ONEOptimal: A *Maple* Package for Generating One-Dimensional Optimal System of Finite Dimensional Lie Algebra*

MIAO Qian (苗倩), HU Xiao-Rui (胡晓瑞), and CHEN Yong (陈勇)†

Shanghai Key Laboratory of Trustworthy Computing, East China Normal University, Shanghai 200062, China

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Abstract *We present a Maple computer algebra package, ONEOptimal, which can calculate one-dimensional optimal system of finite dimensional Lie algebra for nonlinear equations automatically based on Olver's theory. The core of this theory is viewing the Killing form of the Lie algebra as an invariant for the adjoint representation. Some examples are given to demonstrate the validity and efficiency of the program.*

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1 Introduction

Symmetry group theory for differential equations built by Sophus Lie[1] plays an important role in constructing explicit solutions for integrable and non-integrable nonlinear equations. For any given subgroup of the full symmetry group, the original nonlinear system can be reduced to a system with fewer independent variables by solving the corresponding characteristic equations. Since there are almost always an infinite amount of such subgroups, it is usually not feasible to list all possible group-invariant solutions to the system. It is anticipated to find those complete but inequivalent group-invariant solutions, that is to say, to classify all the group-invariant solutions. For this problem, some effective and systematic methods have been developed by Ovsiannikov[2] and Olver[3] respectively, which introduce the concept of "optimal system" for group-invariant solutions. More details on how to perform the classification of subgroup under the adjoint action are clarified in Ref. [3]. The problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras, so people always concentrate on the latter. The classification of symmetry subalgebras for many important partial differential equations (PDEs) in physics can be found in [4–14]. However, the operation for the classification of subalgebras shows quite complicated and the infinitesimal techniques do not seem to be overly useful. To the best of our knowledge, despite these numerous results, there is no literature on the process mechanization by the aid of computer.

Different algorithms and packages in computer algebra systems have also been developed implementing Lie symmetry computations and related methods: SPDE by Schwarz,[15] LIE and BIGLIE by Head[16−17] in MUMATH and MATHLIE by Baumann[18] in *Mathematica*. For *Maple* there are also some useful packages: PDEtools by Cheb–Terrab[19] which is distributed since Release 11, DES-OLV by Vu and Carminati,[20−21] and LieAlgebras provided in the built-in DifferentialGeometry package.

Here we devote to constructing one-parameter optimal system of finite dimensional Lie algebra on the computer. Then based on the one-dimensional case, higher-dimensional optimal system can be constructed. Even in the one-dimensional case, it still requires a lot of mechanical and monotonous calculations by rule of thumb, so it must be a significant job to implement the process mechanization.

In this paper, we present one *Maple* package named ONEOptimal to construct one-dimensional optimal system of Lie algebra for nonlinear systems. For a given Lie algebra, the package ONEOptimal is used to find the centers of the vector fields, generate the commutator table as well as the adjoint representation table and give out one invariant (i.e. Killing-form). Then, the function $Classify$ can carry out classification and simplification according to the Killing-form automatically. Our program provides a basis for many possible applications.

This paper is arranged as follows. In Sec. 2, a brief review of the methods to construct one-dimensional optimal system for Lie algebra is given. In Sec. 3, a systematic computational algorithm based on Olver's method is established. In Sec. 4, the programm commands in the Maple package ONEOptimal are explained. In Sec. 5, some different types of examples are given to illustrate and verify the effectiveness of our program. Finally, a brief conclusion is given in Sec. 6.

2 Theoretical Methods

Optimal System Let G be a Lie group. An optimal

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†E-mail: ychen@sei.ecnu.edu.cn

system of s-parameter subgroups is a list of conjugacy inequivalent s-parameter subgroups with the property that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of s-parameter subalgebras forms an optimal system if every s-parameter subalgebra of g is equivalent to a unique member of the list under some element of the adjoint representation $\tilde{\ell} = \mathrm{Ad}\ g(\ell), g \in G$.

The method to construct one-dimensional optimal system of Lie algebra was first proposed by Ovisiannikov,[2] taking advantage of the global matrix of adjoint representation. Then a lightly different way was adopted in [3] to deal with one-dimensional subalgebras, making use of the adjoint representation table. It is also pointed out that for one-dimensional subalgebras, the problem of finding an optimal system is essentially the same as the problem of classifying the orbits of the adjoint transformations. The essence of this method is that the Killing form of the Lie algebra is an "invariant" for the adjoint representation. Based on the sign of the Killing form, the representatives for each equivalence class were obtained. In this paper, we will apply this method to develop our *Maple* package.

For m-dimensional Lie algebra $\mathscr{G}$, its one-dimensional optimal system is computed by the naïve approach of taking a general element v in $\mathscr{G}$ and subjecting it to various adjoint transformations so as to "simplify" it as much as possible. Given a nonzero vector

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_m v_m\,, \tag{1}$$

the key task is to simplify as many of the coefficients a_i as possible through judicious applications of adjoint maps to v, where v_i $(i = 1, \ldots, m)$ are m generators in the vector fields of $\mathscr{G}$. The adjoint representation is

$$\mathrm{Ad}_{\exp(\epsilon v_1)}(v_2) = v_2 - \epsilon[v_1, v_2] + \frac{\epsilon}{2!}[v_1,[v_1,v_2]] + \cdots\,, \tag{2}$$

for $\forall v_1, v_2 \in \mathscr{G}$. In this process, the detection of an invariant is important since it places restrictions on how far we can expect to simplify v. A real function ϕ defined on $\mathscr{G}$ is called an invariant if $\phi(\mathrm{Ad}_g(v)) = \phi(v)$ for all v in $\mathscr{G}$ and g in the Lie group G generated by $\mathscr{G}$. Usually, the famous Killing form[10] is computed as an invariant to simplify v.

The general steps developed by Olver to construct one-dimensional optimal system of subalgebras are:

Step 1 For a nonlinear PDE, get the Lie point symmetry with its generators by the classical Lie symmetry method.

Step 2 Work out the commutator table (ignoring the infinite-dimensional subalgebras which contain arbitrary functions) and the corresponding adjoint representation table for the m-dimensional subalgebra $\mathscr{G}$.

Step 3 Calculate the Killing form from the commutator table, i.e. finding an invariant of $\mathscr{G}$.

Step 4 For the nonzero vector field (1), on the basis of the Killing form and adjoint representation table calculated in Step 2, select the appropriate group generated by v_k to act on v to cancel some coefficients a_i as many as possible.

One remark is given as follows:

Remark 1 It should be noted that, Olver did not mention the concept of "center" in his method. For simplicity, we have taken the centers of $\mathscr{G}$ into account in our algorithm. For the Lie subalgebras $\mathscr{G}$, v_1 is known as the center if the results of commutator to v_1 with all other generators are zero. Then, if all the elements except the center v_1 can form a subalgebra $\mathscr{G}_1$ of $\mathscr{G}$, we only need to consider the one-dimensional optimal system os_1 of $\mathscr{G}_1$, and construct one-dimensional optimal system os of $\mathscr{G}$ by adding cv_1 to each element in os_1, where c is an arbitrary constant. Otherwise, the center v_1 should not be removed from $\mathscr{G}$.

3 Key Algorithm for Constructing One-Dimensional Optimal System

On the basis of the process presented in Sec. 2, we have designed the corresponding mechanization algorithm. Since there have been a lot of software packages to get Lie point symmetries in Step 1, we no longer study it here and start from the obtained Lie algebra instead of the original PDE. For the m-dimensional Lie algebra $\mathscr{G}$, the algorithm to construct one-dimensional optimal system can be divided into six main steps:

Step 1 Single out the centers of the given generators vs and delete the centers, which have no effect on the closure of $\mathscr{G}$.

Step 2 Obtain the commutator table of $\mathscr{G}$ through computation. Here we define the function of commutator operator, and the calculation result is returned in a linear combination form of each generator. The corresponding expression to each generator is also pointed out in the output.

Step 3 Give out the corresponding adjoint representation table using the Lie series (2) in conjunction with the commutator table.

Step 4 Referring to the definition of Killing form, calculate the invariant from the commutator table.

Step 5 Acting on the general non-zero vector field (1) by the groups generated by every element v_i $(i = 1, 2, \ldots, m)$, it results in

$$\mathrm{Ad}_{\exp(\epsilon v_i)} v = \tilde{a}_{i1} v_1 + \tilde{a}_{i2} v_2 + \cdots + \tilde{a}_{im} v_m\,, \tag{3}$$

with $\tilde{a}_{ij} = \tilde{a}_{ij}(a_{ij}, \epsilon)$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, m)$. Take the elements $\tilde{a}_{ij}$ to constitute a coefficient matrix named Cm_k, and work out the corresponding solution matrix named So_k with respect to ϵ. Here, k specifies the current steps.

Step 6 Depending on the sign of the invariant Killing form, there are several different cases according to the expression form of Killing form. For each case, enter into next step.

Step 7 For each reference variable a_i in current case, two subcases ($a_i = 0$ and $a_i \neq 0$) are considered at most.

Step 7.1 If $a_i = 0$, we make the coefficient, which contains a_i vanish.

Step 7.2 If $a_i \neq 0$, we check whether there are coefficients whose denominator contains a_i and make it vanish, if any.

Step 8 Check the current solution matrix So_k to verify whether there are some coefficients that can be eliminated, that is to say, whether there is any new reference variable a_i. If any such a_i exists, enter into Step 7. If no coefficient can be eliminated further, the current case terminates.

All the steps above can be completely mechanized by computer.

4 Maple Package ONEOptimal

Based on the above algorithm, we have developed an automated *Maple* package ONEOptimal on *Maple* versions 13 and above. The package is initialized by the command with (ONEOptimal). Here we briefly describe some input-output parameters and package commands available in ONEOptimal. In Table 1, the abbreviations are used for the input parameters. In Table 2, the abbreviations are used for the output parameters.

Table 1 Input parameters for package ONEOptimal.

Parameter name	Parameter description
vs	Set of generators for the given Lie algebra $\mathscr{G}$. You need only to write vs in vector form if you do not want to delete the centers.
cs	Set of constants specified in the generators.
pf	The print symbol is an optional argument. If you set pf to 0, the detailed process (Cm_k and So_k) will not show and vice versa. The default value is 0.
var	The defined function name of all variables and dependent variables.
xs	Set of all variables.
vf	The vector fields written in the form of vector.
order	The order up to which a Lie series expansion truncates. In general, order = 10 is enough already.
level	The level number of classification.
nonzero_set	Set of index of coefficient, which can not be 0 in each case.
column_set	Set of index i of a_i, which is specified to be eliminated.
case_elements	Set of coefficient a_i, which is used to be a reference variable for classification.
flag	The classification symbol of reference variable a_i (0 means that only the case with $a_i = 0$ is considered, 1 means that only the case with $a_i \neq 0$ is considered, 2 means that both above cases should be considered). The default value is 2.

Table 2 Output parameters for package ONEOptimal.

Parameter name	Parameter description
CT	The commutator table of $\mathscr{G}$.
AdT	The adjoint representation table of $\mathscr{G}$.
KF	The Killing form of $\mathscr{G}$.
Cm_k	The coefficient matrix in Step k constituted by the elements $\tilde{a}_{ij}$ in formula (3).
So_k	The solution matrix to each coefficient matrix Cm_k in Step k.
Result	The adjoint representation result in current case.

Some main package commands and corresponding inputs are given in the following listing. In ONEOptimal, the main routine is Get_Optimal(vs, cs, pf). This procedure calls six sub-procedures:

find_center(vs, xs, var): Singles out the centers from vs with respect to xs and var, and deletes the centers, which have no effect on the closure of $\mathscr{G}$.

commutator_table(vf, xs, var): Generates commutator table for $\mathscr{G}$ with the (i,j)-th entry indicating $[V_i, V_j]$.

ad_table(vf, order): Computes the adjoint representation table of $\mathscr{G}$ with the (i,j)-th entry indicating $\mathrm{Ad}_{\exp(\epsilon V_i)}(V_j)$. For simplicity, the truncated power series expansions up to order in the calculation result are all replaced by the original series name.

K_form(X): Computes the Killing form on the basis of the commutator table X.

Classify(kf, C): Classifies the original system to several cases according to Killing form kf and executes optimization.

show_optimal(): Prints out the optimized results.

Other package commands and corresponding inputs are given in the following listing:

LinearCo(expr, vf, var): Writes the expression expr as a linear combination of the generators in vf with respect to var.

lie_bracket_cal(a, b, var): Computes the commutator of a pair of vectors a and b. Here, both a and b are single generators, while the result is returned as a linear combination expression.

Lie_bracket(a, b): Computes the commutator of a and b. Here, a and b can be linear combination of generators.

ad_operator(a, b, order): Acts by adjoint maps generated by a and b up to order order. Here both a and b are single generators.

ad(a, b): Acts by adjoint maps generated respectively by a and b. Here a and b can be linear combination of generators. Command ad(V_m, Eqn) can also be used to observe and adjust the coefficient in Eqn.

replace(result, j, ex, order): Replaces the truncated series expansion trse in the coefficient of V_j in result with the name of the original series. Here, ex represents the coefficient of ϵ determined from current trse.

coefficient_obtain(eq): Computes the coefficient matrix Cm_k obtained by adjoint maps generated respectively by all generators V_i and eq.

coefficient_solve(C): Solves out ε from every element (expression about ϵ) in the coefficient matrix C, and provides the solution matrix to C.

deno_reduce(d, C): Eliminates some V_i according to the specified generator d from the equation corresponding to matrix C. Here, for d is nonzero, it is possible to eliminate V_i if d appears in the denominator of column i in solution matrix corresponding to C.

reduce(column_set, C): Eliminates a best V_i in column_set from Eq_k corresponding to C in current step k.

reduce_all(column_set, C): Eliminates all V_i in column_set from Eq_k corresponding to C in current step k.

case_classify(case_elements, C, level, nonzero_set, flag): Classifies current case to n subcases for each reference variable a_i in coefficient set case_elements: $a_i = 0$ (when flag $= 0, 2$) and $a_i \neq 0$ (when flag $= 1, 2$).

check_column(s, level, nonzero_set): Checks So_s whether the current case can be simplified further, that is to say, whether there are some columns that can be eliminated. Here, s represents the step number.

check_row(s): Picks out the best row number whose corresponding adjoint representation has most ε in Cm_s.

optimal_calculate(s, nonzero_set): Calculates the reduced adjoint representation result according to nonzero_set for the case in Step s.

5 Illustrative Examples

In this section, several different kinds of examples are given to illustrate the effectiveness of our package ONEOptimal.

5.1 *Examples with One Variable in Killing Form*

Example 1 Consider the four-dimensional symmetry algebra g of the Korteweg-de Vries equation

$$u_t + 6uu_x + u_{xxx} = 0\,, \tag{4}$$

which is generated by the vector fields

$$V_1 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad V_2 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}\,,$$
$$V_3 = \frac{\partial}{\partial t}, \quad V_4 = \frac{\partial}{\partial x}\,. \tag{5}$$

From Table 3

Table 3 The commutator table for Example 1.

	V_1	V_2	V_3	V_4
V_1	0	$-2V_1$	$-V_4$	0
V_2	$2V_1$	0	$-3V_3$	$-V_4$
V_3	V_4	$3V_3$	0	0
V_4	0	V_4	0	0

the Killing form is obtained

$$KF = 14\,a_2^2. \tag{6}$$

An optimal system of one-dimensional subalgebras of this algebra is those spanned by:

$$\text{(a) } V_2, \quad \text{(b) } V_1 + \mathrm{e}^{5\,\epsilon} a_3 V_3, \quad \text{(c) } V_3, \quad \text{(d) } V_4\,. \tag{7}$$

Depending on the sign of a_3, we can make the coefficient of V_3 either $+1, -1$ or 0. Thus the result is consistent with the result given by Olver[3]

$$\text{(a) } V_2, \quad \text{(b) } V_3, \quad \text{(c) } V_3 + V_1\,,$$
$$\text{(d) } V_3 - V_1, \quad \text{(e) } V_1, \quad \text{(f) } V_4\,. \tag{8}$$

The average running time for this example is 0.2622 seconds.

Example 2 Consider the four-dimensional symmetry algebra g of the Navier–Stokes equation

$$\psi_{xxt} + \psi_{yyt} + \psi_x\psi_{xxy} + \psi_x\psi_{yyy} - \psi_y\psi_{xxx} - \psi_y\psi_{xyy}$$
$$- \gamma(\psi_{xxxx} + 2\psi_{xxyy} + \psi_{yyyy}) = 0\,, \tag{9}$$

which is generated by the vector fields

$$V_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \quad V_2 = \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + t\frac{\partial}{\partial t}\,,$$
$$V_3 = -yt\frac{\partial}{\partial x} + xt\frac{\partial}{\partial y} + \frac{x^2+y^2}{2}\frac{\partial}{\partial u}, \quad V_4 = \frac{\partial}{\partial t}\,. \tag{10}$$

From Table 4

Table 4 The commutator table for Example 2.

	V_1	V_2	V_3	V_4
V_1	0	0	0	0
V_2	0	0	V_3	$-V_4$
V_3	0	$-V_3$	0	$-V_1$
V_4	0	V_4	V_1	0

the Killing form is obtained

$$KF = 2\,a_2^2. \tag{11}$$

An optimal system of one-dimensional subalgebras is provided by those generated by:

$$\text{(a) } V_2+a_1V_1, \quad \text{(b) } V_3+a_4\,\mathrm{e}^{2\epsilon}V_4, \quad \text{(c) } V_4, \quad \text{(d) } V_1\,. \tag{12}$$

Depending on the sign of a_4, we can make the coefficient of V_4 either $+1, -1$ or 0. Thus the result is consistent with the result given by Hu[22]

$$\begin{aligned}&\text{(a) } V_2+\alpha V_1, \quad \text{(b) } V_3, \quad \text{(c) } V_3+V_4\,,\\ &\text{(d) } V_3-V_4, \quad \text{(e) } V_4, \quad \text{(f) } V_1\,,\end{aligned} \tag{13}$$

where α is an arbitrary constant.

The average running time for this example is 0.2356 seconds.

5.2 *Examples with Two Variables in Killing Form*

Example 3 Consider the seven-dimensional symmetry algebra g of the Zakharove–Kuznetsov equation[23]

$$u_t+auu_{xx}+bu_{xxx}+u_{xyy}+u_{xzz}=0\,, \tag{14}$$

which is generated by the vector fields

$$\begin{aligned}&V_1=z\frac{\partial}{\partial y}-y\frac{\partial}{\partial z}, \quad V_2=at\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\,,\\ &V_3=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+3t\frac{\partial}{\partial t}-2u\frac{\partial}{\partial u}\,,\\ &V_4=\frac{\partial}{\partial t}, \quad V_5=\frac{\partial}{\partial x}, \quad V_6=\frac{\partial}{\partial y}, \quad V_7=\frac{\partial}{\partial z}\,.\end{aligned} \tag{15}$$

From Table 5

Table 5 The commutator table for Example 3.

	V_1	V_2	V_3	V_4	V_5	V_6	V_7
V_1	0	0	0	0	0	V_7	$-V_6$
V_2	0	0	$-2V_2$	$-aV_5$	0	0	0
V_3	0	$2V_2$	0	$-3V_4$	$-V_5$	$-V_6$	$-V_7$
V_4	0	aV_5	$3V_4$	0	0	0	0
V_5	0	0	V_5	0	0	0	0
V_6	$-V_7$	0	V_6	0	0	0	0
V_7	V_6	0	V_7	0	0	0	0

the Killing form is obtained

$$KF=16\,a_3^2-2\,a_1^2. \tag{16}$$

An optimal system of one-dimensional subalgebras is provided by those generated by:

$$\begin{aligned}&\text{(a) } a_1V_1+V_3+a_7\,\mathrm{e}^{\epsilon}V_7\,,\\ &\text{(b) } V_1+a_3V_3+a_7\,\mathrm{e}^{\epsilon}V_7,\ a_3\neq 0\,,\\ &\text{(c) } V_1+a_4\,\mathrm{e}^{3\epsilon}V_4,\ a_4\neq 0\,,\\ &\text{(d) } V_1+a_2\,\mathrm{e}^{-2\epsilon}V_2+a_4\,\mathrm{e}^{3\epsilon}V_4,\ a_2\neq 0\,,\\ &\text{(e) } V_1+a_5\,\mathrm{e}^{\epsilon}V_5, \quad (\mathrm{f}_1)\ V_2+\mathrm{e}^{5\epsilon}a_4V_4+\mathrm{e}^{3\epsilon}a_6V_6\,,\\ &(\mathrm{f}_2)\ V_2+\mathrm{e}^{5\epsilon}a_4V_4+\mathrm{e}^{3\epsilon}a_7V_7\,,\\ &(\mathrm{g}_1)\ V_4+\mathrm{e}^{-2\epsilon}a_6V_6, \quad (\mathrm{g}_2)\ V_4+\mathrm{e}^{-2\epsilon}a_7V_7\,,\\ &(\mathrm{h}_1)\ a_5\,\mathrm{e}^{\epsilon}V_5+a_6\,\mathrm{e}^{\epsilon}V_6\,,\\ &(\mathrm{h}_2)\ a_5\,\mathrm{e}^{\epsilon}V_5+a_7\,\mathrm{e}^{\epsilon}V_7\,.\end{aligned} \tag{17}$$

Depending on the sign of a_4, a_5, a_6, a_7, we can adjust the coefficient of V_4, V_5, V_6, V_7 to suitable value. Thus this result is consistent with Ref. [24].

$$\begin{aligned}&\text{(a) } V_3+a_1V_1, \quad \text{(b) } V_1+V_4+\alpha V_2\,,\\ &\text{(c) } V_1+V_2, \quad (\mathrm{d}_1)\ V_1+V_5, \quad (\mathrm{d}_2)\ V_1\,,\\ &\text{(e) } V_4+V_7+\alpha V_2, \quad \text{(f) } V_4+V_6+\alpha V_2\,,\\ &(\mathrm{g}_1)\ V_4+V_2, \quad (\mathrm{g}_2)\ V_4, \quad \text{(h) } V_2+V_7\,,\\ &(\mathrm{i}_1)\ V_2+V_6, \quad (\mathrm{i}_2)\ V_2, \quad \text{(j) } V_7+\alpha V_5\,,\\ &\text{(k) } V_6+\alpha V_5, \quad \text{(l) } V_5\,.\end{aligned} \tag{18}$$

where α is an arbitrary constant.

The average running time for this example is 1.2043 seconds.

Example 4 Consider the seven-dimensional symmetry algebra g of the two layers of atmosphere model equation

$$\begin{aligned}&\psi_{1xxt}+\psi_{1yyt}-\psi_{1y}\psi_{1xxx}-\psi_{1y}\psi_{1xyy}+\psi_{1x}\psi_{1xxy}\\ &\quad+\psi_{1x}\psi_{1yyy}-\lambda^2\psi_{1t}+\lambda^2\psi_{2t}+\frac{2\lambda^2\gamma}{f_0}Q=0\,,\end{aligned} \tag{19}$$

$$\begin{aligned}&\psi_{2xxt}+\psi_{2yyt}-\psi_{2y}\psi_{2xxx}-\psi_{2y}\psi_{2xyy}+\psi_{2x}\psi_{2xxy}\\ &\quad+\psi_{2x}\psi_{2yyy}+\lambda^2\psi_{2t}-\lambda^2\psi_{2t}-\frac{2\lambda^2\gamma}{f_0}Q=0\,,\end{aligned} \tag{20}$$

which is generated by the vector fields

$$\begin{aligned}&V_1=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\,,\\ &V_2=-t\frac{\partial}{\partial t}+\psi_1\frac{\partial}{\partial\psi_1}+\psi_2\frac{\partial}{\partial\psi_2}\,,\\ &V_3=\frac{\partial}{\partial t}, \quad V_4=\frac{\partial}{\partial x}, \quad V_5=\frac{\partial}{\partial y}\,,\\ &V_6=\frac{\partial}{\partial\psi_1}, \quad V_7=\frac{\partial}{\partial\psi_2}\,.\end{aligned} \tag{21}$$

From Table 6

Table 6 The commutator table for Example 4.

	V_1	V_2	V_3	V_4	V_5	V_6	V_7
V_1	0	0	0	V_5	$-V_4$	0	0
V_2	0	0	V_3	0	0	$-V_6$	$-V_7$
V_3	0	$-V_3$	0	0	0	0	0
V_4	$-V_5$	0	0	0	0	0	0
V_5	V_4	0	0	0	0	0	0
V_6	0	V_6	0	0	0	0	0
V_7	0	V_7	0	0	0	0	0

the Killing form is obtained

$$KF=3\,a_2^2-2\,a_1^2. \tag{22}$$

A one-dimensional optimal system is those spanned by:

$$\begin{aligned}&\text{(a) } V_1+a_2V_2, \quad a_2\neq 0\,,\\ &\text{(b) } a_3\,\mathrm{e}^{-\epsilon}V_3+a_4V_4+a_5V_5+a_6\,\mathrm{e}^{\epsilon}V_6+a_7\,\mathrm{e}^{\epsilon}V_7,\\ &\qquad a_4^2+a_5^2\neq 0\,,\\ &\text{(c) } V_2+a_4V_4+a_5V_5, \quad a_4^2+a_5^2\neq 0\,,\\ &\text{(d) } V_1+a_3\,\mathrm{e}^{-\epsilon}V_3+a_6\,\mathrm{e}^{\epsilon}V_6+a_7\,\mathrm{e}^{\epsilon}V_7\,.\end{aligned} \tag{23}$$

This is consistent with Ref. [25] after adjusting the coefficients which contain ϵ.

$$\begin{aligned}&\text{(a) } V_1+\lambda V_2, \quad \text{(b) } V_1+\alpha V_3+\beta V_6+\mu V_7\,,\\ &\text{(c) } V_5+\alpha V_4+\lambda V_2, \quad \text{(d) } V_5+V_3+\alpha V_6+\beta V_7+\mu V_4\,,\end{aligned}$$

(e) $V_5+V_6+\alpha V_7+\beta V_4$, (f) $V_5+V_7+\alpha V_4$,
(g) $V_5+\alpha V_4$, (h) $V_4+\lambda V_2$,
(i) $V_4+V_3+\alpha V_6+\beta V_7$, (g) $V_4+V_6+\alpha V_7$,
(k) V_4+V_7, (l) V_4, (m) $V_3+\alpha V_6+\beta V_7$, (24)

where $\lambda\neq 0,\alpha,\beta,\mu$ are arbitrary constants.

The average running time for this example is 0.7489 seconds.

5.3 *Examples with Three Variables in Killing Form*

Example 5 Consider the six-dimensional algebra g of the heat equation

$$u_t=u_{xx}\,, \tag{25}$$

which is generated by the vector fields

$$V_1=u\frac{\partial}{\partial u},\quad V_2=2t\frac{\partial}{\partial x}-xu\frac{\partial}{\partial u}\,,\quad V_3=x\frac{\partial}{\partial x}+2t\frac{\partial}{\partial t}\,,$$
$$V_4=4tx\frac{\partial}{\partial x}+4t^2\frac{\partial}{\partial t}-(x^2+2t)u\frac{\partial}{\partial u}\,,$$
$$V_5=\frac{\partial}{\partial t},\quad V_6=\frac{\partial}{\partial x}\,. \tag{26}$$

From Table 7

Table 7 The commutator table for Example 5.

	V_1	V_2	V_3	V_4	V_5	V_6
V_1	0	0	0	0	0	0
V_2	0	0	$-V_2$	0	$-2V_6$	$-V_1$
V_3	0	V_2	0	$2V_4$	$-2V_5$	$-V_6$
V_4	0	0	$-2V_4$	0	$2V_1-4V_3$	$-2V_2$
V_5	0	$2V_6$	$2V_5$	$-2V_1+4V_3$	0	0
V_6	0	$-V_1$	V_6	$2V_2$	0	0

the Killing form is obtained

$$KF=10\,a_3^2-40\,a_5a_4\,. \tag{27}$$

A one-dimensional optimal system is:

(a) V_2, (b) V_6, (c) $V_2+a_5\,\mathrm{e}^{3\,\epsilon}V_5,\ a_5\neq 0$,
(d) $a_1V_1+V_3$, (e) $\mathrm{e}^{-2\,\epsilon}a_1V_1+V_5$,
(f) $\mathrm{e}^{2\,\epsilon}a_1V_1+V_4+\mathrm{e}^{4\,\epsilon}a_5V_5,\ a_5\neq 0$, (g) V_1. (28)

This result is consistent with Refs. [3, 11] after adjusting the coefficients which contain ϵ.

(a) V_3+aV_1, (b) $V_4+V_5+bV_1$, (c) $V_2\pm V_5$,
(d) $V_5\pm V_1$, (e) V_5, (f) V_6, (g) V_1, (29)

where a, b are arbitrary constants.

The average running time for this example is 0.7427 seconds.

Example 6 Consider six-dimensional algebra g of the quasilinear equation[26]

$$u_x^2u_{xx}+2u_xu_yu_{xy}+u_y^2u_{yy}=0\,, \tag{30}$$

which is generated by the vector fields

$$V_1=u\frac{\partial}{\partial u}\phi,\quad V_2=x\frac{\partial}{\partial x}\phi+y\frac{\partial}{\partial y}\phi\,,$$
$$V_3=y\frac{\partial}{\partial x}\phi-x\frac{\partial}{\partial y}\phi,\quad V_4=\frac{\partial}{\partial u}\phi\,,$$
$$V_5=\frac{\partial}{\partial x}\phi,\quad V_6=\frac{\partial}{\partial y}\phi\,. \tag{31}$$

From Table 8

Table 8 The commutator table for Example 6.

	V_1	V_2	V_3	V_4	V_5	V_6
V_1	0	0	0	$-V_4$	0	0
V_2	0	0	0	0	$-V_5$	$-V_6$
V_3	0	0	0	0	V_6	$-V_5$
V_4	V_4	0	0	0	0	0
V_5	0	V_5	$-V_6$	0	0	0
V_6	0	V_6	V_5	0	0	0

the Killing form is obtained

$$KF=a_1^2+2\,a_2^2-2\,a_3^2. \tag{32}$$

A one-dimensional optimal system is provided by:

(a) $V_1+a_2V_2,\ a_2\neq 0$, (b) $V_1+a_3V_3,\ a_3\neq 0$,
(c) $V_2+a_4\,\mathrm{e}^{\epsilon}V_4$, (d) $V_3+a_4\,\mathrm{e}^{\epsilon}V_4$,
(e) $V_1+a_5\,\mathrm{e}^{\epsilon}V_5+a_6\,\mathrm{e}^{\epsilon}V_6$,
(f) $a_4V_4+a_5\,\mathrm{e}^{\epsilon}V_5+a_6\,\mathrm{e}^{\epsilon}V_6$,
(g$_1$) $V_1+a_2V_2+a_3V_3+a_5\,\mathrm{e}^{\epsilon}V_5,\ a_2a_3\neq 0$,
(g$_2$) $V_1+a_2V_2+a_3V_3+a_6\,\mathrm{e}^{\epsilon}V_6,\ a_2a_3\neq 0$,
(h$_1$) $V_2+a_3V_3+a_4V_4+a_5\,\mathrm{e}^{\epsilon}V_5,\ a_3\neq 0$,
(h$_2$) $V_2+a_3V_3+a_4V_4+a_6\,\mathrm{e}^{\epsilon}V_6,\ a_3\neq 0$. (33)

This result is in accordance with Ref. [27] after adjusting the coefficients which contain ϵ.

(a) V_5, (b) V_6, (c) V_3, (d) V_2,
(e) $V_2+a_3V_3,\ a_3\neq 0$ (f) $V_4+a_3V_3,\ a_3\in R$,
(g) $V_1+a_3V_3,\ a_3\in R$, (h) $V_1+a_2V_2,\ a_2\in R$,
(i) $V_4+a_2V_2,\ a_2\in R$, (j) $a_5V_5+a_6V_6,\ a_5a_6\neq 0$,
(k) $V_2+a_3V_3+a_6V_6,\ a_3a_6\neq 0$,
(l) $V_1+a_5V_5+a_6V_6,\ a_5,a_6\in R$,
(m) $V_4+a_5V_5+a_6V_6,\ a_5,a_6\in R$
(n) $V_1+a_2V_2+a_3V_3+a_5V_5,\ a_2a_3\neq 0$,
(o) $V_1+a_2V_2+a_3V_3+a_6V_6,\ a_2a_3\neq 0$,
(p) $V_4+a_2V_2+a_3V_3+a_5V_5,\ a_2a_3\neq 0$,
(q) $V_4+a_2V_2+a_3V_3+a_6V_6,\ a_2a_3\neq 0$. (34)

The average running time for this example is 1.1356 seconds.

5.4 *More Examples of Real Two-, Three- and Four-Dimensional Lie Algebras*

Consider some nontrivial real Lie algebras listed in the first column of Table 9, which have appeared in Ref. [28]. Based on the existing nonzero commutation relations presented in the second column, the Killing forms, classification results from our program and running times are listed in the last three columns, respectively. Depending on the sign of a_1,a_2,a_3,a_4, we can adjust the coefficients of V_1, V_2, V_3, V_4, which contain ϵ to suitable value such as -1, 1, 0. In comparison, the output from our program is consistent with Ref. [28].

Table 9 One-dimensional optimal systems of real two-, three-, and four-dimensional Lie algebras. Parameter description: a_1, a_2, a_3, a_4 are arbitrary constants.

Algebra	Nonzero commutation relations	Killing form	One-dimensional optimal system	Run time (second)
A_2	$[V_1,V_2]=V_2$	a_1^2	$\{V_1\}$, $\{V_2\}$	0.093
$A_{3,3}$	$[V_1,V_3]=V_1$, $[V_2,V_3]=V_2$	$2a_3^2$	$\{V_3\}$, $\{a_1\,\mathrm{e}^{\epsilon}V_1+a_2\,\mathrm{e}^{\epsilon}V_2\}$	0.109
$A_{3,4}$ $(e(1,1))$	$[V_1,V_3]=V_1$, $[V_2,V_3]=-V_2$	$2a_3^2$	$\{V_3\}$, $\{V_2\}$, $\{V_1+a_2\,\mathrm{e}^{-2\epsilon}V_2\}$	0.109
$A_{3,5}^{a}$ $(0<\lvert a\rvert<1)$	$[V_1,V_3]=V_1$, $[V_2,V_3]=aV_2$	$a_3^2(a^2+1)$	$\{V_3\}$, $\{V_2\}$, $\{V_1+a_2\,\mathrm{e}^{(a-1)\epsilon}V_2\}$	0.110
$A_{3,8}$ $(\mathrm{su}(1,1))$	$[V_1,V_2]=V_1$, $[V_2,V_3]=V_3$, $[V_1,V_3]=-2V_2$	$2(a_2^2+4a_1a_3)$	$\{V_2\}$, $\{V_1\}$, $\{V_1-V_3\}$	0.125
$A_{3,9}$ $(\mathrm{su}(2))$	$[V_1,V_2]=V_3$, $[V_2,V_3]=V_1$, $[V_1,V_3]=-V_2$	$-2(a_1^2+a_2^2+a_3^2)$	$\{V_1\}$	0.156
$A_2\oplus 2A_1$	$[V_1,V_2]=V_2$	a_1^2	$\{V_1+a_3V_3+a_4V_4\}$, $a_3V_3+a_4V_4\}$, $\{V_2+\mathrm{e}^{\epsilon}(a_3V_3+a_4V_4\}$	0.125
$2A_2$	$[V_1,V_2]=V_2$, $[V_3,V_4]=V_4$	$a_1^2+a_3^2$	$\{V_4\}$, $\{V_2+a_4\,\mathrm{e}^{-\epsilon}V_4\}$, $\{V_1+a_3V_3,\ a_3\neq 0\}$, $\{V_1+a_4\,\mathrm{e}^{-\epsilon}V_4\}$, $\{V_3+a_2\,\mathrm{e}^{-\epsilon}V_2\}$	0.187
$A_{3,3}\oplus A_1$	$[V_1,V_3]=V_1$, $[V_2,V_3]=V_2$	$2a_3^2$	$\{V_3+a_4V_4\}$, $\{a_1V_1+a_2V_2\}$, $\{V_4+\mathrm{e}^{\epsilon}(a_1V_1+a_2V_2)\}$	0.125
$A_{3,4}\oplus A_1$	$[V_1,V_3]=V_1$, $[V_2,V_3]=-V_2$	$2a_3^2$	$\{V_3+a_4V_4\}$, $\{V_2+a_4\,\mathrm{e}^{\epsilon}V_4\}$, $\{V_1+a_2\,\mathrm{e}^{-2\epsilon}V_2+a_4\,\mathrm{e}^{-\epsilon}V_4\}$	0.140
$A_{3,5}^{a}\oplus A_1$ $(0<\lvert a\rvert<1)$	$[V_1,V_3]=V_1$, $[V_2,V_3]=aV_2$	$a_3^2(a^2+1)$	$\{V_3+a_4V_4\}$, $\{V_2+a_4\,\mathrm{e}^{-a\epsilon}V_4\}$, $\{V_1+a_2\,\mathrm{e}^{(a-1)\epsilon}V_2+a_4\,\mathrm{e}^{-\epsilon}V_4\}$	0.156
$A_{3,8}\oplus A_1$	$[V_1,V_2]=V_1$, $[V_2,V_3]=V_3$, $[V_1,V_3]=-2V_2$	$2(a_2^2+4a_1a_3)$	$\{V_1+a_4\,\mathrm{e}^{\epsilon}V_4\}$, $\{V_2+a_4V_4\}$, $\{V_4\}$, $\{V_1-V_3+a_4V_4\}$	0.203
$A_{4,3}$	$[V_1,V_4]=V_1$, $[V_3,V_4]=V_2$	a_4^2	$\{V_2\}$, $\{V_1+a_2\,\mathrm{e}^{-\epsilon}V_2\}$, $\{V_3+a_1V_1\}$, $\{V_4+a_3V_3\}$	0.156
$A_{4,5}^{a,a}$ $(-1\leq a<1,\ a\neq 0)$	$[V_1,V_4]=V_1$, $[V_2,V_4]=aV_2$, $[V_3,V_4]=aV_3$	$a_4^2(2a^2+1)$	$\{V_4\}$, $\{a_2V_2+a_3V_3\}$, $\{V_1+\mathrm{e}^{(a-1)\epsilon}(a_2V_2+a_3V_3)\}$	0.140
$A_{4,5}^{a,1}$ $(-1\leq a<1,\ a\neq 0)$	$[V_1,V_4]=V_1$, $[V_2,V_4]=aV_2$, $[V_3,V_4]=V_3$	$a_4^2(a^2+2)$	$\{V_4\}$, $\{a_1V_1+a_3V_3\}$, $\{V_2+\mathrm{e}^{(1-a)\epsilon}(a_1V_1+a_3V_3)\}$	0.172
$A_{4,5}^{1,1}$	$[V_1,V_4]=V_1$, $[V_2,V_4]=V_2$, $[V_3,V_4]=V_3$	$3a_4^2$	$\{V_4\}$, $\{V_3\}$, $\{V_2+a_3V_3\}$, $\{V_1+a_2V_2+a_3V_3\}$	0.156
$A_{4,8}$	$[V_2,V_3]=V_1$, $[V_2,V_4]=V_2$, $[V_3,V_4]=-V_3$	$2a_4^2$	$\{V_1\}$, $\{V_3\}$, $\{V_4+a_1V_1\}$, $\{V_2+a_3\,\mathrm{e}^{-2\epsilon}V_3\}$	0.203
$A_{4,9}^{b}$ $(0<\lvert b\rvert<1)$	$[V_2,V_3]=V_1$, $[V_2,V_4]=V_2$, $[V_1,V_4]=(1+b)V_1$, $[V_3,V_4]=bV_3$	$2a_4^2(b^2+b+1)$	$\{V_1\}$, $\{V_3\}$, $\{V_4\}$, $\{V_2+a_3\,\mathrm{e}^{-(b+1)\epsilon}V_3\}$	0.234
$A_{4,9}^{1}$	$[V_2,V_3]=V_1$, $[V_1,V_4]=2V_1$, $[V_2,V_4]=V_2,[V_3,V_4]=V_3$	$6a_4^2$	$\{V_1\}$, $\{V_3\}$, $\{V_4\}$ $\{V_2+a_3V_3\}$	0.187
$A_{4,9}^{0}$	$[V_2,V_3]=V_1$, $[V_1,V_4]=V_1$, $[V_2,V_4]=V_2$	$2a_4^2$	$\{V_1\}$, $\{V_2+a_3\,\mathrm{e}^{-\epsilon}V_3\}$ $\{V_3\}$, $\{V_4+a_3V_3\}$	0.156

5.5 *Examples of Some Classical Lie Algebras*

Finally, we also consider some classical Lie algebras using our program. Their corresponding nonzero commutation relations, Killing forms, classification results, and running times are given in Table 10.

Table 10 One-dimensional optimal systems of classical Lie algebras.

Algebra	Nonzero commutation relations	Killing form	One-dimensional optimal system	Run time /second
$\mathrm{sl}(2,R)$	$[V_1,V_2]=V_3$, $[V_2,V_3]=2V_2$, $[V_1,V_3]=-2V_1$	$8(a_3^2+a_1a_2)$	$\{V_2\}$, $\{V_3\}$, $\{V_1-V_2\}$	0.125
$\mathrm{su}(1,1)$ ($\sim \mathrm{so}(1,2)\sim \mathrm{sp}(2,R)$)	$[V_1,V_2]=V_1$, $[V_2,V_3]=V_3$, $[V_1,V_3]=-2V_2$	$2(a_2^2+4a_1a_3)$	$\{V_2\}$, $\{V_1\}$, $\{V_1-V_3\}$	0.125
$\mathrm{su}(2)$ ($\sim \mathrm{so}(3)$)	$[V_1,V_2]=V_3$, $[V_2,V_3]=V_1$, $[V_1,V_3]=-V_2$	$-2(a_1^2+a_2^2+a_3^2)$	$\{V_1\}$	0.156
$\mathrm{so}(n,R)(n\geq 3)$		$-2(n-2)(a_1^2+a_2^2+\cdots+a_{n(n-1)/2}^2)$	$\{V_1\}$	–

6 Conclusions

In this paper, we have presented and clarified the *Maple* package ONEOptimal to construct one-dimensional optimal system of Lie algebra based on Olver's method. The program ONEOptimal can compute the commutator table, adjoint representation table and Killing form automatically, while it can also execute the optimization process step by step. ONEOptimal is very easy to perform as it requires minimal user input and the output with instructions is easy to understand. The program will play a significant role in the search of group invariant solutions. How to involve the cases with more variables in the Killing form and realize the mechanization of high-dimensional optimal system of subalgebras is worthy of our further study.

Appendix: The Detailed Usage of Package ONEOptimal

The package ONEOptimal will work on *Maple* 13 or higher version. In the following two classical examples (Examples 1 and 5 in Sec. 5) are given to illustrate how to use this package. The detailed input and output are demonstrated as follows.

Example 1 Consider the KdV equation (4), one can proceed as follows:

```
#Import the package ONEOptimal
> with(ONEOptimal):
#Defination of the function with variables
> alias(phi=phi(x,t,u));
#Defination of the vector fields
> kdv:={diff(phi,x), diff(phi,t), t*diff(phi,x)+diff(phi,u),
      x*diff(phi,x)+3*t*diff(phi,t)-2*u*diff(phi,u)};
#Run the main routine Get_Optimal
> Get_Optimal(kdv,{},0);  #{} -- There is no constant in this algebra.
                           0 -- Do not show the detailed matrices Cm and So.
```

The output is:

There is no center.

The commutator table for this algebra is:

$$\mathrm{CT}=\begin{bmatrix}0 & -2V_1 & -V_4 & 0\\ 2V_1 & 0 & -3V_3 & -V_4\\ V_4 & 3V_3 & 0 & 0\\ 0 & V_4 & 0 & 0\end{bmatrix},$$

with the generators:

$$V_1=t\phi_x+\phi_u,\quad V_2=x\phi_x+3t\phi_t-2u\phi_u,$$
$$V_3=\phi_t,\quad V_4=\phi_x.$$

The adjoint representation table is constructed

$$\mathrm{AdT}=\begin{bmatrix}V_1 & V_2+2\epsilon V_1 & V_3+\epsilon V_4 & V_4\\ \mathrm{e}^{-2\epsilon}V_1 & V_2 & \mathrm{e}^{3\epsilon}V_3 & \mathrm{e}^{\epsilon}V_4\\ V_1-\epsilon V_4 & V_2-3\epsilon V_3 & V_3 & V_4\\ V_1 & V_2-\epsilon V_4 & V_3 & V_4\end{bmatrix}.$$

The Killing form is:

$$\mathrm{KF}=14a_2^2.$$

Step 1

$$\mathrm{Eq}_1=a_1V_1+a_2V_2+a_3V_3+a_4V_4.$$

The coefficient matrix is:

$$\mathrm{Cm}_1=\begin{bmatrix} a_1+2\,a_2\epsilon & a_2 & a_3 & a_3\epsilon+a_4 \\ a_1\,\mathrm{e}^{-2\,\epsilon} & a_2 & a_3\,\mathrm{e}^{3\,\epsilon} & a_4\,\mathrm{e}^{\epsilon} \\ a_1 & a_2 & -3\,a_2\epsilon+a_3 & -a_1\epsilon+a_4 \\ a_1 & a_2 & a_3 & -a_2\epsilon+a_4 \end{bmatrix}.$$

The corresponding solution matrix is:

$$\mathrm{So}_1=\begin{bmatrix} \{-\frac{1}{2}\frac{a_1}{a_2}\} & \{\} & \{\} & \{-\frac{a_4}{a_3}\} \\ \{\} & \{\} & \{\} & \{\} \\ \{\} & \{\} & \{\frac{1}{3}\frac{a_3}{a_2}\} & \{\frac{a_4}{a_1}\} \\ \{\} & \{\} & \{\} & \{\frac{a_4}{a_2}\} \end{bmatrix}.$$

Case 1

$$a_2\neq 0,\quad a_1=0,\quad a_3=0,\quad a_4=0\,,$$
$$\mathrm{Result}=V_2,\quad a_{\{2\}}\neq 0\,.$$

Case 2

$$a_2=0\,.$$

Case 2.1

$$a_1\neq 0,\quad a_4=0\,,$$
$$\mathrm{Result}=V_1+\mathrm{e}^{5\,\epsilon}a_3V_3,\quad a_{\{1\}}\neq 0\,.$$

Case 2.2

$$a_1=0\,.$$

Case 2.2.1

$$a_3\neq 0,\quad a_4=0\,,$$
$$\mathrm{Result}=V_3,\quad a_{\{3\}}\neq 0\,.$$

Case 2.2.2

$$a_3=0,\quad \mathrm{Result}=a_4\,\mathrm{e}^{\epsilon}V_4\,.$$

In this example, depending on the sign of a_3, we can make the coefficient of V_3 either $+1,-1$ or 0 in Case 2.1. Summarize the above cases, the one-dimensional optimal system equals to: $V_2, V_1+V_3, V_1-V_3, V_1, V_3, V_4$.

Example 5 Consider the heat equation (25), one can proceed as follows:

```
#Import the package ONEOptimal
> with(ONEOptimal):
#Defination of the function with variables
> alias(phi=phi(x,t,u));
#Defination of the vector fields
> heat:={u*diff(phi,u), 2*t*diff(phi,x)-x*u*diff(phi,u),
       x*diff(phi,x)+2*t*diff(phi,t), 4*t*x*diff(phi,x)
       +4*t^2*diff(phi,t)-(x^2+2*t)*u*diff(phi,u),
       diff(phi,t),diff(phi,x)};
#Run the main routine Get_Optimal
> Get_Optimal(heat,{},0); #{}-- There is no constant in this algebra.
                            0-- Do not show the detailed matrices Cm and So.
```

The output is:
The center of the algebra is:

$$\left\{u\frac{\partial}{\partial u}\phi\right\}.$$

The centers can not be deleted.
The commutator table for this algebra is:

$$\mathrm{CT}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -V_2 & 0 & -2\,V_6 & V_1 \\ 0 & V_2 & 0 & 2\,V_4 & -2\,V_5 & -V_6 \\ 0 & 0 & -2\,V_4 & 0 & 2\,V_1-4\,V_3 & -2\,V_2 \\ 0 & 2\,V_6 & 2\,V_5 & -2\,V_1+4\,V_3 & 0 & 0 \\ 0 & -V_1 & V_6 & 2\,V_2 & 0 & 0 \end{bmatrix},$$

with the generators:

$$V_1=u\phi_u,\quad V_2=2t\phi_x-xu\phi_u,\quad V_3=x\phi_x+2t\phi_t\,,$$
$$V_4=4tx\phi_x+4t^2\phi_t-(x^2+2t)u\phi_u,\quad V_5=\phi_t,\quad V_6=\phi_x\,.$$

The adjoint representation table is constructed:

$$\mathrm{AdT}=\begin{bmatrix} V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \\ V_1 & V_2 & V_3+\epsilon V_2 & V_4 & V_5+2\epsilon V_6-\epsilon^2 V_1 & V_6-\epsilon V_1 \\ V_1 & \mathrm{e}^{-\epsilon}V_2 & V_3 & \mathrm{e}^{-2\epsilon}V_4 & \mathrm{e}^{2\epsilon}V_5 & \mathrm{e}^{\epsilon}V_6 \\ V_1 & V_2 & V_3+2\epsilon V_4 & V_4 & -2\epsilon V_1+V_5+4\epsilon V_3+4\epsilon^2 V_4 & V_6+2\epsilon V_2 \\ V_1 & V_2-2\epsilon V_6 & V_3-2\epsilon V_5 & 2\epsilon V_1+V_4-4\epsilon V_3+4\epsilon^2 V_5 & V_5 & V_6 \\ V_1 & V_2+\epsilon V_1 & V_3-\epsilon V_6 & V_4-2\epsilon V_2-\epsilon^2 V_1 & V_5 & V_6 \end{bmatrix}.$$

The Killing form is:

$$\mathrm{KF}=10a_3^2-40a_4a_5\,.$$

Step 1

$$\mathrm{Eq}_1=a_1V_1+a_2V_2+a_3V_3+a_4V_4+a_5V_5+a_6V_6\,.$$

The coefficient matrix is:

$$\mathrm{Cm}_1=\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_1-a_6\epsilon-a_5\epsilon^2 & a_2+a_3\epsilon & a_3 & a_4 & a_5 & 2\,a_5\epsilon+a_6 \\ a_1 & a_2\mathrm{e}^{-\epsilon} & a_3 & a_4\,\mathrm{e}^{-2\epsilon} & a_5\mathrm{e}^{2\epsilon} & a_6\,\mathrm{e}^{\epsilon} \\ a_1-2\,a_5\epsilon & a_2+2\,a_6\epsilon & a_3+4\,a_5\epsilon & a_4+2\,a_3\epsilon+4a_5\epsilon^2 & a_5 & a_6 \\ a_1+2\,a_4\epsilon & a_2 & a_3-4\,a_4\epsilon & a_4 & a_5-2\,a_3\epsilon+4\,a_4\epsilon^2 & -2\,a_2\epsilon+a_6 \\ a_1+a_2\epsilon-a_4\epsilon^2 & a_2-2\,a_4\epsilon & a_3 & a_4 & a_5 & -a_3\epsilon+a_6 \end{bmatrix}.$$

The corresponding solution matrix is:

$$\mathrm{So}_1=\begin{bmatrix} \{\} & \{\} & \{\} & \{\} & \{\} & \{\} \\ \{\frac{-a_6\pm\sqrt{a_6^2+4\,a_5a_1}}{2a_5}\} & \{\frac{-a_2}{a_3}\} & \{\} & \{\} & \{\} & \{\frac{-a_6}{2a_5}\} \\ \{\} & \{\} & \{\} & \{\} & \{\} & \{\} \\ \{\frac{a_1}{2a_5}\} & \{\frac{-a_2}{2a_6}\} & \{\frac{-a_3}{4a_5}\} & \{\frac{-a_3\pm\sqrt{a_3^2-4\,a_5a_4}}{4a_5}\} & \{\} & \{\} \\ \{-\frac{a_1}{2a_4}\} & \{\} & \{\frac{a_3}{4a_4}\} & \{\} & \{\frac{a_3\pm\sqrt{a_3^2-4a_4a_5}}{4a_4}\} & \{\frac{a_6}{2a_2}\} \\ \{\frac{a_2\pm\sqrt{a_2^2+4\,a_4a_1}}{2}a_4\} & \{\frac{a_2}{2a_4}\} & \{\} & \{\} & \{\} & \{\frac{a_6}{a_3}\} \end{bmatrix}.$$

Case 1

$$0<a_5a_4,\quad a_4\neq 0,\quad a_5\neq 0\,,$$
$$a_3=0,\quad a_2=0,\quad a_6=0\,,$$
$$\mathrm{Result}=V_4+\,\mathrm{e}^{2\epsilon}a_1V_1+\,\mathrm{e}^{4\epsilon}a_5V_5,\ a_{\{4,5t\}}\neq 0\,.$$

Case 2

$$a_4=0\,.$$

Case 2.1

$$a_3\neq 0,\quad a_5=0,\quad a_2=0,\quad a_6=0\,,$$
$$\mathrm{Result}=a_1V_1+V_3,\quad a_{\{3\}}\neq 0\,.$$

Case 3

$$a_4=0,\quad a_3=0\,.$$

Case 3.1

$$a_5\neq 0,\quad a_6=0\,.$$

Case 3.1.1

$$a_2\neq 0,\quad a_1=0\,,$$
$$\mathrm{Result}=V_2+a_5\,\mathrm{e}^{3\epsilon}V_5,\quad a_{\{2,5\}}\neq 0\,.$$

Case 3.1.2

$$a_2=0\,,$$
$$\mathrm{Result}=\,\mathrm{e}^{-2\epsilon}a_1V_1+V_5,\quad a_{\{5\}}\neq 0\,.$$

Case 4

$$a_3=0,\quad a_4=0,\quad a_5=0\,.$$

Case 4.1

$$a_2\neq 0,\quad a_6=0,\quad a_1=0\,,$$
$$\mathrm{Result}=V_2,\quad a_{\{2\}}\neq 0\,.$$

Case 4.2

$$a_2=0\,.$$

Case 4.2.1

$$a_6\neq 0,\quad a_1=0,\quad \mathrm{Result}=V_6,\ a_{\{6\}}\neq 0\,.$$

Case 4.2.2

$$a_6=0,\quad \mathrm{Result}=a_1V_1\,.$$

In this example, depending on the sign of a_5, we can make the coefficient of V_5 only $+1$ in Case 1 and either $+1$ or -1 in Case 3.1.1. The coefficient of V_1 can be set either $+1,-1$ or 0 in Case 3.1.2, but arbitrary in both Case 1 and Case 2.1. Summarize the above cases, this one-dimensional optimal system equals to: $V_4+V_5+bV_1$, V_3+aV_1, V_2+V_5, V_2-V_5, V_5+V_1, V_5-V_1, V_5, V_2, V_6, V_1, where $a,b\in R$.

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