group invariant solutions of (10) are not so important since at that time $J = 0$ only generates the group invariant solutions of the KdV equation itself.

Example 7 illustrates the difficulty to seek general high-order second integrals.

3. Proof of Theorem 2

Let us first determine the rough structure of the cofactor operator α for the KdV equation.

Lemma 1. *The α for (5) is of form $\alpha = g_0[u] + g_1[u]\partial + g_2[u]\partial^2 + g_3[u]\partial^3$.*

Proof. Let the second integral be $J = J(u, u^{(1)}, \cdots, u^{(N)})$, where $u^{(j)} = \frac{\partial^j u}{\partial x^j}$. Then $\frac{d}{dt}J = \sum_{j=0}^N \frac{\partial J}{\partial u^{(j)}}\frac{du^{(j)}}{dt} = \sum_{j=0}^N \frac{\partial J}{\partial u^{(j)}}\frac{d^j(u_{xxx}+6uu_x)}{dx^j}$. So the highest order of $\frac{d}{dt}J$ is $N+3$. Suppose $\alpha = \sum_{j=0}^{N_\alpha} g_j[u]\partial^j$. Then the highest order of αJ is $N + N_\alpha$ by Remark 2. So $N + 3 = N + N_\alpha$, i.e. $N_\alpha = 3$. Therefore α must have form $\alpha = g_0[u] + g_1[u]\partial + g_2[u]\partial^2 + g_3[u]\partial^3$. □

Lemma 2. $g_3[u] = 1$.

Proof. Still suppose $J = J(u, u^{(1)}, \cdots, u^{(N)})$. Then the highest order term of $\frac{d}{dt}J$ is $\frac{\partial J}{\partial u^{(N)}}u^{(N+3)}$. The highest order term of αJ is $g_3[u]\frac{\partial J}{\partial u^{(N)}}u^{(N+3)}$. So $g_3[u] = 1$. □

Let $J = J(u, u^{(1)}, \cdots, u^{(N)})$. Then

$$\frac{dJ}{dx} = \sum_{i=0}^N \frac{\partial J}{\partial u^{(i)}}u^{(i+1)},$$

$$\frac{d^2J}{dx^2} = \sum_{i=0}^{N}\sum_{j=0}^{N}\frac{\partial^2 J}{\partial u^{(i)}\partial u^{(j)}}u^{(i+1)}u^{(j+1)} + \sum_{i=0}^{N}\frac{\partial J}{\partial u^{(i)}}u^{(i+2)},$$

$$\begin{aligned}\frac{d^3J}{dx^3} &= \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N}\frac{\partial^3 J}{\partial u^{(i)}\partial u^{(j)}\partial u^{(k)}}u^{(i+1)}u^{(j+1)}u^{(k+1)}\\ &\quad+\sum_{i=0}^{N}\sum_{j=0}^{N}\frac{\partial^2 J}{\partial u^{(i)}\partial u^{(j)}}\left(u^{(i+2)}u^{(j+1)}+u^{(i+1)}u^{(j+2)}\right)\\ &\quad+\sum_{i=0}^{N}\sum_{j=0}^{N}\frac{\partial^2 J}{\partial u^{(i)}\partial u^{(j)}}u^{(i+2)}u^{(j+1)}+\sum_{i=0}^{N}\frac{\partial J}{\partial u^{(i)}}u^{(i+3)},\end{aligned}$$

and

$$\frac{dJ}{dt} = \sum_{i=0}^{N}\frac{\partial J}{\partial u^{(i)}}\frac{d^i(u_{xxx}+6uu_x)}{dx^i}.$$

Now we can prove Theorem 2.

Proof. By Lemma 1, Lemma 2 and the assumption of Theorem 2, we know $\alpha_0 = g_0(u) + g_1(u)\partial + g_2(u)\partial^2 + \partial^3$. By comparing the first few highest order coefficients of $\frac{dJ}{dt}$ and $\alpha_0 J$, we get

$$\begin{aligned} u^{(N+3)}: &\quad \frac{\partial J}{\partial u^{(N)}} = \frac{\partial J}{\partial u^{(N)}},\\ u^{(N+2)}u^{(N+1)}: &\quad 0 = \frac{\partial^2 J}{\partial u^{(N)}\partial u^{(N)}},\end{aligned} \tag{11}$$

$$u^{(N+2)}: \quad \frac{\partial J}{\partial u^{(N-1)}} = \frac{\partial J}{\partial u^{(N-1)}} + 3\sum_{j=0}^{N-1}\frac{\partial^2 J}{\partial u^{(N)}\partial u^{(j)}}u^{(j+1)} + g_2\frac{\partial J}{\partial u^{(N)}}, \tag{12}$$

$$\begin{aligned} \left(u^{(N+1)}\right)^3: &\quad 0 = \frac{\partial^3 J}{\partial u^{(N)}\partial u^{(N)}\partial u^{(N)}},\\ \left(u^{(N+1)}\right)^2: &\quad 0 = 3\frac{\partial^2 J}{\partial u^{(N)}\partial u^{(N-1)}},\\ u^{(N+1)}: &\quad \frac{\partial J}{\partial u^{(N-2)}} + 6u\frac{\partial J}{\partial u^{(N)}} = \frac{\partial J}{\partial u^{(N-2)}} + 3\sum_{j=0}^{N-1}\frac{\partial^2 J}{\partial u^{(N-1)}\partial u^{(j)}}u^{(j+1)}\\ &\quad +3\sum_{j=0}^{N-2}\frac{\partial^2 J}{\partial u^{(N)}\partial u^{(j)}}u^{(j+2)} + 3\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\frac{\partial^3 J}{\partial u^{(N)}\partial u^{(j)}\partial u^{(k)}}u^{(j+1)}u^{(j+1)}\\ &\quad +g_2\times\left(\frac{\partial J}{\partial u^{(N-1)}} + 2\sum_{j=0}^{N-1}\frac{\partial^2 J}{\partial u^{(N)}\partial u^{(j)}}u^{(j+1)}\right) + g_1\times\frac{\partial J}{\partial u^{(N)}}.\end{aligned} \tag{13}$$

By (11), $J = \theta_1(u, u^{(1)}, \cdots, u^{(N-1)})u^{(N)} + \theta_0(u, u^{(1)}, \cdots, u^{(N-1)})$. By (12), $\theta_1(u, u^{(1)}, \cdots, u^{(N-1)}) = C_{\theta_1},\ g_2(u) = 0$, where $C_{\theta_1} \neq 0$ is a constant. By (13), $\theta_0(u, u^{(1)}, \cdots, u^{(N-1)}) = C_{\theta_{01}}u^{(N-1)} + \theta_{00}(u, u^{(1)}, \cdots, u^{(N-2)})$ and $g_1(u) = 6u$. Then we can compare the $u^{(N)}$ coefficients of $\frac{dJ}{dt}$ and $\alpha_0 J$, which gives $g_0(u) = 0$. So $\alpha_0 = 6u\partial + \partial^3$. In addition, such special class second integrals exist since $\frac{d}{dt}\rho_{2j} = (6u\partial + \partial^3)\rho_{2j}$, where $\rho_{2j} = \rho_{2j}[u]$ are the conserved quantities of the KdV equation, is a well-known fact for the KdV equation. □

Remark 5. Applying the above procedure to Conjecture 1 until to the $u^{(N)}$ term can not determine the cofactor operator α_1.

4. Proof of Theorem 3

Recall d_0 and α_2 are defined by (6) and (7). We should prove that such α_2-class exists for the KdV equation.
For simplicity, we denote

$$\begin{aligned}&\mathrm{SKdV}_0 = u,\\ &\mathrm{SKdV}_{2j} = \left(\partial^3 + 4u - 2\int u_x\right)\mathrm{SKdV}_{2j-2},\end{aligned}$$

$$\mathrm{SKdV}_1 = u_x,$$
$$\mathrm{SKdV}_{2j+1} = \left(\partial^3 + 4u + 2u_x \int\right)\mathrm{SKdV}_{2j-1}$$

with all integral constants being 0.

4.1. Some second integrals of α_2-class

$$J_{00} = \frac{u + c_2}{d_0},$$
$$J_{01} = 1 - c_1 \frac{u + c_2}{d_0} 2u,$$
$$J_2 = (u_{xx} + 3u^2) - \frac{c_1(u + c_2)}{d_0}(u_x^2 + 2u^3),$$
$$J_4 = (u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3) - \frac{c_1(u + c_2)}{d_0}(2u_x u_{xxx} - u_{xx}^2 + 10uu_x^2 + 5u^4),$$
$$J_6 = \left(u^{(6)} + 14uu'''' + 28u'u''' + 21(u'')^2 + 70u^2u'' + 70(u')^2u + 35u^4\right) - c_1\frac{c_2 + u}{d_0}$$
$$\times\left(2u'u^{(5)} - 2u''u^{(4)} + (u''')^2 + 28uu'u''' - 14(u'')^2u + 28(u')^2u'' + 70(u')^2u^2 + 14u^5\right).$$

The general expression of J_{2n}, $n \geq 1$ is

$$J_{2n} = \mathrm{SKdV}_{2n} - c_1\frac{u + c_2}{d_0}\int 2u_x \mathrm{SKdV}_{2n}\, dx. \tag{14}$$

4.2. The proof

Let us begin with the Wahlquist–Estabrook Lax pair [21] of the KdV equation:

$$\frac{\partial y_2}{\partial x} = -e^{2y_3}, \qquad \frac{\partial y_2}{\partial t} = -2e^{2y_3}(u + 2\lambda),$$
$$\frac{\partial y_3}{\partial x} = -y_8, \qquad \frac{\partial y_3}{\partial t} = u_x - 2(u + 2\lambda)y_8,$$
$$\frac{\partial y_8}{\partial x} = \lambda - u - y_8^2, \quad \frac{\partial y_8}{\partial t} = 2u_x y_8 - 2(u + 2\lambda)(u + y_8^2 - \lambda) - u_{xx}. \tag{15}$$

The main difference between Lax pairs (15) and the well-known Lax pair [22]

$$\hat{L}_0 = \partial^2 + u,$$
$$\hat{P}_0 = 4\partial^3 + 3(u\partial + \partial u), \tag{16}$$

is that (15) has an additional potential y_2, which is nonlocal with respect to u, y_3 and y_8. It can be checked directly that

$$J^A(\lambda) = -\mathrm{e}^{-2y_3}y_8^2R_1 + R_2,$$
$$J^B(\lambda) = (\mathrm{e}^{-y_3}y_8 - \mathrm{e}^{y_3}/y_2)\mathrm{e}^{-y_3}y_8y_2R_1 - y_2R_2,$$
$$J^C(\lambda) = -(\mathrm{e}^{-y_3}y_8 - \mathrm{e}^{y_3}/y_2)^2y_2^2R_1 + y_2^2R_2, \tag{17}$$

where

$$R_1 = 2c_1(u + c_2)/d_0,$$
$$R_2 = \mathrm{e}^{-2y_3}(3 + c_1c_2^2 + 2c_1c_2\lambda + 2c_1\lambda u - c_1u^2)/d_0$$
$$= \mathrm{e}^{-2y_3}\left(1 + 2c_1(\lambda - u)(u + c_2)/d_0\right)$$

belong to the α_2-class.

Remark 6. If $c_1 = 0$, then $R_1 = 0$, $R_2 = \mathrm{e}^{-2y_3}$. Then J_x^A is the nonlocal symmetries generated by the Bäcklund transformation of the KdV equation [23,24].

To prove Theorem 3, we need to expand J^B at $\lambda = \infty$. Let

$$\lambda = k^2, \quad \phi = \mathrm{e}^{-2y_3}y_2.$$

Then

$$\frac{d\phi}{dx} = 2y_8\phi - 1. \tag{18}$$

Now J^B is expressed by ϕ and y_8

$$J^B = -\phi - 2c_1 \frac{u+c_2}{d_0}\left(y_8'\phi + y_8\right). \tag{19}$$

By (15), we get the expansion of y_8

$$y_8(x) = k - \frac{k^{-1}}{2}u + \frac{k^{-2}}{4}u' - \frac{k^{-3}}{8}(u^2+u'') + \frac{k^{-4}}{16}(4uu'+u''')$$
$$-\frac{k^{-5}}{32}\left(2u^3+5(u')^2+6uu''+u^{(4)}\right) + \frac{k^{-6}}{64}\left(16u^2u' + 18u'u'' + 8uu''' + u^{(5)}\right) + \cdots. \tag{20}$$

Substituting (20) to (18), the expansion of ϕ is obtained as

$$\phi(x) = \frac{k^{-1}}{2} + \frac{k^{-3}}{4}u + \frac{k^{-5}}{16}(3u^2+u'') + \frac{k^{-7}}{64}\left(10u^3+5(u')^2+10uu''+u^{(4)}\right)$$
$$+\frac{k^{-9}}{256}\left(35u^4+70u(u')^2+70u^2u''+21(u'')^2+28u'u'''+14uu^{(4)}+u^{(6)}\right)+\cdots. \tag{21}$$

At last, substituting (20) and (21) to (19), we get

$$J^B = \frac{1}{d_0}\Bigg(-2c_1(u+c_2)k - \frac{k^{-1}}{2}(3+c_1c_2^2-c_1u^2) - \frac{k^{-3}}{4}(3+c_1c_2^2+c_1c_2u)u$$
$$+\frac{k^{-5}}{16}\left((9+3c_1c_2^2+4c_1c_2u+c_1u^2)u^2 + c_1(u+c_2)(u')^2 - d_0u''\right)+\cdots\Bigg). \tag{22}$$

The k, k^{-1} and k^{-3} terms of (22) give J_{00} and J_{01} in Section 4.1. The k^{-5} term gives J_2, and the k^{-7} term gives J_4, and the k^{-9} term gives J_6, and so on and so forth.

Remark 7. It would be much complicated to prove Theorem 3 directly from the general expression (14).

5. Constraining the KdV equation to $J = 0$ of α_2-class

Example 8. Let $J = J_2 + \beta_0 J_{00} + \beta_1 J_{01}$, where J_2, J_{00} and J_{01} are defined in Section 4.1. This J belongs to the α_2 class since the second integrals in the same class can be superposed. Let us consider the solutions of $J = 0$:

$$u_{xx} + 3u^2 - c_1\frac{u+c_2}{d_0}(2u^3+u_x^2) + \beta_0\frac{u+c_2}{d_0} + \beta_1\left(1 - c_1\frac{u+c_2}{d_0}2u\right) = 0. \tag{23}$$

It is easy to verify that

$$\bar{\beta}_0 = \frac{3}{d_0}\left(\beta_0 - 2\beta_1c_1u - 2c_1u^3 - c_1(u')^2\right)$$

is a first integral for (23). So the solving of (23) is reduced to solve

$$u_{xx} + 3u^2 + \frac{\bar{\beta}_0}{3}(u+c_2) + \beta_1 = 0. \tag{24}$$

Constraining the KdV equation with condition (24), we get nothing but the traveling wave solution of the KdV equation.

Remark 8. Let $J = \beta_0 J_{00} + \beta_1 J_{01} + \sum_{i=1}^{n} \beta_{2i}J_{2i}$. By (14),

$$\bar{\beta}_0 = \frac{3}{d_0}\left(\beta_0 - 2\beta_1c_1u - c_1\sum_{i=1}^{n}\beta_{2i}\int 2u_x\mathrm{SKdV}_{2i}\,dx\right)$$

is a movement constant of $J = 0$. Thus $J = 0$ is reduced to

$$\frac{\bar{\beta}_0}{3}(u+c_2) + \beta_1 + \sum_{i=1}^{n}\beta_{2i}\mathrm{SKdV}_{2i} = 0.$$

Let us explain Remark 8 by a more concrete example. Suppose $n = 2$ and $J = J_4 + 3J_2 + J_{01}$ with $c_1 = 1,\ c_2 = 1$. The ODE $J = 0$ with initial values $u(0) = 1,\ u_x(0) = 2,\ u_{xx}(0) = 3,\ u_{xxx}(0) = 4$ determines a unique curve l_A. The ODE $\mathrm{SKdV}_4 + 3\mathrm{SKdV}_2 - \frac{72}{7}u - \frac{65}{7} = 0$ with $u(0) = 1,\ u_x(0) = 2,\ u_{xx}(0) = 3,\ u_{xxx}(0) = 4$ determines a unique curve l_B. It can be verified that the curves l_A and l_B are the same.

The $J = 0$ of α_2-class is an example to mix an ODE with its first integral. Here is a much simpler example, showing the mixture. $u_{xx} + 3u^2 = C$ is the first integral of $u_{xxx} + 6uu_x = 0$. Then $u_{xxx} + 6uu_x + (u_{xx} + 3u^2) = 0$ will be reduced to solving $u_{xx} + 3u^2 = ke^{-x}$, where k is a constant.

Example 9. A nonlocal constraint. Since J^A, J^B and J^C all belong to α_2-class, we can examine their combination with J_{00}, J_{01} and J_{2n}. Here we investigate the simplest:

$$J_{00} + \beta_1 J_{01} + \beta_{A1} J^A(\lambda_1) + \beta_{B1} J^B(\lambda_1) + \beta_{C1} J^C(\lambda_1) = 0. \tag{25}$$

Combining (25) and (15), eliminating y_2, y_3 and y_8, we get a complicated second order ODE for u:

$$\begin{aligned}
&2\beta_1 c_1\big(2\beta_1 c_1\lambda_1 - 1\big)u_{xx}^2 + (\beta_{B1}^2 - 4\beta_{A1}\beta_{C1})\big(3 + c_1(c_2 + 4\lambda_1 - 3u)(c_2 + u) - c_1 u_{xx}\big)^2 \\
&-2\Big(c_2 + u + \beta_1(3 + c_1(c_2 - 3u)(u + c_2)) - 2\beta_1^2 c_1(3 + c_1c_2^2 - c_1u^2)u\Big)u_{xx} \\
&-\big(2\beta_1 c_1(2\lambda_1 - u) - 1\big)^2 u_x^2 + 4(u - \lambda_1)\big(\beta_1(3 + c_1c_2^2 - c_1u^2) + c_2 + u\big)^2 = 0.
\end{aligned} \tag{26}$$

Fortunately, (26) is solved by the first order ODE

$$\begin{aligned}
&\frac{1}{2}u_x^2 + u^3 + \frac{1}{2}\kappa u^2 - \left(c_2^2 - \left(\frac{1}{2\beta_1 c_1} - 2\lambda_1\right)\kappa + \frac{3\beta_1 + c_2 - 2\lambda_1}{\beta_1 c_1} + 8\lambda_1^2\right)u \\
&+\left(\frac{1}{4\beta_1 c_1} - \frac{\lambda_1}{2}\right)\kappa^2 - \frac{\beta_{B1}^2 - 4\beta_{A1}\beta_{C1}}{2\beta_1^2}\left(c_2 - 2\lambda_1 - \frac{\kappa}{2}\right)^2 - 2\lambda_1(c_2^2 + 4\lambda_1^2) \\
&-\left(\frac{c_2 - 4\lambda_1}{2\beta_1 c_1} + \frac{3}{2c_1} + \frac{c_2^2}{2} + 4\lambda_1^2\right)\kappa + \frac{2(2\lambda_1 - c_2 - 3\beta_1)\lambda_1}{\beta_1 c_1} = 0.
\end{aligned} \tag{27}$$

In (27), κ acts as the movement constant.

6. A special class of nonlocal second integrals

Let

$$\hat{L}_s = \partial + u\partial^{-1} \tag{28}$$

and $\tilde{J}_n$ is defined recursively by

$$\begin{aligned}
&\tilde{J}_1 = u, \\
&\tilde{J}_n = \hat{L}_s \tilde{J}_{n-1}.
\end{aligned} \tag{29}$$

Then we have the following relation.

Theorem 4.

$$\tilde{J}_{nt} = \big(\partial^3 + 3\partial u\big)\tilde{J}_n. \tag{30}$$

Define

$$\alpha_s = \partial^3 + 3\partial u. \tag{31}$$

Theorem 4 means that we have a special α_s-class second integrals, which are nonlocal in general. In fact, only $\tilde{J}_1 = u$ is local. All other $\tilde{J}_n$ $(n \neq 1)$ are nonlocal, for example $\tilde{J}_2 = u_x + u\int u dx$ is obviously nonlocal.

The recursion formula (29) and the α_s class formula (30) lead to a Lax pair $\frac{d}{dt}\hat{L}_s = [\alpha_s, \hat{L}_s]$ for the KdV equation. This less popular Lax pair was found early in the study of the constraints of the KP hierarchy [25]. It is also used to investigate the bi-Hamiltonian structure of the constrained KP hierarchy [26], as well as to deduce the recursion operator for the KdV equation [27]. Now the Lax pair can be explained in the framework of special second integral: the operator $\hat{L}_s$ is a 'recursion operator' in the special α_s-class second integrals and α_s defines the special class. The $\tilde{J} = 0$ solution in this α_s-class is called soliton-type since the nonsingular ones are just the soliton or multi-soliton solutions of the KdV equation [20].

The nonlocal second integrals can be converted to local ones. For example, let us consider $\tilde{J}_2$. Define $\tilde{\tilde{J}}_2 = \tilde{J}_2/u = u_x/u + \int u dx$. Then $\tilde{\tilde{J}}_2$ should be a second integral. Next $\check{J}_2 = \frac{d}{dx}\tilde{\tilde{J}}_2 = u_{xx}/u - u_x^2/u^2 + u$ is a second integral too. At last $\bar{J}_2 = u\check{J}_2 = u_{xx} - u_x^2/u + u^2$ is a second integral in the 'standard' form. In this way, the $\tilde{J}_n$ defined by (29) can be written into the standard form:

$$\begin{aligned}
&\bar{J}_1 = u, \\
&\bar{J}_2 = u_{xx} - \frac{u_x^2}{u} + u^2, \\
&\bar{J}_n = \big(\ln\bar{J}_{n-1}\big)_{xx}\bar{J}_{n-1} + \frac{\bar{J}_{n-1}^2}{\bar{J}_{n-2}}.
\end{aligned} \tag{32}$$

The second integrals defined by (32) are not in one special class. In fact we have:

Theorem 5.

$$\bar{J}_{nt} = \bar{J}_{nxxx} + 6\frac{\bar{J}_n \bar{J}_{nx}}{\bar{J}_{n-1}}. \tag{33}$$

We suspect that the discrete systems (32) and (33) are both integrable.

7. Discussions

First we would like to stress the algorithm behind Theorem 2. Given a general PDE of form (3), there should be no algorithm to determine all its second integrals. But it will be fairly easy to check if the PDE has special classes of second integrals. The algorithm first determines the cofactor operators by examining the relative low orders of second integrals. Then it checks the existence of relatively high order second integrals of that class. If the two processes succeed, we should prove there are indeed infinitely many second integrals of that class. The proof is most likely to begin with a Lax pair just as the proof of Theorem 3. The Lax pair is almost known: the operator $\hat{P}$ is just the cofactor operator and the operator $\hat{L}$ is the recursion operator for that special class of second integrals. Section 6 shows that the algorithm lacks the ability to seek special class of nonlocal second integrals.

Second we give a conjecture that generalizes the examples in Section 5. The two examples in Section 5 are both solved by $\mathrm{SKdV}_2 + \kappa\,\mathrm{SKdV}_0 = c$. Numerical experiments support the following general result:

Conjecture 2. *Let J_{00}, J_{01}, J_{2i}, J^A, J^B J^C be defined in Section 5. Then*

$$\beta_1 J_{01} + \beta_0 J_{00} + \sum_{i=1}^{M} \beta_{2i} J_{2i} + \sum_{i=1}^{N} \Big(\beta_{Ai} J^A(\lambda_i) + \beta_{Bi} J^B(\lambda_i) + \beta_{Ci} J^C(\lambda_i) \Big) = 0 \tag{34}$$

is solvable by

$$\sum_{i=0}^{M+N} \kappa_{2i} \mathrm{SKdV}_{2i} = c. \tag{35}$$

The specific case $c_1 = c_2 = M = \beta_{Bi} = \beta_{Ci} = 0$ is already clear, known as the nonlinearization of Lax pair [28]. By Conjecture 2, (34) has many symmetries since there are so many redundant parameters in (34). So one would wonder what the symmetries are. It maybe has relation to the ones found in [23,24] for u and the spectral functions. It is quite possible that the symmetries induced by c_1 and c_2 are simpler. Examples in Section 5 also show that the redundant parameters relate directly with the first integrals. Let $M = 0$, the redundant parameters in (34) are c_1, c_2, β_{Bi} and β_{Ci}. Their total number is $2N + 2$, which is quite close to the order of ODE (35). This provides another possible explanation for the integrability of (35).

The third is to generalize the method to multi-component integrable PDEs. The situations of multi-component systems are much more complex. For example, (4) has two special classes of nonlocal second integrals [6]: $(-\partial^3 + 3qr\partial + 3q_x r)$-class and $(-\partial^3 + 3qr\partial + 3qr_x)$-class. Until now, the relation between the special classes of nonlocal second integrals and the local ones for (4) is still unknown.

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