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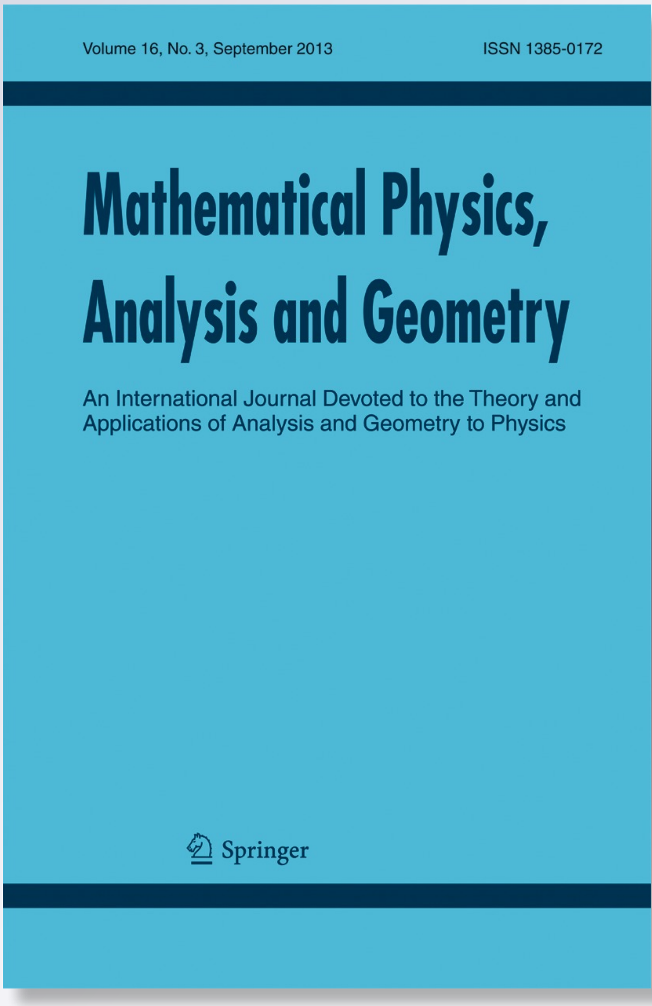

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Long-time Asymptotic for the Derivative Nonlinear Schrödinger Equation with Step-like Initial Value

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Abstract We study long-time asymptotics of the solution to the Cauchy problem for the Gerdjikov-Ivanov type derivative nonlinear Schrödinger equation

$$iq_t + q_{xx} - iq^2\bar{q}_x + \frac{1}{2}|q|^4 q = 0$$

with step-like initial data $q(x, 0) = 0$ for $x \leqslant 0$ and $q(x, 0) = Ae^{-2iBx}$ for $x > 0$, where $A > 0$ and $B \in \mathbb{R}$ are constants. We show that there are three regions in the half-plane $\{(x, t)| -\infty < x < \infty, t > 0\}$, on which the asymptotics has qualitatively different forms: a slowly decaying self-similar wave of Zakharov-Manakov type for $x > -4tB$, a plane wave region: $x < -4t\left(B + \sqrt{2A^2\left(B + \frac{A^2}{4}\right)}\right)$, an elliptic region: $-4t\left(B + \sqrt{2A^2\left(B + \frac{A^2}{4}\right)}\right) < x < -4tB$. Our main tools include asymptotic analysis, matrix Riemann-Hilbert problem and Deift-Zhou steepest descent method.

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1 Introduction

The classical, mathematical model for non-linear pulse propagation in the picosecond time scale with the anomalous dispersion regime in an isotropic, homogeneous, lossless, non-amplifying, polarization-preserving single-mode optical fibre is the nonlinear Schrödinger (NLS) equation [1, 2]. However, experiments and theories on the propagation of high-power ultrashort pulses in long monomode optical fibres have shown that the NLS equation is no longer valid in the subpicosecond-femtosecond time scale. In this case, some additional non-linear terms and higher-order linear dispersion should be taken into account and this model can be described by the following non-linear evolution equation [38]

$$iu_\xi + \frac{1}{2}u_{\tau\tau} + |u|^2u + is(|u|^2u)_\tau = -i\tilde{\Gamma}u + i\tilde{\delta}u_{\tau\tau\tau} + \frac{\tau_n}{\tau_0}u(|u|^2)_\tau, \tag{1.1}$$

where u is the slowly varying amplitude of the complex field envelope, ξ is the propagation distance along the fibre length, τ is the time measured in a frame of reference moving with the pulse at the group velocity, $s > 0$ governs the effects due to the intensity dependence of the group velocity, $\tilde{\Gamma}$ is the intrinsic fibre loss, $\tilde{\delta}$ governs the effects of the third-order linear dispersion, and $\frac{\tau_n}{\tau_0}$, where τ_0 is the normalized input pulsewidth and τ_n is related to the slope of the Raman gain curve and governs the soliton self-frequency [5, 6] shift effect.

By setting the right-hand side of (1.1) to zero, we obtain the following equation

$$iu_\xi + \frac{1}{2}u_{\tau\tau} + |u|^2u + is(|u|^2u)_\tau = 0, \tag{1.2}$$

which is related to the Kaup–Newell equation

$$iq_t(x,t) = -q_{xx}(x,t) + (\bar{q}q^2)_x, \tag{1.3}$$

by change of variables

$$u(\xi,\tau) = q(x,t)\mathrm{e}^{i\left(\frac{t}{4s^4}-\frac{x}{2s^2}\right)}, \qquad \xi = \frac{t}{2s^2}, \qquad \tau = -\frac{x}{2s} + \frac{t}{2s^3}.$$

And we note that if we replace [7] x by $-x$, equation (1.3) becomes

$$iq_t(x,t) = -q_{xx}(x,t) - (\bar{q}q^2)_x. \tag{1.4}$$

We find that if we formulate a Riemann-Hilbert problem for solutions of the inverse spectral [8] problem of the equation (1.4), the solutions cannot approach 2×2 identity matrix $\mathbb{I}$ as $k \to \infty$. It is well-known that there are three kinds of celebrated derivative nonlinear Schrödinger (DNLS) equations, including Kaup-Newell equation (1.4), Chen-Lee-Liu equation [39]

$$iq_t + q_{xx} + i|q|^2q_x = 0,$$

and Gerdjikov-Ivanov(GI) equation [40, 42]

$$iq_t + q_{xx} - iq^2\bar{q}_x + \frac{1}{2}|q|^4 q = 0. \tag{1.5}$$

It has been found that they may be transformed into each other by gauge transformations [40, 41]. it is shown that the solutions of its spectral problem for the GI equation approach the 2×2 identity matrix $\mathbb{I}$ as $k \to \infty$ [42]. So, we consider long-time asymptotics for the GI equation which is named as DNLS equation in the following of this paper.

Initial value problems for nonlinear evolution equations with step-like initial data have attracted much attention since the early 1970s [16–19], but only a few rigorous results concerning the long-time behavior of solutions of such problems were available. In 1980s–1990s, a considerable progress was achieved following the development of the theory of Whitham deformations [20] and the analysis of matrix Riemann-Hilbert problem representations of solutions of initial value problems [21–23]. Most complete results,obtained by using this approach,were related to integrable equations, for which linear operators from the associated [14, 15] Lax pair were self-adjoint and thus their spectrum was real. In [22], Bikbaev considered the case of the focusing nonlinear Schrödinger equation, which required the development of a much more complicated complex form of the theory of Whitham deformations.

A completely rigorous approach for studying asymptotics of solutions of integrable nonlinear equations was introduced by Deift and Zhou [9]. This approach was inspired by earlier works of Manakov [3, 4, 24] and Its [25]; see [10] for a detailed historical review and further extended by Deift, Venakides, and Zhou [26, 27]. This approach is based on the development of the nonlinear steepest descent method for Riemann-Hilbert problems associated with integrable nonlinear equations. Being originally introduced for studying initial value problems with decaying initial data, this approach was recently adapted by Buckingham and Venakides to problems with shock-type oscillating initial data for focusing nonlinear Schrödinger equation [28]. A central role in this development is played by the so-called [12] g-function mechanism allowing to deform the original Riemann-Hilbert problem to a form that can be asymptotically treated with the help of associated singular integral equations.

The Riemann-Hilbert problem approach to initial value problems with non-decaying step-like initial data shares many issues with the adaptation of this approach for studying initial-boundary value problems with non-decaying boundary data [29, 31, 33, 34]. However, there is an important difference: in the latter case, the construction of the associated Riemann-Hilbert problem normally requires the knowledge of spectral functions associated with over-specified initial and boundary data, which leads to a conditional character on the asymptotic results, see [29]). As for the initial value problems considered in this paper, the Riemann-Hilbert construction requires only initial data, the issue of over determination does not arise.

In this paper,we consider a pure step-like initial value problem for the DNLS equation

$$iq_t + q_{xx} - iq^2\bar{q}_x + \frac{1}{2}|q|^4 q = 0, \qquad x \in \mathbb{R}, t > 0, \tag{1.6a}$$

$$q(x,0) = q_0(x) = \begin{cases} 0 & \text{if } x \geqslant 0, \\ A\mathrm{e}^{-2iBx} & \text{if } x < 0, \end{cases} \tag{1.6b}$$

where $A > 0$ and $B \in \mathbb{R}$ are some constants. Kitaev and Vartanian got the leading order long-time asymptotic for the KN-type DNLS equation with the decaying initial value [36, 37], and the higher order long-time asymptotic in [38].

Since the DNLS equation (1.6a) has a plane wave solution

$$q^p(x,t) = A\mathrm{e}^{-2iBx+2i\omega t}, \tag{1.7}$$

with

$$\omega := A^2 B - 2B^2 + \frac{A^4}{4}, \tag{1.8}$$

which is consistent with (1.6b) for $x < 0$, that is, $q^p(x,0) = q_0(x)$. We assume that the solution $q(x,t)$ of the initial value problem (1.6a) evaluated at any $t > 0$ has the following behavior

$$q(x,t) = o(1), \qquad x \to +\infty, \tag{1.9}$$

$$q(x,t) = q^p(x,t) + o(1), \qquad x \to -\infty, \tag{1.10}$$

where $o(1)$ means sufficiently fast decay to zero. This assumption can be justified a posteriori, by evaluating the large-x behavior of the solution of the Riemann-Hilbert problem formulated in Section 3.

Recently, Monvel, Kotlyarov, and Shepelsky considered the long-time dynamics of the initial value problem for the focusing nonlinear Schrödinger equation with step-like data [32]. The strategy of the Riemann-Hilbert problem deformations that we adopt in this paper is similar, though not identical, to that in [28]. In particular, the realization of the g-function mechanism is different as well as the resulting asymptotic picture.

As we have already mentioned, the main tool available now for studying rigorously the long-time asymptoitcs of solutions of initial value problem for integrable nonlinear equations is the asymptotic analysis associated Riemann-Hilbert problems, whose construction involves dedicated solutions of the system of two linear equations, the Lax pair associated with the integrable nonlinear equation.

For the DNLS equation (1.6a), a Lax pair is as follows [36]:

$$\begin{aligned} \Psi_x(x,t;k) &= M(x,t;k)\Psi(x,t;k), \\ \Psi_t(x,t;k) &= N(x,t;k)\Psi(x,t;k), \end{aligned} \tag{1.11}$$

where

$$\begin{aligned} M(x,t;k) &= -ik^2\sigma_3 + kQ + \frac{i}{2}|q|^2\sigma_3, \\ N(x,t;k) &= -2ik^4\sigma_3 + 2k^3 Q + ik^2|q|^2\sigma_3 - ikQ_x\sigma_3 \\ &\quad + \frac{i}{4}|q|^4\sigma_3 + \frac{1}{2}(q\bar{q}_x - \bar{q}q_x)\sigma_3, \end{aligned} \tag{1.12}$$

with $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $\Psi(x, t; k)$ is a 2×2 matrix-value function, $k \in \mathbb{C}$ is a spectral parameter, and the matrix coefficient Q is expressed in terms of a scalar function q

$$Q = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}. \tag{1.13}$$

It is well-known that this over-determined system of equations (1.11) is compatible if and only if $q(x, t)$ solves the DNLS equation (1.6a) [36].

In Section 2, we present spectral functions associated the Lax pair (1.11). These functions are then used in Section 3 for constructing a basic Riemann-Hilbert problem, whose solution gives the solution of the initial value problem (1.6). Section 4, we focus on the asymptotic analysis of this Riemann-Hilbert problem leading to asymptotic formulas for the solution of the original Cauchy problem (1.6).

2 Eigenfunctions

Let Q^p be defined by (1.13) with q^p instead of q. A particular solution of the system (1.11) with Q^p instead of Q, is given by (see Appendix A)

$$\Psi^p(x, t; k) = \mathrm{e}^{i(\omega t - Bx)\sigma_3} E(k) \mathrm{e}^{-i(xX(k)+t\Omega(k))\sigma_3}, \tag{2.1}$$

where

$$X(k) = \sqrt{\left(k^2 - B - \frac{A^2}{2}\right)^2 + k^2 A^2}, \tag{2.2}$$

$$\Omega(k) = 2(k^2 + B)X(k), \tag{2.3}$$

$$E(k) = \frac{1}{2}\begin{pmatrix} \varphi(k) + \frac{1}{\varphi(k)} & \varphi(k) - \frac{1}{\varphi(k)} \\ \varphi(k) - \frac{1}{\varphi(k)} & \varphi(k) + \frac{1}{\varphi(k)} \end{pmatrix}, \tag{2.4}$$

and

$$\varphi(k) = \left(\frac{k^2 - B - \frac{A^2}{2} - ikA}{k^2 - B - \frac{A^2}{2} + ikA}\right)^{\frac{1}{4}}. \tag{2.5}$$

The branch cut for X and φ is taken along the segment

$$\gamma \cup \bar{\gamma} := \{k \in \mathbb{C} | k_1^2 - k_2^2 = B, k_1^2 \leqslant C^2\}, \tag{2.6}$$

where $\gamma = \{k \in \mathbb{C} | k_1^2 - k_2^2 = B, k_1^2 \leqslant C^2, \mathrm{Im}k^2 > 0\}$, $C^2 = B + \frac{A^2}{4}$, $k_1 = \mathrm{Re}k$ and $k_2 = \mathrm{Im}k$. And the branches are fixed by the asymptotics:

$$X(k) = k^2 - B + O\left(\frac{1}{k^2}\right), \qquad \text{as } k \to \infty,$$

$$\varphi(k) = 1 + O\left(\frac{1}{k}\right), \qquad \text{as } k \to \infty.$$

We find that $\Omega(k) = 2k^4 + \omega + O\left(\frac{1}{k}\right)$, as $k \to \infty$. We also find that $\mathrm{Im}X(k) = 0$ is

$$k_1 k_2 \left(k_1^2 - k_2^2 - B\right) = 0, \tag{2.7}$$

which is on

$$\Sigma := \mathbb{R} \cup i\mathbb{R} \cup \gamma \cup \bar{\gamma}. \tag{2.8}$$

Thus, for any $t \geqslant 0$, $\Psi^p(x, t; k)$ is bounded in x if and only if $k \in \Sigma$ (Fig. 1).

Let $q(x,t)$ be a solution of the Cauchy problem (1.6) satisfying the asymptotic conditions (1.9), (1.10); $Q(x,t)$ and $Q^p(x,t)$ be defined by (1.13), in terms of q and q^p, respectively. Define the 2×2 matrix-value functions $\mu_j(x,t;k)$, $j = 1, 2$, $-\infty < x < \infty$, $0 \leqslant t < \infty$, as the solutions of the Volterra integral equations

$$\mu_1(x,t;k) = \mathbb{I} + \int_{+\infty}^{x} \mathrm{e}^{ik^2(y-x)\sigma_3} \left(\left(kQ + \tfrac{i}{2}|q|^2\sigma_3\right)\mu_1\right)(y,t;k)\mathrm{e}^{-ik^2(y-x)\sigma_3} dy, \quad k^2 \in \mathbb{R}, \tag{2.9}$$

$$\begin{aligned} \mu_2(x,t;k) = {} & \mathrm{e}^{i(\omega t - Bx)\hat{\sigma}_3} E(k) \\ & + \int_{-\infty}^{x} \Gamma^p(x,y,t,k) \left[k(Q - Q^p) + \frac{i}{2}|q|^2\sigma_3 - \frac{i}{2}A^2\sigma_3\right](y,t) \\ & \mu_2(y,t,k)\mathrm{e}^{-i(X(k)+B)(y-x)\sigma_3} dy, \qquad k \in \Sigma, \end{aligned} \tag{2.10}$$

where

$$\Gamma^p(x,y,t,k) := \Psi^p(x,t,k)[\Psi^p(y,t,k)]^{-1}.$$

Note that Γ^p can be written in the form

$$\Gamma^p(x,y,t,k) = \mathrm{e}^{i(\omega t - Bx)\sigma_3} G^p(x,y,k) \mathrm{e}^{-i(\omega t - By)\sigma_3},$$

where

$$G^p(x,y,k) = \begin{pmatrix} \alpha + i\left(k^2 - B - \frac{A^2}{2}\right)\beta & -kA\beta \\ kA\beta & \alpha - i\left(k^2 - B - \frac{A^2}{2}\right)\beta \end{pmatrix},$$

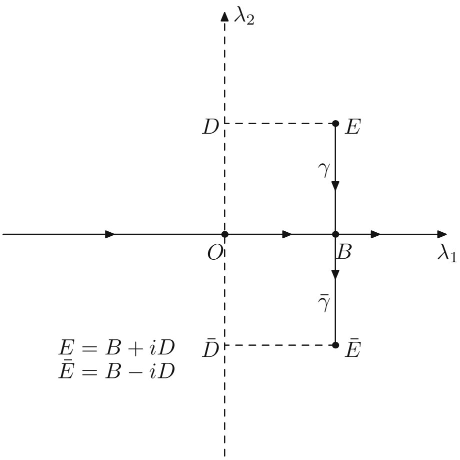


Fig. 1 The oriented contour $\Sigma = \mathbb{R} \cup \gamma \cup \bar{\gamma}$

with

$$\alpha = \cos[(y-x)X(k)], \qquad \beta = \frac{\sin[(y-x)X(k)]}{X(k)}.$$

For any $(x, y) \in \mathbb{R}^2$, $G^p(x, y, k)$ is an entire function of k with asymptotic behavior

$$G^p(x, y, k) = \mathrm{e}^{i(y-x)\left(k^2-B-\frac{A^2}{2}\right)\sigma_3}\left[\mathbb{I} + O\left(\frac{1}{k}\right)\right], \qquad \text{as } k \to \infty, \quad \mathrm{Im}k^2 = 0.$$

We denote the first and second columns of $\mu_j(x, t; k)$ by $\mu_j^{(1)}(x, t, k)$ and $\mu_j^{(2)}(x, t, k)$, then from (2.9) and (2.10), $j = 1, 2$, analytic properties of the matrices $\mu_j(x, t; k)$ can be summarized as follows.

Proposition 2.1 *The matrices $\mu_1(x, t; k)$ and $\mu_2(x, t; k)$ have the following properties*

(i) *$det\mu_1(x, t, k) = \mu_2(x, t; k) = 1$.*

(ii) *The functions $\Phi(x, t, k)$ and $\Psi(x, t, k)$ defined by*

$$\Psi(x, t, k) := \mu_1(x, t, k)\mathrm{e}^{-ik^2x\sigma_3 - 2ik^4t\sigma_3},$$
$$\Phi(x, t, k) := \mu_2(x, t; k)\mathrm{e}^{-ix(X(k)+B)\sigma_3 - it(\Omega(k)-\omega)\sigma_3}$$

satisfy the Lax pair equations (1.11).

(iii) *$\mu_1^{(1)}(x, t, k)$ is analytic in $\mathrm{Im}k^2 < 0$ and*

$$\mu_1^{(1)}(x, t, k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O\begin{pmatrix} \frac{1}{k} \end{pmatrix}, \text{ as } k \to \infty, \quad \mathrm{Im}k^2 \leqslant 0.$$

(iv) *$\mu_1^{(2)}(x, t, k)$ is analytic in $\mathrm{Im}k^2 > 0$ and*

$$\mu_1^{(2)}(x, t, k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O\begin{pmatrix} \frac{1}{k} \end{pmatrix}, \text{ as } k \to \infty, \quad \mathrm{Im}k^2 \geqslant 0.$$

(v) *$\mu_2^{(1)}(x, t, k)$ is analytic in $\mathrm{Im}k^2 > 0 \backslash \gamma$, has a jump across γ, and*

$$\mu_2^{(1)}(x, t, k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O\begin{pmatrix} \frac{1}{k} \end{pmatrix}, \text{ as } k \to \infty, \quad \mathrm{Im}k^2 \geqslant 0.$$

(vi) *$\mu_2^{(2)}(x, t, k)$ is analytic in $\mathrm{Im}k^2 < 0 \backslash \bar{\gamma}$, has a jump across $\bar{\gamma}$, and*

$$\mu_2^{(2)}(x, t, k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O\begin{pmatrix} \frac{1}{k} \end{pmatrix}, \text{ as } k \to \infty, \quad \mathrm{Im}k^2 \leqslant 0.$$

(vii) *Moreover,*

$$\mu_j(x, t, k) = \mathbb{I} + \frac{\tilde{\mu}(x, t)}{ik} + o\left(\frac{1}{k}\right)$$

as $k \to \infty$ along curves transversal to the real and image axis, where

$$[\sigma_3, \tilde{\mu}(x, t)] = \begin{pmatrix} 0 & q(x, t) \\ -\bar{q}(x, t) & 0 \end{pmatrix}$$

(viii) *$\mu_2^{(1)}(x, t, k)(k^2 - E)^{\frac{1}{4}}$ is bounded near $k^2 = E$ and $\mu_2^{(2)}(x, t, k)(k^2 - \bar{E})^{\frac{1}{4}}$ is bounded near $k^2 = \bar{E}$.*

Proof From (2.9)–(2.10) and the relation between Ψ, Φ and μ_1, μ_2, we have

$$\Psi(x,t,k) = \mathrm{e}^{-i(k^2x+2k^4t)\sigma_3} + \int_{+\infty}^{x} \mathrm{e}^{ik^2(y-x)\sigma_3}\left(\left(kQ+\frac{i}{2}|q|^2\sigma_3\right)\Psi\right)(y,t;k)dy \tag{2.11}$$

$$\begin{aligned}\Phi(x,t,k) \;=\; \Psi^p(x,t,k) + \int_{-\infty}^{x} \Gamma^p(x,y,t,k)\Big[k(Q-Q^p)(y,t)\\ + \frac{i}{2}|q|^2(y,t)\sigma_3 - \frac{i}{2}A^2\sigma_3\Big]\Phi(y,t,k)dy\end{aligned} \tag{2.12}$$

We just prove Φ satisfy the Lax pair equations (1.11), and the case for Ψ is simple.

$$\begin{aligned}\Phi_x(x,t,k) = \Psi_x^p(x,t,k) + \left[k(Q-Q^p)(x,t)+\frac{i}{2}|q|^2(x,t)\sigma_3 - \frac{i}{2}A^2\sigma_3\right]\Phi(x,t,k)\\ + \int_{-\infty}^{x}\Gamma_x^p(x,y,t,k)\left[k(Q-Q^p)(y,t)+\frac{i}{2}|q|^2(y,t)\sigma_3\right.\\ \left. - \frac{i}{2}A^2\sigma_3\right]\Phi(y,t,k)dy,\end{aligned} \tag{2.13}$$

where $\Gamma_x^p(x,y,t,k) = \Psi_x^p(x,t,k)(\Psi^p(y,t,k))^{-1}$. As $\Psi^p(x,t,k)$ satisfies $\Psi_x^p(x,t,k) = M^p(x,t,k)\Psi^p(x,t,k)$, *with* $M^p(x,t,k) = -ik^2\sigma_3 + kQ^p + \frac{i}{2}|q^p|^2\sigma_3$ Substituting this equation into (2.13),

$$\Phi_x(x,t,k) = M^p\Psi^p(x,t,k) + (M-M^p)\Phi(x,t,k) + M^p[\Phi(x,t,k)-\Psi^p(x,t,k)]$$

thus we prove $\Phi(x,t,k)$ satisfies the x-equation of the Lax pair equations (1.11). The t-equation is similar.

Now we prove the large k asymptotics of μ_j in (vii). From the relation between Ψ and μ_1, we have

$$\mu_{1x} + ik^2[\sigma_3,\mu_1] = \left(kQ+\frac{i}{2}|q|^2\sigma_3\right)\mu_1.$$

Substituting $\mu_1 = D_0 + \frac{D_1}{k} + \frac{D_2}{k^2} + \cdots$, $k\to\infty$ into the above equation, we have

$$\begin{aligned}&O(k^2): \quad i[\sigma_3,D_0]=0,\\ &O(k): \quad i[\sigma_3,D_1]=QD_0,\\ &O(1): \quad D_{0x}+i[\sigma_3,D_2]=QD_1+\frac{i}{2}|q|^2\sigma_3D_0.\end{aligned}$$

We can get D_0 is a diagonal matrix from $O(k^2)$, then from $O(k)$ we have

$$i\begin{pmatrix}0 & 2D_{12}^{(1)}\\ -2D_{21}^{(1)} & 0\end{pmatrix} = \begin{pmatrix}0 & qD_{22}^{(0)}\\ -\bar{q}D_{11}^{(0)} & \end{pmatrix},$$

so together with these equations yields the following equation from $O(1)$

$$D_{0x} = 0.$$

Thus from (2.9), we know that $D_0 = \mathbb{I}$. Hence, we prove (vii) for μ_1.

From the relation between Φ and μ_2, we have

$$\mu_{2x} + ik^2\sigma_3\mu_2 - i(X(k) + B)\mu_2\sigma_3 = \left(kQ + \frac{i}{2}|q|^2\sigma_3\right)\mu_2.$$

we recall that $X(k) = k^2 - B + o\left(\frac{1}{k^2}\right)$ as $k \to \infty$. If we set $\mu_2 = D_0' + \frac{D_1'}{k} + \frac{D_2'}{k^2} + \cdots$, $k \to \infty$, we can get the same equations about $O(k^2)$, $O(k)$ and $O(1)$, then we can also get $D_{0x}' = 0$. Now from (2.10) and $E(k) \to \mathbb{I}$, $k \to \infty$, we get $D_0' = \mathbb{I}$. So, we prove (vii) for μ_2. □

Remark 2.2 From our proof, we know that though the initial data given in (1.6b) are not continuous, this lack of regularity does not affect the asymptotic expansions of the μ_j's as stated in the proposition.

Since the eigenfunctions $\Psi(x, t, k)$ and $\Phi(x, t, k)$ satisfy both equations of the Lax pair (1.11), we have

$$\Phi(x, t, k) = \Psi(x, t, k)S(k), \qquad k^2 \in \mathbb{R}, \tag{2.14}$$

where $S(k)$ is independent of (x,t). It follows from (2.9) and (2.10) that

$$\Psi(x, 0, k) = \mathrm{e}^{-ik^2x\sigma_3}, \qquad \text{for } x \geqslant 0,$$
$$\Phi(x, 0, k) = \mathrm{e}^{-iBx\sigma_3}E(k)\mathrm{e}^{-ixX(k)\sigma_3}, \qquad \text{for } x \leqslant 0,$$

which lead to

$$S(k) = \Psi^{-1}(0, 0, k)\Phi(0, 0, k) = \Phi(0, 0, k) = E(k). \tag{2.15}$$

Thus, we have

$$S(k) = \begin{pmatrix} \bar{a}(\bar{k}) & b(k) \\ -\bar{b}(\bar{k}) & a(k) \end{pmatrix} = \begin{pmatrix} a(k) & b(k) \\ b(k) & a(k) \end{pmatrix}, \tag{2.16}$$

where

$$a(k) = \bar{a}(\bar{k}) = \frac{1}{2}\left[\varphi(k) + \frac{1}{\varphi(k)}\right],$$
$$b(k) = -\bar{b}(\bar{k}) = \frac{1}{2}\left[\varphi(k) - \frac{1}{\varphi(k)}\right]. \tag{2.17}$$

3 The Basic Riemann-Hilbert Problem

The scattering relation (2.14) involving the eigenfunctions $\Psi(x, t, k)$ and $\Phi(x, t, k)$ can be rewritten in the form of conjugation of boundary values of a piecewise analytic matrix-value function on a contour in the complex k-plane, namely:

$$M_+(x, t, k) = M_-(x, t, k)J(x, t, k), \qquad k \in \Sigma, \tag{3.1}$$

where $M_{\pm}(x,t,k)$ denote the boundary vales of $M(x,t,k)$ according to a chosen orientation of Σ, and $\Sigma = \mathbb{R} \cup i\mathbb{R} \cup \gamma \cup \bar{\gamma}$.

Indeed, let us write (2.14) in the vector form:

$$\begin{aligned} \frac{\Phi^{(1)}(x,t,k)}{a(k)} &= \Psi^{(1)}(x,t,k) + r(k)\Psi^{(2)}(x,t,k), \\ \frac{\Phi^{(2)}(x,t,k)}{a(k)} &= r(k)\Psi^{(1)}(x,t,k) + \Psi^{(2)}(x,t,k), \end{aligned} \tag{3.2}$$

where

$$r(k) := \frac{b(k)}{a(k)} = \frac{i}{kA}\left[k^2 - B - \frac{A^2}{2} - X(k)\right], \tag{3.3}$$

and define the matrix $M(x,t,k)$ as follows:

$$M(x,t,k) = \begin{cases} \begin{pmatrix} \frac{\Phi^{(1)}(x,t,k)}{a(k)} e^{it\theta(k)} & \Psi^{(2)}(x,t,k)e^{-it\theta(k)} \end{pmatrix}, & k \in \{k \in \mathbb{C} | \mathrm{Im}k^2 > 0 \backslash \gamma\}, \\ \begin{pmatrix} \Psi^{(1)}(x,t,k)e^{it\theta(k)} & \frac{\Phi^{(2)}(x,t,k)}{a(k)} e^{-it\theta(k)} \end{pmatrix}, & k \in \{k \in \mathbb{C} | \mathrm{Im}k^2 < 0 \backslash \bar{\gamma}\}, \end{cases} \tag{3.4}$$

where

$$\theta(k) := 2k^4 + \frac{x}{t}k^2, \tag{3.5}$$

Then the boundary values $M_+(x,t,k)$ and $M_-(x,t,k)$ relative to Σ are related by (3.1),where

$$J(x,t,k) = \begin{cases} \begin{pmatrix} 1 - r^2(k) & -r(k)e^{-2it\theta(k)} \\ r(k)e^{2it\theta(k)} & 1 \end{pmatrix}, & k^2 \in \mathbb{R}, \\ \begin{pmatrix} 1 & 0 \\ f(k)e^{2it\theta(k)} & 1 \end{pmatrix}, & k^2 \in \gamma, \\ \begin{pmatrix} 1 & f(k)e^{-2it\theta(k)} \\ 0 & 1 \end{pmatrix}, & k^2 \in \bar{\gamma}, \end{cases} \tag{3.6}$$

with

$$f(k) := r_+(k) - r_-(k). \tag{3.7}$$

The jump relation (3.1) considered together with the properties of the eigenfunctions listed in Proposition 1 suggests a way of representing the solution to the Cauchy problem (1.6a) and (1.6b) in terms of the solution of the Riemann-Hilbert problem, which is specified by the initial conditions (1.6b) via the associated spectral function $r(k)$.

The solution $q(x,t)$ of the initial value problem (1.6a) and (1.6b) can be expressed in terms of the solution of the basic Riemann-Hilbert problem as follows:

$$q(x,t) = 2i \lim_{k\to\infty} (kM(x,t,k))_{12}. \tag{3.8}$$

where M is the solution of the following Riemann-Hilbert problem:

Basic Riemann-Hilbert problem I.

Given $r(k)$, $k^2 \in \mathbb{R}$ and $f(k) = r_+(k) - r_-(k)$, $k^2 \in \gamma \cup \bar{\gamma}$, and $\Sigma = \mathbb{R} \cup i\mathbb{R} \cup \gamma \cup \bar{\gamma}$, find a 2×2 matrix-value function $M(x,t,k)$ such that

(i) $M(x,t,k)$ is analytic in $k \in \mathbb{C} \backslash \Sigma$.

(ii) $M(x,t,k)$ is bounded at the end points E and $\bar{E}$.
(iii) The boundary value $M_{\pm}(x,t,k)$ at Σ satisfy the jump condition

$$M_+(x,t,k) = M_-(x,t,k)J(x,t,k), \quad k \in \Sigma$$

where the jump matrix $J(x,t,k)$ is defined in terms of $r(k)$ and $f(k)$ by (3.6).
(iv) Behavior at ∞

$$M(x,t,k) = \mathbb{I} + O\left(\frac{1}{k}\right), \qquad \text{as } k \to \infty.$$

If we try to analysis the long-time asymptotic behavior of the DNLS equation (1.6a) and (1.6b) with step-like initial value problem, this Riemann-Hilbert problem has a contradiction in the plane wave region. To overcome this contradiction, we try to derive a new Riemann-Hilbert problem, which is similar to the type of nonlinear Schrödinger equation. That means we arrive at the following Riemann-Hilbert problem.

We define

$$N(x,t,k) = k^{-\frac{\hat{\sigma}_3}{2}} M(x,t,k), \tag{3.9}$$

then the jump condition for N is

$$N_+(x,t,k) = N_-(x,t,k)\mathrm{e}^{-i(k^2x+2k^4t)\hat{\sigma}_3} J_N(x,t,k), \tag{3.10}$$

introducing $\lambda = k^2$ and control the branch of k as $Sign\mathrm{Im}k = Sign\mathrm{Im}\lambda$, and define the modified scattering data $\rho(\lambda) = \frac{r(k)}{k}$, [13].

Then

$$X(\lambda) = \sqrt{\left(\lambda - B - \frac{A^2}{2}\right)^2 + \lambda A^2} = \sqrt{(\lambda - B)^2 + \frac{A^4}{4} + A^2 B}, \tag{3.11}$$

$$\Omega(\lambda) = 2(\lambda + B)X(\lambda), \tag{3.12}$$

and the segment

$$\gamma \cup \bar{\gamma} := \{\lambda \in \mathbb{C} | \lambda_1 = B, \lambda_2^2 \leqslant D^2\}, \tag{3.13}$$

where $\gamma = \{k \in \mathbb{C} | \lambda_1 = B, \lambda_2^2 \leqslant D^2, \mathrm{Im}\lambda_2 > 0\}$, $D^2 = A^2 B + \frac{A^4}{4}$, $\lambda_1 = \mathrm{Re}\lambda$ and $\lambda_2 = \mathrm{Im}\lambda$. Let $E = B + iD$, then $\gamma = [E, B]$ and $\bar{\gamma} = [B, \bar{E}]$. And the jump condition for N is

$$N_+(x,t,\lambda) = N_-(x,t,\lambda)\mathrm{e}^{-i(\lambda x+2\lambda^2 t)\hat{\sigma}_3} J_N(x,t,\lambda), \tag{3.14}$$

where

$$J_N(x,t,\lambda) = \begin{cases} \begin{pmatrix} 1-\lambda\rho(\lambda)^2 & -\rho(\lambda)\mathrm{e}^{-2it\theta(\lambda)} \\ \lambda\rho(\lambda)\mathrm{e}^{2it\theta(\lambda)} & 1 \end{pmatrix}, & \lambda \in \mathbb{R}, \\ \begin{pmatrix} 1 & 0 \\ \lambda f(\lambda)\mathrm{e}^{2it\theta(\lambda)} & 1 \end{pmatrix}, & \lambda \in \gamma, \\ \begin{pmatrix} 1 & f(\lambda)\mathrm{e}^{-2it\theta(\lambda)} \\ 0 & 1 \end{pmatrix}, & \lambda \in \bar{\gamma}, \end{cases} \tag{3.15}$$

where

$$f(\lambda) = \rho(\lambda)_+ - \rho(\lambda)_-. \tag{3.16}$$

In other word, we have the following basic Riemann-Hilbert problem

Basic Riemann-Hilbert problem II.

Given $\rho(\lambda), \lambda \in \mathbb{R}$ and $f(\lambda) = \rho(\lambda)_+ - \rho(\lambda)_-, \lambda \in \gamma \cup \bar{\gamma}$, and $\Sigma = \mathbb{R} \cup \gamma \cup \bar{\gamma}$, find a 2×2 matrix-value function $N(x, t, \lambda)$ such that

(i) $N(x, t, \lambda)$ is analytic in $\lambda \in \mathbb{C} \backslash \Sigma$.
(ii) $N(x, t, \lambda)$ is bounded at the end points E and $\bar{E}$.
(iii) The boundary value $N_{\pm}(x, t, \lambda)$ at Σ satisfy the jump condition

$$N_+(x, t, \lambda) = N_-(x, t, \lambda) J_N(x, t, \lambda), \quad \lambda \in \Sigma \backslash \{E, \bar{E}, B\},$$

where the jump matrix $J_N(x, t, k)$ is defined in terms of $\rho(\lambda)$ and $f(\lambda)$ by (3.15).

(iv) Behavior at ∞

$$N(x, t, \lambda) = \mathbb{I} + O\left(\frac{1}{\lambda}\right), \qquad \text{as} \lambda \to \infty.$$

4 Long-time Asymptotics

The representation of the solution $q(x, t)$ of the initial value problem (1.6) in terms of the solution of an associated basic Riemann-Hilbert problem allows using the ideas of the asymptotic analysis of oscillating Riemann-Hilbert problems [9–11, 28, 32] for studying the long-time asymptotics of $q(x, t)$. The key fact leading to different asymptotics in different regions of the (x,t) half-plane is that the behavior of the jump matrix of the basic Riemann-Hilbert problem as a function of the large parameter t is different in these regions. Indeed, as seen on (3.15), this behavior is governed by the sign of $\mathrm{Im}\theta(\lambda)$, which itself depends on $\xi = \frac{x}{4t}$. As we have already written, three regions are to be distinguished (Fig. 2) :

(i) A Zakharov-Manakov region: $\xi > -B$.
(ii) A plane wave region: $\xi < -\sqrt{2}D - B$.
(iii) An elliptic wave region: $-\sqrt{2}D - B < \xi < -B$.

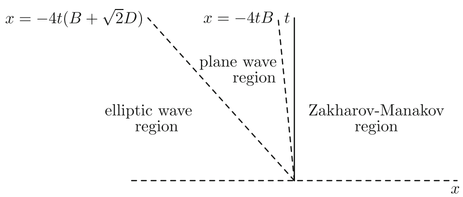


Fig. 2 The different regions of the (x,t)-plane

4.1 The Zakharov-Manakov Region: $\xi > -B$

In this region $\xi > -B$, we have $\mathrm{Im}\theta(\lambda) > 0$ for all $\lambda \in \gamma$ and $\mathrm{Im}\theta(\lambda) < 0$ for all $\lambda \in \bar{\gamma}$. Therefore, the exponentials in the jump matrix J_N, see (3.15), are decaying as $t \to +\infty$ for $\lambda \in \Sigma \backslash \mathbb{R}$.

This implies that one can follow the technique of asymptotic analysis proposed for the first time in [9]. The basic step of the procedure is a deformation of the original Riemann-Hilbert problem, with the help of the solution of an appropriate scalar Riemann-Hilbert problem, in order to obtain an equivalent Riemann-Hilbert problem whose jump matrix decays, in t, to a constant (in λ) matrix. This leads to model Riemann-Hilbert problems whose solutions can be given explicitly.

A particular feature of the Riemann-Hilbert problem under consideration is that the contour of the modified Riemann-Hilbert problem contains neither the real axis, where the jump matrix for the original Riemann-Hilbert problem oscillates with t, see (3.15), nor the finite parts γ and $\bar{\gamma}$. This happens due to the pure step-like initial conditions, which in turn implies that the associated spectral functions $\rho(\lambda)$ and $\lambda\rho(\lambda)$ can be analytically extended from the contour to the whole λ-plane.

4.1.1 First Transformation

The first transform is as usual:

$$N^{(1)}(x,t,\lambda) = N(x,t,\lambda)\delta^{-\sigma_3}(\lambda), \tag{4.1}$$

where ([43])

$$\delta(\lambda) = \exp \frac{1}{2\pi i} \int_{-\infty}^{\lambda_0} \frac{\log(1-\lambda'\rho(\lambda')^2)}{\lambda'-\lambda} d\lambda', \tag{4.2}$$

is the solution of the following scalar Riemann-Hilbert problem:

- $\delta(\lambda)$ is analytic in $\mathbb{C}\backslash(-\infty, \lambda_0]$,
- $\delta(\lambda) \to 1$ as $\lambda \to \infty$,
- $\delta(\lambda)$ satisfies the jump relation

$$\delta_+(\lambda) = \delta_-(\lambda)(1-\lambda\rho^2(\lambda)), \qquad \lambda \in (-\infty, \lambda_0). \tag{4.3}$$

Here, λ_0 is the stationary point of the phase function $\theta(\lambda) = 2\lambda^2 + 4\xi\lambda$, that is, $\theta'(\lambda_0) = 0$:

$$\lambda_0 = -\xi = \frac{-x}{4t}.$$

Then $N^{(1)}(x,t,\lambda)$ satisfies the jump condition

$$N_+^{(1)}(x,t,\lambda) = N_-^{(1)}(x,t,N) J_N^{(1)}(x,t,\lambda), \\ \lambda \in \Sigma^{(1)} = \Sigma, \tag{4.4}$$

where

$$J_N^{(1)}(x,t,\lambda) = \delta_-^{\sigma_3} J_N \delta_+^{-\sigma_3},$$

that is

$$J_N^{(1)}(x,t,\lambda)=\begin{cases} e^{-it\theta\hat{\sigma}_3}\begin{pmatrix} \frac{\delta_-}{\delta_+}(1-\lambda\rho(\lambda)^2) & -\rho\delta_+\delta_- \\ \frac{\lambda\rho}{\delta_+\delta_-} & \frac{\delta_+}{\delta_-}\end{pmatrix}, & \lambda\in\mathbb{R}, \\ \begin{pmatrix} \frac{\delta_-}{\delta_+} & 0 \\ \frac{\lambda f}{\delta_+\delta_-}e^{2it\theta\sigma_3} & \frac{\delta_+}{\delta_-}\end{pmatrix}, & \lambda\in\gamma, \\ \begin{pmatrix} \frac{\delta_-}{\delta_+} & f\delta_+\delta_- e^{-2it\theta\sigma_3} \\ 0 & \frac{\delta_-}{\delta_+}\end{pmatrix}, & \lambda\in\bar{\gamma}. \end{cases} \tag{4.5}$$

From the Riemann-Hilbert problem of the δ, we can find

$$J_N^{(1)}(x,t,\lambda)=\begin{cases} e^{-it\theta\hat{\sigma}_3}\begin{pmatrix} 1-\lambda\rho^2 & -\rho\delta^2 \\ \frac{\lambda\rho}{\delta^2} & 1\end{pmatrix}, & \lambda>\lambda_0, \\ e^{-it\theta\hat{\sigma}_3}\begin{pmatrix} 1 & \frac{-\rho}{1-\lambda\rho^2}\delta_-^2 \\ \frac{\lambda\rho}{1-\lambda\rho^2}\frac{1}{\delta_+^2} & 1-\lambda\rho^2\end{pmatrix}, & \lambda<\lambda_0, \\ \begin{pmatrix} 1 & 0 \\ \frac{\lambda f}{\delta^2}e^{2it\theta} & 1\end{pmatrix}, & \lambda\in\gamma, \\ \begin{pmatrix} 1 & f\delta^2 e^{-2it\theta} \\ 0 & 1\end{pmatrix}, & \lambda\in\bar{\gamma}. \end{cases} \tag{4.6}$$

4.1.2 *Second Transformation*

The next transformation is:

$$N^{(2)}(x,t,\lambda)=N^{(1)}(x,t,\lambda)G(\lambda), \tag{4.7}$$

where

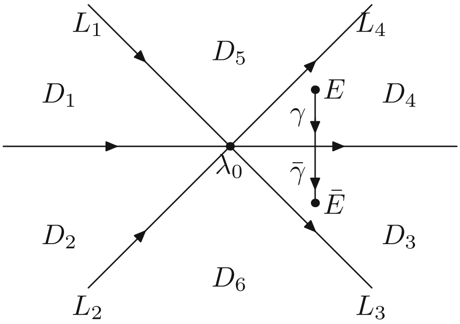
$$G(\lambda)=\begin{cases} \begin{pmatrix} 1 & \frac{\rho}{1-\lambda\rho^2}\delta_-^2 e^{-2it\theta} \\ 0 & 1\end{pmatrix}, & \lambda\in D_1, \\ \begin{pmatrix} 1 & 0 \\ \frac{\lambda\rho}{1-\lambda\rho^2}\frac{1}{\delta_+^2}e^{2it\theta} & 1\end{pmatrix}, & \lambda\in D_2, \\ \begin{pmatrix} 1 & -\rho\delta^2 e^{-2it\theta} \\ 0 & 1\end{pmatrix}, & \lambda\in D_3, \\ \begin{pmatrix} 1 & 0 \\ \frac{-\lambda\rho}{\delta^2}e^{2it\theta} & 1\end{pmatrix}, & \lambda\in D_4, \\ \mathbb{I}, & \lambda\in D_5\cup D_6. \end{cases} \tag{4.8}$$

The domains $D_1,\ldots D_6$ are shown on the following (Fig. 3).

This new function $N^{(2)}$ solves the equivalent Riemann-Hilbert problem:

$$N_+^{(2)}(x,t,\lambda)=N_-^{(2)}(x,t,\lambda)J_N^{(2)}(x,t,\lambda),$$
$$\lambda\in\Sigma^{(2)},$$

Fig. 3 The oriented contour $\Sigma^{(2)} = L_1 \cup L_2 \cup L_3 \cup L_4$



where

$$J_N^{(2)}(x,t,\lambda) = \begin{cases} \begin{pmatrix} 1 & \frac{-\rho}{1-\lambda\rho^2}\delta_-^2 \mathrm{e}^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & \lambda \in L_1, \\ \begin{pmatrix} 1 & 0 \\ \frac{\lambda\rho}{1-\lambda\rho^2}\frac{1}{\delta_+^2}\mathrm{e}^{2it\theta} & 1 \end{pmatrix}, & \lambda \in L_2, \\ \begin{pmatrix} 1 & -\rho\delta^2 \mathrm{e}^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & \lambda \in L_3, \\ \begin{pmatrix} 1 & 0 \\ \frac{\lambda\rho}{\delta^2}\mathrm{e}^{2it\theta} & 1 \end{pmatrix}, & \lambda \in L_4. \end{cases} \tag{4.9}$$

4.1.3 The Last Transformation

Now $J_N^{(2)}(x,t,\lambda)$ decays exponentially fast to the identity matrix, as $t \to +\infty$, and uniformly outside any neighborhood of $\lambda = \lambda_0$. Thus, we are in a situation where the asymptotic analysis of [43] works. Particularly,

$$N^{(2)}(x,t,\lambda) = Z(x,t,\lambda)N^{as}(x,t,\lambda),$$

where $N^{as}(x,t,\lambda)$ is a solution of the model problem explicitly given in terms of parabolic cylinder functions whereas $Z(x,t,\lambda)$ can be estimated:

$$Z(x,t,\lambda) = \mathbb{I} + O\left(\frac{logt}{t^{\frac{1}{2}}}\right).$$

Therefore, the final asymptotic result is as in [43] giving the main term of the asymptotic in terms of the modified reflection coefficient $\rho(\lambda)$:

Theorem 4.1 *(The Zakharov-Manakov region) In the region $x > -4tB$, the asymptotics, as $t \to +\infty$, of the solution $q(x,t)$ of the initial value problem (1.6) is described by the Zakharov-Manakov type formula*

$$q(x,t) = q_{as}(x,t) + O\left(\frac{\log t}{t}\right) \tag{4.10}$$

where

$$q_{as} = \frac{1}{\sqrt{t}}\alpha(\lambda_0)e^{\frac{ix^2}{4t} - i\nu(\lambda_0)\log t},$$
$$|\alpha(\lambda_0)|^2 = \frac{\nu(\lambda_0)}{2} = -\frac{1}{4\pi}\log(1-\lambda_0|\rho(\lambda_0)|^2),$$
$$\arg\alpha(\lambda_0) = -3\nu\log 2 - \frac{\pi}{4} + \arg\Gamma(i\nu) - \arg r(\lambda_0) + \frac{1}{\pi}\int_{-\infty}^{\lambda_0}\log|\lambda-\lambda_0|d,$$
$$\log(1-\lambda|\rho(\lambda)|^2),\ \lambda_0 = -\frac{x}{4t}. \tag{4.11}$$

4.2 The Plane Wave Region: $\xi < -\sqrt{2}D - B$

For $x < -4t\left(B + \sqrt{2A^2\left(B + \frac{A^2}{4}\right)}\right)$, that means, Im$\theta(\lambda)$ is negative on γ and positive on $\bar{\gamma}$, which implies that the exponentials in (3.15) increase with t. Thus, the jump matrix J_N for the Riemann-Hilbert problem does not converge to a reasonable limit as $t \to \infty$.

To bypass this difficulty, one deforms the Riemann-Hilbert problem in such a way that the phase Im$\theta(\lambda)$ is replaced by another function, $g(\lambda)$, providing suitable behavior of the modified jump matrix. The extension of the nonlinear steepest descent method for Riemann-Hilbert problems, involving the "g-function mechanism" was first proposed by Deift, Venakides, and Zhou, see [26, 27].

4.2.1 The g Function

A natural choice for a g-function appropriate for the region adjacent to the half-axis $x < 0$, $t = 0$, is the phase appearing in the explicit expression for the eigenfunction Ψ^p, see (2.1), associated with the "potential" q^p. Setting

$$g(x,t,\lambda) = xX(\lambda) + t\Omega(\lambda), \tag{4.12}$$

where $X(\lambda)$ and $\Omega(\lambda)$ are defined in (3.11) and (3.12),we have

$$\Psi^p(x,t,k) = e^{i(\omega t - Bx)\sigma_3}E(\lambda)e^{-ig(x,t,\lambda)\sigma_3}. \tag{4.13}$$

The "signature table" for Im$g(\lambda;\xi)$ is the partition of the λ-plane into maximal domains where the sign of Im$g(\lambda;\xi)$ is constant. Its form can be controlled by the zeros of the differential $dg(\lambda)$. Indeed,

$$dg(\lambda) = 4\frac{(\lambda-\mu_+)(\lambda-\mu_-)}{X(\lambda)}d\lambda, \tag{4.14}$$

where

$$\mu_{\pm} = \frac{B-\xi}{2} \pm \sqrt{\frac{(B+\xi)^2}{4} - \frac{\frac{A^4}{4}+A^2B}{2}}, \tag{4.15}$$

Thus, for $\xi < -\left(B+\sqrt{2A^2\left(B+\frac{A^2}{4}\right)}\right)$, $\mu_{\pm}$ are both real. Moreover,

$$B < \mu_- < \mu_+ < -\xi.$$

In what follows the signature table of the function $\text{Im}g(\lambda)$ for different values of ξ plays a very important role. The lines of separation between the different domains are the real axile

$$\lambda_2 = 0,$$

and the algebraic curve

$$\lambda_2^2(\lambda_1+\xi) = (\lambda_1+B+2\xi)\left[(\lambda_1-B)(\lambda_1+\xi)+\frac{\frac{A^4}{4}+A^2B}{2}\right], \tag{4.16}$$

They are indeed given by $\text{Im}g(\lambda)=0$. Because of

$$\text{Im}g(\lambda) = 4\lambda_2\left\{(\lambda_1+B+2\xi)\left[(\lambda_1-B)(\lambda_1+\xi)+\frac{\frac{A^4}{4}+A^2B}{2}\right]-\lambda_2^2(\lambda_1+\xi)\right\},$$

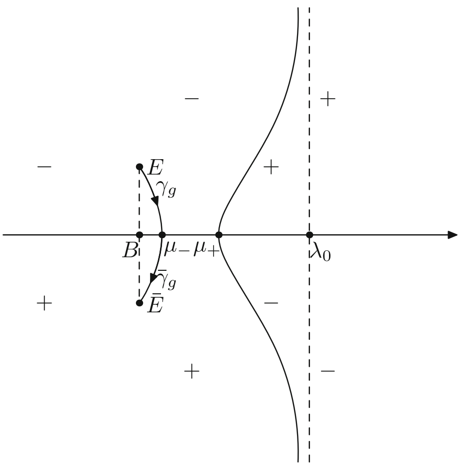
The equation (4.16) can be written:

$$\lambda_2^2(\lambda_1+\xi) = (\lambda_1+B+2\xi)[(\lambda_1-\mu_+)(\lambda_1-\mu_-)].$$

And the signature table of the function $\text{Im}g(\lambda)$ is shown in the following Fig. 4.

The advantage of the signature table shown in Fig. 4 is that there is a finite arc connecting the branch points E and $\bar{E}$ such that $\text{Im}g(\lambda)=0$ for all λ along this arc. Since the jump matrix depends on t via exponentials of type $e^{\pm ig(\lambda)}$, it is oscillatory along an arc where $\text{Im}g(\lambda)=0$.

Fig. 4 The curves of $\text{Im}g(\lambda)=0$ for $x < -4t\left(B+\sqrt{2A^2\left(B+\frac{A^2}{4}\right)}\right)$



This suggests to deform the original contour $\gamma \cup \bar{\gamma}$ of the basic Riemann-Hilbert problem to a new contour $\gamma_g \cup \bar{\gamma}_g$ which depends on ξ and where $\mathrm{Im} g(\lambda) = 0$, and to view $X(\lambda)$, thus also $g(\lambda)$ as functions with branch cut $\gamma_g \cup \bar{\gamma}_g$.

Another important feature of $g(\lambda; \xi)$ is that it has, up to a constant, the same large λ asymptotic behavior as the phase function $\theta(\lambda)$:

$$g(\lambda; \xi) = t(2\lambda^2 + 4\xi\lambda + g(\infty; \xi)) + O\left(\frac{1}{\lambda}\right), \qquad \lambda \to \infty, \tag{4.17}$$

where

$$g(\infty; \xi) = (\omega - 4B\xi). \tag{4.18}$$

4.2.2 The First Transformation

We put

$$N^{(1)}(x, t, \lambda) = \mathrm{e}^{-itg(\infty,\xi)\sigma_3} N(x, t, \lambda) \mathrm{e}^{-i(\lambda x + 2\lambda^2 t - g(\lambda))\sigma_3},$$

Then the matrix-value function $N^{(1)}(x, t, \lambda)$ satisfies the following Riemann-Hilbert problem:

$$N_+^{(1)}(x, t, \lambda) = N_-^{(1)}(x, t, \lambda) J_N^{(1)}(x, t, \lambda), \qquad \lambda \in \Sigma^{(1)} = \mathbb{R} \cup \gamma_g \cup \bar{\gamma}_g,$$

with the jump matrix

$$J_N^{(1)}(x, t, \lambda) = \begin{cases} \begin{pmatrix} 1 - \lambda\rho^2(\lambda) & -\rho(\lambda)\mathrm{e}^{-2ig(\lambda)} \\ \lambda\rho(\lambda)\mathrm{e}^{2ig(\lambda)} & 1 \end{pmatrix}, & \lambda \in \mathbb{R}, \\ \begin{pmatrix} \mathrm{e}^{-2ig_-(\lambda)} & 0 \\ \lambda f(\lambda) & \mathrm{e}^{2ig_-(\lambda)} \end{pmatrix}, & \lambda \in \gamma_g, \\ \begin{pmatrix} \mathrm{e}^{-2ig_-(\lambda)} & f(\lambda) \\ 0 & \mathrm{e}^{2ig_-(\lambda)} \end{pmatrix}, & \lambda \in \bar{\gamma}_g. \end{cases} \tag{4.19}$$

Here $g_\pm(\lambda)$ are boundary values of g on $\gamma_g \cup \bar{\gamma}_g$, and they are real. We also use the equation $g_+(\lambda) = -g_-(\lambda)$.

4.2.3 The Second Transformation

The next transformation is similar to the first transformation applied in the Zakharov-Manakov region, see Section 4.1.1. It involves the solution $\delta(\lambda)$ of the scalar Riemann-Hilbert problem (4.3) but with μ_+ instead of λ_0, where μ_+ is the stationary point of the new phase function $g(\lambda)$. With this new scalar function $\delta(\lambda)$, we set

$$N^{(2)}(x, t, \lambda) = N^{(1)}(x, t, \lambda)\delta^{-\sigma_3}(\lambda),$$

Then the matrix-value function $N^{(2)}(x, t, \lambda)$ satisfies the following Riemann-Hilbert problem

$$N_+^{(2)}(x, t, \lambda) = N_-^{(2)}(x, t, \lambda) J_N^{(2)}(x, t, \lambda), \qquad \lambda \in \Sigma^{(2)} = \Sigma^{(1)}, \tag{4.20}$$

where $J_N^{(2)}(x,t,\lambda)$ is defined as follows:

$$J_N^{(2)}(x,t,\lambda)=\begin{cases} e^{-ig\hat{\sigma}_3}\begin{pmatrix} 1-\lambda\rho^2 & -\rho\delta^2 \\ \frac{\lambda\rho}{\delta^2} & 1 \end{pmatrix}, & \lambda>\mu_+, \\ e^{-ig\hat{\sigma}_3}\begin{pmatrix} 1 & \frac{-\rho}{1-\lambda\rho^2}\delta_-^2 \\ \frac{\lambda\rho}{1-\lambda\rho^2}\frac{1}{\delta_+^2} & 1-\lambda\rho^2 \end{pmatrix}, & \lambda<\mu_+, \\ \begin{pmatrix} e^{-2ig_-(\lambda)} & 0 \\ \frac{\lambda f}{\delta^2} & e^{2ig_-(\lambda)} \end{pmatrix}, & \lambda\in\gamma_g, \\ \begin{pmatrix} e^{-2ig_-(\lambda)} & f\delta^2 \\ 0 & e^{2ig_-(\lambda)} \end{pmatrix}, & \lambda\in\bar{\gamma}_g. \end{cases} \tag{4.21}$$

4.2.4 The Third Transformation

The subsequent transformation

$$N^{(3)}(x,t,\lambda)=N^{(2)}(x,t,\lambda)G(\lambda),$$

involves $G(\lambda)$ defined similarly to (4.8), with $t\theta$ replaced by g and λ_0 replaced by μ_+. Then $N^{(3)}(x,t,\lambda)$ satisfies the jump relation

$$N_+^{(3)}(x,t,\lambda)=N_-^{(3)}(x,t,\lambda)J_N^{(3)}(x,t,\lambda),$$

across to the contour

$$\Sigma^{(3)}=L_1\cup L_2\cup L_3\cup L_4\cup\gamma_g\cup\bar{\gamma}_g,$$

shown in Fig. 5.

And we notice that

1. For $\lambda\in L_1\cup L_2\cup L_3\cup L_4$ the jump matrix $J_N^{(3)}(x,t,\lambda)$ decays to the identity matrix, as $t\to\infty$, exponentially fast and uniformly outside any neighborhood of $\lambda=\mu_+$.

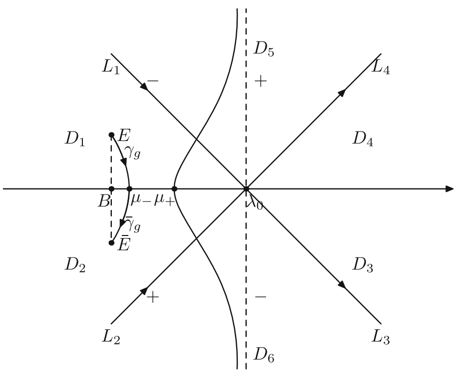


Fig. 5 The contour $\Sigma^{(3)}=L_1\cup L_2\cup L_3\cup L_4\cup\gamma_g\cup\bar{\gamma}_g$ of the Riemann-Hilbert problem for $N^{(3)}$ for $x<-4t\left(B+\sqrt{2A^2\left(B+\frac{A^2}{4}\right)}\right)$

2. For $\lambda \in \gamma_g$, the jump matrix $J_N^{(3)}(x,t,\lambda)$ factorizes as

$$\begin{pmatrix} 1 & \left(\frac{-\rho}{1-\lambda\rho^2}\right)_- \delta^2 \mathrm{e}^{-2ig_-(\lambda)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-2ig(\lambda)} & 0 \\ \lambda f(\lambda)\delta^{-2}(\lambda) & \mathrm{e}^{2ig_-0(\lambda)} \end{pmatrix} \begin{pmatrix} 1 & \left(\frac{\rho}{1-\lambda\rho^2}\right)_+ \delta^2 \mathrm{e}^{2ig_-(\lambda)} \\ 0 & 1 \end{pmatrix} \tag{4.22}$$

3. For $\lambda \in \bar{\gamma}_g$, the jump matrix $J_N^{(3)}(x,t,\lambda)$ factorizes as

$$\begin{pmatrix} 1 & 0 \\ \left(\frac{-\lambda\rho}{1-\lambda\rho^2}\right)_- \delta^{-2}\mathrm{e}^{2ig_-(\lambda)} & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-2ig_-(\lambda)} & f(\lambda)\delta^2(\lambda) \\ 0 & \mathrm{e}^{2ig_-(\lambda)} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \left(\frac{\lambda\rho}{1-\lambda\rho^2}\right)_+ \delta^{-2}\mathrm{e}^{2ig_-(\lambda)} & 1 \end{pmatrix} \tag{4.23}$$

4. Using the identities

$$1+\lambda f\left(\frac{-\rho}{1-\lambda\rho^2}\right)_- = 0,$$

$$1+f\left(\frac{\lambda\rho}{1-\lambda\rho^2}\right)_+ = 0,$$

we find

$$J_N^{(3)}(x,t,k) = \begin{cases} \begin{pmatrix} 0 & -(\lambda f)^{-1}(\lambda)\delta^2(\lambda) \\ \lambda f(\lambda)\delta^{-2}(\lambda) & 0 \end{pmatrix}, & \lambda \in \gamma_g, \\ \begin{pmatrix} 0 & f(\lambda)\delta^2(\lambda) \\ -f^{-1}(\lambda)\delta^{-2}(\lambda) & 0 \end{pmatrix}, & \lambda \in \bar{\gamma}_g, \end{cases} \tag{4.24}$$

In order to arrive at a Riemann-Hilbert problem whose jump matrix does not depend on λ, we introduce a factorization involving a scalar function $F(\lambda)$ to be defined;

$$J_N^{(3)}(x,t,\lambda) = \begin{pmatrix} F_+^{-1}(\lambda) & 0 \\ 0 & F_+(\lambda) \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} F_-(\lambda) & 0 \\ 0 & F_-^{-1}(\lambda) \end{pmatrix}, \tag{4.25}$$

in such a way that the boundary values $F_\pm(\lambda)$ of $F(\lambda)$ along the two sides of $\gamma_g \cup \bar{\gamma}_g$ satisfy

$$F_-(\lambda)F_+(\lambda) = \begin{cases} -i\lambda f(\lambda)\delta^{-2}(\lambda) & \lambda \in \gamma_g, \\ if^{-1}(\lambda)\delta^{-2}(\lambda) & \lambda \in \bar{\gamma}_g. \end{cases} \tag{4.26}$$

Indeed, once (4.25) is satisfied, one can absorb the diagonal factors into a new piecewise analytic function whose jump across $\gamma_g \cup \bar{\gamma}_g$ is only the constant middle factor in (4.25).

Thus, we arrive at the folloeing scalar Riemann-Hilbert problem:

Scalar Riemann-Hilbert problem Find a scalar function $F(\lambda)$ such that

- $F(\lambda)$ and $F^{-1}(\lambda)$ are analytic in $\mathbb{C}\backslash\{\gamma_g \cup \bar{\gamma}_g\}$.
- $F(\lambda)$ satisfies the jump relation:

$$F_+(\lambda)F_-(\lambda) = \begin{cases} -i\lambda f(\lambda)\delta^{-2}(\lambda) = a_+^{-1}(\lambda)a_-^{-1}(\lambda)\sqrt{\lambda}\delta^{-2}(9\lambda), & \lambda \in \gamma_g, \\ if^{-1}(\lambda)\delta^{-2}(\lambda) = a_+(\lambda)a_-(\lambda)\sqrt{\lambda}\delta^{-2}(\lambda), & \lambda \in \bar{\gamma}_g. \end{cases} \tag{4.27}$$

where the contour $\gamma_g \cup \bar{\gamma}_g$ is oriented from E to $\bar{E}$, and

- $F(\lambda)$ is bounded at $\lambda = \infty$.

Introducing

$$H(\lambda)=\begin{cases}F(\lambda)a(\lambda), & \lambda\in\mathbb{C}_+\backslash\gamma_g,\\ \frac{F(\lambda)}{a(\lambda)}, & \lambda\in\mathbb{C}_-\backslash\bar{\gamma}_g.\end{cases} \tag{4.28}$$

then the jump relation (4.27) transforms to

$$\left[\frac{\log H(\lambda)}{X(\lambda)}\right]_+-\left[\frac{\log H(\lambda)}{X(\lambda)}\right]_-=\begin{cases}\frac{\log\sqrt{\lambda}\delta^{-2}(\lambda)}{X(\lambda)_+}, & \lambda\in\gamma_g\cup\bar{\gamma}_g,\\ \frac{\log a^2(\lambda)}{X(\lambda)}, & \lambda\in\mathbb{R}.\end{cases} \tag{4.29}$$

The Sokhotski-Plemelj formula shows that this last jump relation is satisfied by

$$H(k)=\exp\left\{\frac{X(\lambda)}{2\pi i}\left[\int_{\gamma_g\cup\bar{\gamma}_g}\frac{\log\sqrt{s}+\log\delta^{-2}(s,\xi)}{s-\lambda}\frac{ds}{X_+(s)}+\int_{\mathbb{R}}\frac{\log ab(s)}{s-\lambda}\frac{ds}{X(s)}\right]\right\} \tag{4.30}$$

Then $F(\lambda)$ is defined in terms of $H(\lambda)$ by (4.28). At $\lambda=\infty$ we find

$$F(\infty)=H(\infty)=\mathrm{e}^{i\phi(\xi)},$$

where

$$\phi(\xi)=\frac{1}{2\pi}\left[\int_{\gamma_g\cup\bar{\gamma}_g}\frac{\log\sqrt{s}\delta^{-2}(s,\xi)}{X_+(s)}ds+\int_{\mathbb{R}}\frac{\log a^2(s)}{X(s)}ds\right] \tag{4.31}$$

with

$$\delta(\lambda,\xi)=\exp\frac{1}{2\pi i}\int_{-\infty}^{\mu_+}\frac{\log(1-\lambda'\rho(\lambda')^2)}{\lambda'-\lambda}d\lambda', \tag{4.32}$$

Using the relation $1-\lambda\rho^2(\lambda)=a^{-2}(\lambda)$, we find a simpler expression for $\phi(\xi)$:

$$\phi(\xi)=\frac{1}{2\pi}\left[\int_{\mu_+}^{+\infty}\log a^2(\lambda)\frac{d\lambda}{X(\lambda)}+\int_{\gamma_g\cup\bar{\gamma}_g}\frac{\log\sqrt{\lambda}}{X_+(\lambda)}d\lambda\right]$$

4.2.5 The Fourth Transformation

The factorization (4.25) suggests a fourth transformation

$$N^{(4)}(x,t,\lambda)=F^{\sigma_3}(\infty,\xi)N^{(3)}(x,t,\lambda)F^{-\sigma_3}(\lambda,\xi),$$

Then we have

$$N_+^{(4)}(x,t,\lambda)=N_-^{(4)}(x,t,\lambda)J_N^{(4)}(x,t,\lambda)$$

For $\lambda\in\gamma_g\cup\bar{\gamma}_g$ the jump matrix $J_N^{(4)}(x,t,\lambda)$ is constant

$$J_N^{(4)}(x,t,\lambda)=J_N^{mod}=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}.$$

1. For $\lambda\in\gamma_g\cup\bar{\gamma}_g$ the jump matrix $J_N^{(4)}(x,t,\lambda)$ is constant:

$$J_N^{(4)}(x,t,\lambda)=J_N^{mod}=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}.$$

2. For $\lambda\in L\cup\bar{L}$, the jump matrix $J_N^{(4)}(x,t,\lambda)$ decays to the identity

$$J_N^{(4)}(x,t,\lambda)=\mathbb{I}+O\left(\frac{1}{\mathrm{e}^{\varepsilon t}}\right).$$

4.2.6 The Final Transformation

Finally, we can express $N^{(4)}$ in the form

$$N^{(4)}(x,t,\lambda) = N^{err}(x,t,\lambda)N^{mod}(x,t,\lambda),$$

where $N^{mod}(x,t,\lambda)$ solves the model problem:

$$N_-^{mod}(x,t,\lambda) = N_+^{(mod)}(x,t,\lambda)J_N^{mod}, \qquad \lambda \in \gamma_g \cup \bar{\gamma}_g, \tag{4.33}$$

with constant jump matrix

$$J_N^{mod} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

and $N^{err}(x,t,\lambda) = \mathbb{I} + O(t^{-\frac{1}{2}})$. As for the model problem, since $\varphi(\lambda)_- = i\varphi(\lambda)_+$ on $\gamma_g \cup \bar{\gamma}_g$, its solution can be given explicitly in terms of $\varphi(\lambda)$:

$$N^{mod}(x,t,\lambda) = \frac{1}{2}\begin{pmatrix} \varphi(\lambda) + \frac{1}{\varphi(\lambda)} & \varphi(\lambda) - \frac{1}{\varphi(\lambda)} \\ \varphi(\lambda) - \frac{1}{\varphi(\lambda)} & \varphi(\lambda) + \frac{1}{\varphi(\lambda)} \end{pmatrix}.$$

4.2.7 Back to the Original Problem

Let $N^*(x,t,\lambda)$, $* =$ (1),(2),(3),(4),mod, denote the solution of the Riemann-Hilbert problem RH^*, and let

$$m_{12}^*(x,t) = \lim_{\lambda\to\infty}(\lambda M^*(x,t,\lambda))_{12},$$

Then, going back to the determination of $q(x,t)$ in terms of the solution of the basic Riemann-Hilbert problem, we have

$$\begin{aligned} q(x,t) &= 2im(x,t)_{12} = 2i\mathrm{e}^{2ig(\infty,\xi)}m^{(1)}(x,t)_{12} \\ &= 2i\mathrm{e}^{2ig(\infty,\xi)}m^{(2)}(x,t)_{12} + O\left(t^{-\frac{1}{2}}\right) \\ &= 2i\mathrm{e}^{2ig(\infty,\xi)}m^{(3)}(x,t)_{12} + O\left(t^{-\frac{1}{2}}\right) \\ &= 2i\mathrm{e}^{2ig(\infty,\xi)}m^{(4)}(x,t)_{12}F^{-2}(\infty,\xi) + O\left(t^{-\frac{1}{2}}\right) \\ &= 2i\mathrm{e}^{2ig(\infty,\xi)}m^{mod}(x,t)_{12}F^{-2}(\infty,\xi) + O\left(t^{-\frac{1}{2}}\right). \end{aligned} \tag{4.34}$$

Taking into account that $g(\infty,\xi) = \omega t - 4Bx$, $2im^{mod}(x,t)_{12} = A$ and $F^{-2}(\infty,\xi) = \mathrm{e}^{-2i\phi(\xi)}$ we arrive at the following theorem:

Theorem 4.2 **(Plane wave region)** *In the region* $x < -4t\left(B + \sqrt{2A^2\left(B + \frac{A^2}{4}\right)}\right)$, *the asymptotics, as* $t \to +\infty$, *of the solution* $q(x,t)$ *of the initial value problem (1.6) takes the form of a plane wave:*

$$q(x,t) = A\mathrm{e}^{2i(\omega t - Bx - \phi(\xi))} + O\left(t^{-\frac{1}{2}}\right), \qquad t \to +\infty. \tag{4.35}$$

Remark 4.3 If we let $\xi \to +\infty$, then $\mu_+ \to +\infty$, then $\phi(\xi) \to \phi$, with $\phi = \frac{1}{2\pi}\int_{\gamma_g\cup\bar{\gamma}_g} \frac{\log\sqrt{\lambda}}{X_+(\lambda)}d\lambda$, and then the above equation (4.62) reduce to $q(x,t) = A\mathrm{e}^{2i(\omega t - Bx - \phi)}$, this is correspondence to our initial condition up to a phase shift.

4.3 The Elliptic Region: $-4t(B+\sqrt{2}D) < x < -4tB$

For the limit case $\xi_0 = -\left(B + \sqrt{2A^2\left(B+\frac{A^2}{4}\right)}\right)$, we have $\mu_+(\xi_0) = \mu_-(\xi_0)$, see Fig. 7, whereas for $\xi > -\left(B + \sqrt{2A^2\left(B+\frac{A^2}{4}\right)}\right)$, μ_+ and μ_- become non-real, complex conjugated numbers. As a result, the g-function mechanism with $g(\lambda;\xi)$ as in the plane wave region fails. This shows that there is a break in the qualitative picture of the asymptotic behavior at $\xi = \xi_0$.

4.3.1 The New g-function

A suitable g-function for $\xi > -\left(B + \sqrt{2A^2\left(B+\frac{A^2}{4}\right)}\right)$ can be obtained as follows. First, we need to introduce a new real stationary point $\mu(\xi)$ which must be a zero of the new differential $d\hat{g}$. On the other hand we have to preserve the asymptotic behavior of the g-function for large λ. To do so we must change the denominator of the differential $d\hat{g}$. Thus the new differential takes the form:

$$d\hat{g}(\lambda,\xi) = 4\frac{(\lambda-\mu(\xi))(\lambda-\mu_-(\xi))(\lambda-\mu_+(\xi))}{\sqrt{(\lambda-E)(\lambda-\bar{E})(\lambda-d(\xi))(\lambda-\bar{d}(\xi))}}d\lambda, \tag{4.36}$$

where $\mu(\xi)$, $\mu_\pm(\xi)$, and $d(\xi)$, $\bar{d}(\xi)$ are to be determined. If $\mu = d = \bar{d}$, then the new differential coincides with the previous one, that is $dg = d\hat{g}$, which is expected to hold for the value ξ_0 of ξ limiting the two adjacent asymptotic regions.

Now we consider $d\hat{g}$ as an Abelian differential of the second kind with poles at $\infty_\pm$ on the Riemann-Hilbert surface of

$$\omega(\lambda) = \sqrt{(\lambda-E)(\lambda-\bar{E})(\lambda-d(\xi))(\lambda-\bar{d}(\xi))},$$

with

$$E = B + iD, \quad d(\xi) = d_1(\xi) + id_2(\xi).$$

The branch of the square root is fixed by the asymptotics on the upper sheet:

$$\omega(\lambda) = \lambda^2 + O(\lambda), \qquad \lambda \to \infty_+.$$

We choose on this Riemann surface a basis $\{a, b\}$ of cycles as follows. The b-cycle is a closed clock-wise oriented simple loop around the arc $\gamma_{E,d}$ joining E and d. The a-cycle starts on the upper sheet from the left side of the cut $\gamma_{E,d}$, goes to the left side of the cut $\gamma_{\bar{d},\bar{E}}$, proceeds to the lower sheet, and then returns to the starting point.We can also write the Abelian differential $d\hat{g}(\lambda)$ in the form:

$$d\hat{g}(\lambda) = 4\frac{\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0}{\omega(\lambda)}d\lambda, \tag{4.37}$$

and normalize it so that its a-period vanishes. This determines c_0:

$$c_0 = -\frac{\int_{\bar{d}}^{d}\left(\lambda^3 + c_2\lambda^2 + c_1\lambda\right)\frac{d\lambda}{\omega(\lambda)}}{\int_{\bar{d}}^{d}\frac{d\lambda}{\omega(\lambda)}} \in \mathbb{R}.$$

We also require that $\hat{g}(\lambda)$ has the same large-λ behavior as the original phase function $\theta\lambda$:

$$\hat{g}(\lambda) = 2\lambda^2 t + 4\lambda x + O(1), \qquad \lambda \to \infty_+.$$

This condition implies

$$c_1 = (B - \xi)d_1 - B\xi + \frac{1}{2}\left(d_2^2 + D^2\right),$$

$$c_2 = \xi - B - d_1,$$

Define $\hat{g}(\lambda)$ as the sum of two Abelian integrals:

$$\hat{g}(\lambda, \xi) = 2\left(\int_E^{\lambda} + \int_{\bar{E}}^{\lambda}\right)\frac{\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0}{\omega(\lambda)}d\lambda. \tag{4.38}$$

Then it evidently has real b-period

$$B_{\hat{g}} = 2\left(\int_E^{d} + \int_{\bar{E}}^{\bar{d}}\right)\frac{\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0}{\omega(\lambda)}d\lambda. \tag{4.39}$$

Now notice that $\hat{g}(\lambda)$ can be written as a single Abelian integral

$$\hat{g}(\lambda) = 4\int_E^{k}\frac{\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0}{\omega(\lambda)}d\lambda$$

and indeed

$$B_{\hat{g}} = \int_b d\hat{g}.$$

The large-λ asymptotics of $\hat{g}(\lambda, \xi)$ can now be specified as

$$\hat{g}(\lambda, \xi) = 2\lambda^2 t + 4\xi\lambda t + \hat{g}(\infty, \xi) + O(\lambda^{-1}).$$

where

$$\hat{g}(\infty, \xi) = t\left(2\left(\int_E^{\infty} + \int_{\bar{E}}^{\infty}\right)\left[\frac{\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0}{\omega(\lambda)} - (\lambda + \xi)\right]d\lambda + 2D^2 - 2B^2 - 4B\xi\right) \tag{4.40}$$

is a real function of ξ.

Remark 4.4 For $\xi = -B$, if we set $\mu(-B) = d_1(-B) = B$ and $d_2(-B) = D$, that is, $d(-B) = E$ and $\bar{d}(-B) = \bar{E}$, then $\hat{g}(\lambda, -B)$ coincide(up to a constant) with $\theta(\lambda, -B)$:

$$\hat{g}(\lambda, -B) = \theta(\lambda, -B) + 2|E|^2.$$

which provides matching at the interface with the Zakharov-Manakov region.

In order to define μ, $\mu_\pm$ and d as functions of ξ, let us compare the forms (4.36) and (4.37) of the differential $d\hat{g}$. This gives ($\mu_\pm = \mu_1 \pm i\mu_2$):

$$\mu + 2\mu_1 - d_1 = B - \xi,$$
$$2\mu\mu_1 + \mu_1^2 + \mu_2^2 + (\xi - B)d_1 - \tfrac{1}{2}d_2^2 = \tfrac{1}{2}D^2 - B\xi,$$
$$\mu\left(\mu_1^2 + \mu_2^2\right) = -c_0(\xi, d_1, d_2).$$

The local expansion of $\hat{g}(\lambda)$ at $\lambda = d$ is of the form

$$\hat{g}(\lambda) = B_{\hat{g}} + g_1(\lambda - d)^{1/2} + g_2(\lambda - d)^{3/2} + \cdots,$$

where $B_{\hat{g}}$ is real. The signature table for $\mathrm{Im}\hat{g}(\lambda)$ must have three branches of the curve $\mathrm{Im}\hat{g}(\lambda) = 0$ going out from the point d, see Fig. 6. Indeed:

- Since $\hat{g}(E) = 0$, one branch should connect d with E.
- There should exist a branch separating the basins of + and - near the real axis.
- Since $\hat{g}(\lambda)$ behaves like $\theta(\lambda)$ for large λ, there should be an infinite branch going to infinity along the asymptotic line $\mathrm{Re}\lambda = -\xi$.

Therefore, we arrive at the requirement $g_1 = 0$, that is

$$(\lambda - d)^{1/2}\hat{g}'(\lambda)|_{\lambda=d} = 4\frac{(d - \mu(\xi))(d - \mu_-(\xi))(d - \mu_+(\xi))}{\sqrt{(\lambda - E)(\lambda - \bar{E})(d - \bar{d})}} = 0,$$

The fact that μ is real implies that $\mu_+ = d$ and $\mu_- = \bar{d}$, which finally leads to the following ansatz for $d\hat{g}(\lambda)$:

$$d\hat{g}(\lambda) = 4(\lambda - \mu(\xi))\sqrt{\frac{(\lambda - d(\xi))(\lambda - \bar{d}(\xi))}{(\lambda - E)(\lambda - \bar{E})}}d\lambda,$$

where $\mu(\xi)$, $d_1(\xi)$ and $d_2(\xi)$ ($d = d_1 + id_2$, $d_2 \geqslant 0$) satisfy the equations:

$$\mu = B - \xi - d_1, \tag{4.41a}$$

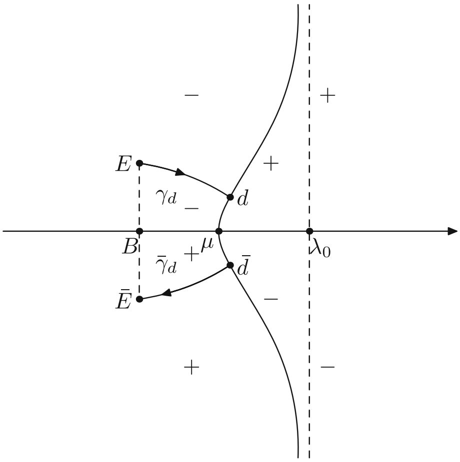


Fig. 6 The curves of $\mathrm{Im}\hat{g}(\lambda) = 0$ for $-4t\left(B + \sqrt{2A^2\left(B + \frac{A^2}{4}\right)}\right) < x < -4tB$

$$d_2^2 = D^2 - 2(B-\mu)(B-d_1), \tag{4.41b}$$

$$\int_{B-iD}^{B+iD} \sqrt{\frac{(\lambda-d_1)^2+d_2^2}{(\lambda-B)^2+D^2}}(\lambda-\mu)d\lambda = 0. \tag{4.41c}$$

Recall that (4.41a) and (4.41b) follow from the requirement that

$$d\hat{g}(\lambda) = (4\lambda + 4\xi + O(\lambda^{-2}))d\lambda, \qquad \text{as } \lambda \to \infty.$$

while (4.41c) is the normalization condition $\int_{\bar{E}}^{E} d\hat{g}(\lambda) = 0$.

Substituting (4.41a) and (4.41b) into (4.41c) yields an equation relating implicitly d_1 and ξ. In terms of the variables u and v, where

$$u = \frac{B-d_1}{D}, \qquad v = \frac{\xi + B}{2D}.$$

this equation reads

$$\mathcal{F}(u,v) = \int_{-1}^{1} \sqrt{\frac{(i\tau+1)^2 + 1 - 4uv + 2u^2}{1-\tau^2}}(i\tau + 2v - u)d\tau = 0. \tag{4.42}$$

which is considered for $0 \leqslant v \leqslant \frac{\sqrt{2}}{2}$ and $u \geqslant 0$. It is easy to check that $\mathcal{F}(0,v) = 4v$(and thus $\mathcal{F}(0,v) > 0$ for $v > 0$), $\mathcal{F}(+\infty, v) < 0$, $\mathcal{F}(0,0) = \mathcal{F}\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = 0$ and $\mathcal{F}_u(u,v) < 0$ for $(u,v) \neq \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. Therefore, (4.42) determines a unique function $u = u(v)$, $v \in \left[0, \frac{\sqrt{2}}{2}\right]$ such that $u(0) = 0$ and $u\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$. Consequently, we have that the system (4.41) determines uniquely $d_1(\xi)$, $d_2(\xi)$ and $\mu(\xi)$, such that $d_1(-B-\sqrt{2}D) = B + \sqrt{2}D$ and $d_1(-B) = B$.

We have now specified a g-function $\hat{g}(\lambda)$ whose signature table is as in Fig. 6. Hence, we can begin deforming the basic Riemann-Hilbert problem.

4.3.2 The First Deformation

We deform the part $\gamma \cup \bar{\gamma}$ of the contour of the basic Riemann-Hilbert problem into a contour $\gamma_{E,\bar{E}}$ connecting E and $\bar{E}$ in such a way that it contains:

(i) Two arcs γ_d and $\bar{\gamma}_d$ connecting, respectively, E with d and $\bar{d}$ and $\bar{E}$, and where $\mathrm{Im}\hat{g}(\lambda) = 0$;
(ii) An arc γ_μ connecting d and $\bar{d}$, passing through μ, and along which $\mathrm{Im}\hat{g}(\lambda) < 0$ for $\mathrm{Im}\lambda < 0$ and $\mathrm{Im}\hat{g}(\lambda) > 0$ for $\mathrm{Im}\lambda > 0$.

Supplying $\gamma_{E,\bar{E}} = \gamma_\mu \cup \gamma_d \cup \bar{\gamma}_d$ with the orientation as going from E to $\bar{E}$, we fix the branch of $\hat{g}(\lambda)$ as having a jump across $\gamma_{E,\bar{E}}$:

$$\begin{aligned}
&\hat{g}(\lambda)_+ + \hat{g}(\lambda)_- = 0, && \lambda \in \gamma_d \cup \bar{\gamma}_d;\\
&\hat{g}(\lambda)_+ - \hat{g}(\lambda)_- = B_{\hat{g}}, && \lambda \in \gamma_\mu,\\
&\text{with } \mathrm{Im}B_{\hat{g}} = 0.
\end{aligned}$$

4.3.3 The Second Transformation

The further series of transformations

$$N(x,t,\lambda) \rightsquigarrow N^{(1)}(x,t,\lambda) \rightsquigarrow N^{(2)}(x,t,\lambda) \rightsquigarrow N^{(3)}(x,t,\lambda),$$

is similar to that for the plane wave region but

(i) with $g(\lambda)$ replaced by $\hat{g}(\lambda)$,
(ii) with μ, which is the real stationary point of $\hat{g}(\lambda)$ instead of μ_+,
(iii) with the partition into domains with boundaries L as shown in Fig. 7.

The jump matrix $J_N^{(3)}(x,t,\lambda)$ is as follows:

- For $\lambda \in L_j$ at a fixed positive distance from the stationary point $\lambda = \mu(\xi)$,

$$J_N^{(3)}(x,t,\lambda) = \mathbb{I} + O\left(\mathrm{e}^{-\varepsilon t}\right) \text{ as } t \to +\infty.$$

- For $\lambda \in \gamma_\mu$ we have

$$J_N^{(3)}(x,t,\lambda) = \begin{cases} \begin{pmatrix} \mathrm{e}^{-itB_{\hat{g}}} & 0 \\ \lambda f(\lambda)\delta^{-2}(\lambda)\mathrm{e}^{it(\hat{g}_+(\lambda)+\hat{g}_-(\lambda))} & \mathrm{e}^{itB_{\hat{g}}} \end{pmatrix}, & \mathrm{Im}\lambda > 0, \\ \begin{pmatrix} \mathrm{e}^{-itB_{\hat{g}}} & f(\lambda)\delta^{2}(\lambda)\mathrm{e}^{-it(\hat{g}_+(\lambda)+\hat{g}_-(\lambda))} \\ 0 & \mathrm{e}^{itB_{\hat{g}}} \end{pmatrix}, & \mathrm{Im}\lambda < 0, \end{cases} \tag{4.43}$$

Thus, away from d,μ and $\bar{d}$ and as $t \to +\infty$, $J_N^{(3)}(x,t,\lambda)$ is close to a diagonal matrix:

$$J_N^{(3)}(x,t,\lambda) = \begin{pmatrix} \mathrm{e}^{-itB_{\hat{g}}} & 0 \\ 0 & \mathrm{e}^{itB_{\hat{g}}} \end{pmatrix} + O\left(\mathrm{e}^{-\varepsilon t}\right), \qquad t \to +\infty. \tag{4.44}$$

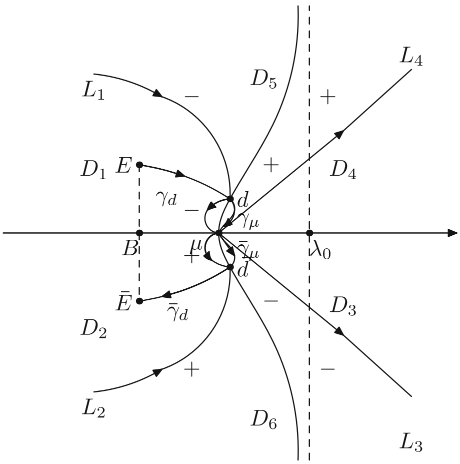


Fig. 7 The contour $\Sigma^{(3)} = L_1 \cup L_2 \cup L_3 \cup L_4 \cup \gamma_d \cup \bar{\gamma}_d \cup \gamma_\mu \cup \bar{\gamma}_\mu$ for $-4t\left(B + \sqrt{2A^2\left(B + \frac{A^2}{4}\right)}\right) < x < -4tB$

- For $\lambda \in \gamma_d \cup \bar{\gamma}_d$, similarly to the plane wave region, $J_N^{(3)}(x,t,\lambda)$ reduces to

$$J_N^{(3)}(x,t,\lambda) = \begin{cases} \begin{pmatrix} 0 & -f^{-1}(\lambda)\delta^2(\lambda) \\ \lambda f(\lambda)\delta^{-2}(\lambda) & 0 \end{pmatrix}, & \lambda \in \gamma_d, \\ \begin{pmatrix} 0 & f(\lambda)\delta^2(\lambda) \\ -\lambda f^{-1}(\lambda)\delta^{-2}(\lambda) & 0 \end{pmatrix}, & \lambda \in \gamma_d, \end{cases} \tag{4.45}$$

In order to arrive at a Riemann-Hilbert problem with a jump matrix independent of λ, we proceed as in the plane wave region.

Scalar Riemann-Hilbert problem We are looking for a scalar function $F(\lambda)$ analytic in $\mathbb{C}\backslash\gamma_d \cup \bar{\gamma}_d$ such that

$$F_-(\lambda)F_+(\lambda) = h(\lambda)\sqrt{\lambda}\delta^{-2}(\lambda), \qquad \lambda \in \gamma_d \cup \bar{\gamma}_d, \tag{4.46}$$

where

$$h(\lambda) = \begin{cases} -i\sqrt{\lambda} f(\lambda), & \lambda \in \gamma_g, \\ i\sqrt{\lambda}^{-1} f^{-1}(k), & \lambda \in \bar{\gamma}_g. \end{cases} \tag{4.47}$$

After solving this scalar problem, $J_N^{(3)}(x,t,\lambda)$ can be factorized as in (4.25). This factorization allows absorbing the diagonal factors into a new Riemann-Hilbert problem with constant jump matrix on $\gamma_d \cup \bar{\gamma}_d$.

However, an important difference with the plane wave region is that now the jump conditions (4.46) for $F(\lambda)$ are specified on two disjoint arcs. This implies that in order to arrive at a jump condition in additive form, we are led to use

$$\omega(\lambda) = \sqrt{(\lambda - E)(\lambda - \bar{E})(\lambda - d(\xi))(\lambda - \bar{d}(\xi))}.$$

Indeed, (4.46) can be rewritten as

$$\left[\frac{\log F(\lambda)}{\omega(\lambda)}\right]_+ - \left[\frac{\log F(\lambda)}{\omega(\lambda)}\right]_- = \frac{\log h(\lambda)}{\omega_+(\lambda)}, \quad \lambda \in \gamma_d \cup \bar{\gamma}_d, \tag{4.48}$$

and thus for $F(\lambda)$, we have

$$F(\lambda) = \exp\left\{\frac{\omega(\lambda)}{2\pi i}\int_{\gamma_d \cup \bar{\gamma}_d} \frac{\log h(s)}{\omega_+(s)}\frac{ds}{s-\lambda}\right\}. \tag{4.49}$$

But now $F(\lambda)$ has an essential singularity at infinity:

$$F(\lambda) = F_\infty e^{i\Delta\lambda}(1 + O(\lambda^{-1})), \qquad \lambda \to \infty.$$

where

$$\Delta = \Delta(\xi) = \frac{1}{2\pi}\int_{\gamma_d \cup \bar{\gamma}_d} \frac{\log h(\lambda)}{\omega_+(\lambda)} d\lambda, \tag{4.50}$$

and

$$F_\infty(\xi) = \exp\left\{\frac{i}{2\pi}\int_{\gamma_d \cup \bar{\gamma}_d} (s - e_1)\frac{\log h(s)}{\omega_+(s)} ds\right\},$$

with

$$e_1 = \frac{E + \bar{E} + d + \bar{d}}{2}. \tag{4.51}$$

To account for this singularity, let us introduce the normalized, that is, its a-period vanishes, Abelian integral $w(\lambda)$ of the second kind with simple poles at $\infty_\pm$:

$$w(\lambda) = \int_E^\lambda \frac{z^2 - \mathrm{e}_1 z + \mathrm{e}_0}{\omega(z)} dz,$$

where e_1 is the same as in (4.51) and e_0 is determined by the condition $\int_a dw(\lambda) = 0$:

$$\mathrm{e}_0 = -\frac{\int_d^{\bar{d}} (z^2 - \mathrm{e}_1 z + \mathrm{e}_0) \frac{dz}{\omega(z)}}{\int_d^{\bar{d}} \frac{dz}{\omega(z)}}.$$

The large-λ expansion of $w(\lambda)$ is of the form

$$w(\lambda) = \lambda + w_\infty(\xi) + O(\lambda^{-1}), \qquad \lambda \to \infty, \tag{4.52}$$

where

$$\begin{aligned} w_\infty &= \int_E^\infty \left[\frac{z^2 - \mathrm{e}_1 z + \mathrm{e}_0}{\omega(z)} - 1 \right] dz - E \\ &= \frac{1}{2} \left(\int_E^\infty + \int_{\bar{E}}^\infty \right) \left[\frac{z^2 - \mathrm{e}_1 z + \mathrm{e}_0}{\omega(z)} - 1 \right] dz - B, \end{aligned} \tag{4.53}$$

The jump conditions for $w(\lambda)$ are as follows:

$$\begin{aligned} w_+(\lambda) + w_-(\lambda) &= 0, \qquad && \lambda \in \gamma_d \cup \bar{\gamma}_d, \\ w_+(\lambda) - w_-(\lambda) &= B_w, && \lambda \in \gamma_\mu. \end{aligned}$$

Here B_w is the b-period of $w(\lambda)$:

$$B_w = \int_b dw = 2 \int_E^d \frac{z^2 - \mathrm{e}_1 z + \mathrm{e}_0}{\omega(z)} dz = \left(\int_E^d + \int_{\bar{E}}^{\bar{d}} \right) \frac{z^2 - \mathrm{e}_1 z + \mathrm{e}_0}{\omega(z)} dz \in \mathbb{R}. \tag{4.54}$$

Now introduce

$$\hat{F}(\lambda) = F(\lambda) \mathrm{e}^{-i\Delta w(\lambda)}, \tag{4.55}$$

This new function is clearly bounded at $\lambda = \infty$:

$$\hat{F}(\infty, \xi) = \mathrm{e}^{i\hat{\phi}(\xi)}, \tag{4.56}$$

with

$$\hat{\phi}(\xi) = \frac{1}{2\pi} \int_{\gamma_d \cup \bar{\gamma}_d} (s - \mathrm{e}_1) \log [h(s) \delta^{-2}(s, \xi)] \frac{ds}{\omega_+(s)} - \Delta(\xi) w_\infty(\xi).$$

Also, $\hat{F}(\lambda)$ has the same jumps as $F(\lambda)$ across γ_d and $\bar{\gamma}_d$. On the other hand, the price for introducing the exponential factor in (4.55) is that $\hat{F}(\lambda)$ has a jump across γ_μ:

$$\frac{\hat{F}_+(\lambda)}{\hat{F}_-(\lambda)} = \mathrm{e}^{-i\Delta B_w}, \qquad \lambda \in \gamma_\mu.$$

Now we can absorb $\hat{F}(\lambda)$ into the Riemann-Hilbert problem for $N^{(4)}(x, t, \lambda)$:

$$N^{(4)}(x, t, \lambda) = \hat{F}^{\sigma_3}(\infty) N^{(3)}(x, t, \lambda) \hat{F}^{-\sigma_3}(\lambda),$$

which leads to the jump conditions

$$N_+^{(4)}(x,t,\lambda) = N_-^{(4)}(x,t,\lambda) J_N^{(4)}(x,t,\lambda),$$

where

$$J_N^{(4)}(x,t,\lambda) = \begin{cases} J_N^{mod} + O(\mathrm{e}^{-\varepsilon t}), & \lambda \in \gamma_d \cup \bar{\gamma}_d \cup \gamma_\mu, \\ \mathbb{I} + O(\mathrm{e}^{-\varepsilon t}), & \lambda \in L \cup \bar{L}. \end{cases}$$

with

$$J_N^{(mod)} = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \lambda \in \gamma_d \cup \bar{\gamma}_d, \\ \begin{pmatrix} \mathrm{e}^{-itB_{\hat{g}} - i\Delta B_w} & 0 \\ 0 & \mathrm{e}^{itB_{\hat{g}} + i\Delta B_w} \end{pmatrix}, & \lambda \in \gamma_\mu. \end{cases} \tag{4.57}$$

4.3.4 The Model Problem

Thus, we arrive at the model Riemann-Hilbert problem:

$$N_+^{mod}(x,t,\lambda) = N_-^{mod}(x,t,\lambda) J_N^{mod}(x,t,\lambda), \qquad \lambda \in \gamma_d \cup \bar{\gamma}_d \cup \gamma_\mu, \tag{4.58a}$$

$$N^{mod}(x,t,\lambda) = \mathbb{I} + O\left(\frac{1}{\lambda}\right), \qquad \lambda \to \infty. \tag{4.58b}$$

The solution of this model Riemann-Hilbert problem approximates $N^{(4)}(x,t,\lambda)$:

$$N^{(4)}(x,t,\lambda) = \left(\mathbb{I} + O\left(t^{-\frac{1}{2}}\right)\right) N^{mod}(x,t,\lambda), \tag{4.59}$$

The model problem (4.58) can be solved in terms of elliptic theta functions. Let

$$U(\lambda) = \frac{1}{c}\int_E^\lambda \frac{dz}{\omega(z)},$$

be the normalized Abelian integral, that is

$$c = 2\int_{\bar{d}}^{d} \frac{dz}{\omega(z)}.$$

Then, define

$$\tau = \tau(\xi) = \frac{2}{c}\int_E^d \frac{dz}{\omega(z)}, \tag{4.60}$$

with $\mathrm{Im}\tau > 0$. Furthermore, the following relations are valid:

$$\begin{aligned} &U_+(\lambda) + U_-(\lambda) = 0, && \lambda \in \gamma_d, \\ &U_+(\lambda) + U_-(\lambda) = -1, && \lambda \in \bar{\gamma}_d, \\ &U_+(\lambda) - U_-(\lambda) = \tau, && \lambda \in \gamma_\mu, \end{aligned} \tag{4.61}$$

Next, define

$$\nu(\lambda) = \left(\frac{(\lambda - E)(\lambda - d)}{(\lambda - \bar{E})(\lambda - \bar{d})}\right)^{\frac{1}{4}},$$

where the branch is fixed by specifying the branch cut $\gamma_{E,\bar{E}}$ and the behavior as $\lambda \to \infty$;

$$\nu(\lambda) = 1 + \frac{D + d_2}{2i\lambda} + O(\lambda^{-2}), \qquad \lambda \to \infty.$$

Along the cut,we have

$$\nu_+(\lambda) = \begin{cases} -i\nu_-(\lambda), & \lambda \in \gamma_d \cup \bar{\gamma}_d, \\ -\nu_-(\lambda), & \lambda \in \gamma_\mu. \end{cases}$$

Finally, introduce the theta function

$$\theta_3(z) = \sum_{m\in\mathbb{Z}} \mathrm{e}^{\pi i \tau m^2 + 2\pi i m z},$$

and define the 2×2 matrix-value function $\Theta(\lambda) = \Theta(t, \xi, \lambda)$ with entries:

$$\Theta_{11}(\lambda) = \frac{1}{2}\left[\nu(\lambda) + \frac{1}{\nu(\lambda)}\right] \frac{\theta_3\left[U(\lambda) - U_0 - \frac{1}{2} - \frac{B_{\hat{g}}t}{2\pi} - \frac{B_w\Delta}{2\pi}\right]}{\theta_3[U(\lambda) - U_0]},$$

$$\Theta_{12}(\lambda) = \frac{1}{2}\left[\nu(\lambda) - \frac{1}{\nu(\lambda)}\right] \frac{\theta_3\left[U(\lambda) + U_0 + \frac{1}{2} + \frac{B_{\hat{g}}t}{2\pi} + \frac{B_w\Delta}{2\pi}\right]}{\theta_3[U(\lambda) + U_0]},$$

$$\Theta_{21}(\lambda) = \frac{1}{2}\left[\nu(\lambda) - \frac{1}{\nu(\lambda)}\right] \frac{\theta_3\left[U(\lambda) + U_0 - \frac{1}{2} - \frac{B_{\hat{g}}t}{2\pi} - \frac{B_w\Delta}{2\pi}\right]}{\theta_3[U(\lambda) + U_0]},$$

$$\Theta_{22}(\lambda) = \frac{1}{2}\left[\nu(\lambda) + \frac{1}{\nu(\lambda)}\right] \frac{\theta_3\left[U(\lambda) - U_0 + \frac{1}{2} + \frac{B_{\hat{g}}t}{2\pi} + \frac{B_w\Delta}{2\pi}\right]}{\theta_3[U(\lambda) - U_0]},$$

where U_0 is to be chosen so that the unique zero of $\theta_3(U(\lambda) - U_0)$, as a function on the Riemann surface, lying on the first sheet is compensated by the zero of $\nu(\lambda) + \frac{1}{\nu(\lambda)}$ where $\theta_3(U(\lambda) + U_0)$ has no zero on this sheet. Setting

$$U_0 = U(E_0) + \frac{1}{2} + \frac{\tau}{2},$$

where

$$E_0 = \frac{Ed - \bar{E}\bar{d}}{E - \bar{E} + d - \bar{d}},$$

satisfies this requirement, and thus $\Theta(\lambda)$ can be viewed as a function analytic in $\mathbb{C}\backslash\gamma_{E,\bar{E}}$. On the other hand, due to the properties of theta function:

$$\theta_3(-z) = \theta_3(z), \quad \theta_3(z+1) = \theta_3(z), \quad \theta_3(z \pm \tau) = \mathrm{e}^{-\pi i \tau \mp 2\pi i z}\theta_3(z),$$

$\Theta(\lambda)$ satisfies the jump conditions (4.58a)–(4.57) of the model Riemann-Hilbert problem. Taking into account the normalization condition (4.58b), the solution of the model Riemann-Hilbert problem is given by

$$N^{mod}(x, t, \lambda) = \Theta^{-1}(t, \xi, \infty)\Theta(t, \xi, \lambda).$$

4.3.5 Back to the Original Problem

Now, following the sequence of equations of type (4.34) (with g and F replaced, respectively, by $\hat{g}$ and $\hat{F}$) and taking into account the equations $\hat{g}$ and $\hat{F}$, and the explicit formula for $n_{12}^{mod}(x, t, \lambda)$

$$2in_{12}^{mod}(x, t, \lambda) = [D + d_2] \frac{\theta_3\left[\frac{B_{\hat{g}}t}{2\pi} + \frac{B_w\Delta}{2\pi} + U_0 + \frac{1}{2} + U(\infty)\right]}{\theta_3\left[\frac{B_{\hat{g}}t}{2\pi} + \frac{B_w\Delta}{2\pi} + U_0 + \frac{1}{2} - U(\infty)\right]} \frac{\theta_3[U_0 - U(\infty)]}{\theta_3[U_0 + U(\infty)]},$$

and $\hat{F}^{-2}(\infty) = \mathrm{e}^{-2i\hat{\phi}(\xi)}$, we obtain the asymptotics in the region $-4t\left(B+\sqrt{2A^2\left(B+\frac{A^2}{4}\right)}\right) < x < -4tB$.

Theorem 4.5 (**Elliptic wave region**) *In the region* $-4t\left(B+\sqrt{2A^2\left(B+\frac{A^2}{4}\right)}\right) < x < -4tB$,*the asymptotics, as* $t \to +\infty$, *of the solution* $q(x,t)$ *of the initial value problem* (1.6) *takes the form of a modulated elliptic wave:*

$$q(x,t)=[D+\mathrm{Im}d(\xi)]\frac{\theta_3\left[\frac{B_{\hat{g}}t}{2\pi}+\frac{B_w\Delta}{2\pi}+V_+(\xi)\right]}{\theta_3\left[\frac{B_{\hat{g}}t}{2\pi}+\frac{B_w\Delta}{2\pi}+V_-(\xi)\right]}\frac{\theta_3[V_-(\xi)-\frac{1}{2}]}{\theta_3[V_+(\xi)-\frac{1}{2}]}+O(t^{-\frac{1}{2}}),\ t\to+\infty. \tag{4.62}$$

Here $B_{\hat{g}}$, B_w *and* Δ *are functions of the variable* $\xi = \frac{x}{4t}$ *defined, respectively, by* (4.39), (4.54) *and* (4.50), *and* $V_\pm(\xi) = U_0 + \frac{1}{2} \pm U(\infty)$. *Furthermore,*

$$\theta_3(z) = \sum_{z\in\mathbb{Z}} \mathrm{e}^{\pi i\tau m^2+2\pi imz},$$

is the theta function of invariant $\tau = \tau(\xi)$ *defined in* (4.60),

$$\hat{g}(\infty,\xi) = t\left(2\left(\int_E^\infty+\int_{\bar{E}}^\infty\right)\left[(z-\mu(\xi))\sqrt{\frac{(z-d(\xi))(z-\bar{d}(\xi))}{(z-E)(z-\bar{E})}}\right.\right.$$
$$\left.\left.-(z+\xi)\right]dz+2D^2-2B^2-4B\xi\right),$$

and the phase shift $\phi(\xi)$ *is given by*

$$\phi(\xi) = \frac{1}{2\pi}\int_{\gamma_d\cup\gamma_{\bar{d}}}\frac{[s-\mathrm{e}_1(\xi)-\omega_\infty(\xi)]\log[h(s)\sqrt{s}\delta^{-2}(s,\xi)]}{[(s-E)(s-\bar{E})(s-d(\xi))(s-\bar{d}(\xi))]^{1/2}}ds,$$

where

$$h(\lambda) = \begin{cases} a_+^{-1}(\lambda)a_-^{-1}(\lambda), & \lambda\in\gamma_d \\ a_+(\lambda)a_-(\lambda), & \lambda\in\gamma_{\bar{d}} \end{cases}$$
$$\delta(\lambda,\xi) = \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\mu(\xi)}\frac{\log(1+\lambda\rho^2(\lambda))}{s-\lambda}ds\right\}.$$

and $\mathrm{e}_1(\xi)$, ω_∞ *and* $\mu(\xi)$ *are defined, respectively, by* (4.51), (4.53) *and* (4.41).

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Appendix A: The details of deriving $\Psi^p(x,t,k)$ in (2.1)

In fact, we have to define matrix valued Jost solutions of the Lax equations (1.11):

$$\Psi(x,t,k) = \mathrm{e}^{-i(k^2x+2k^4t)\sigma_3} + o(1), \quad x \to +\infty, \qquad \mathrm{Im}k^2 = 0, \tag{A1}$$

$$\Phi(x,t,k) = \Psi^p(x,t,k) + o(1), \quad x \to -\infty, \qquad \mathrm{Im}k^2 = 0, \tag{A2}$$

Here, $\Psi^p(x,t,k)$ is the solution of the linear differential equations:

$$\begin{aligned} \Psi^p_x(x,t,k) &= M^p(x,t,k)\Psi^p(x,t,k), \\ \Psi^p_t(x,t,k) &= N^p(x,t,k)\Psi^p(x,t,k), \end{aligned} \tag{A3}$$

where $M^p(x,t,k)$ and $N^p(x,t,k)$ are defined by (1.12) with q^p instead of q.

To prove (2.1) with the $X(k)$, $\Omega(k)$, $E(k)$ and $\varphi(k)$ defined in (2.2), (2.3), (2.4) and (2.5), respectively, we look for the solution of the x-equation in (A3) in the form

$$\Psi^p(x;k) = \mathrm{e}^{-iBx\sigma_3}E(k)\mathrm{e}^{-ixX(k)\sigma_3}, \tag{A4}$$

where the function $X(k)$ and the matrix $E(k)$ have to be determined. Consider logarithmic derivative

$$\Psi^p_x(x;k)(\Psi^p(x;k))^{-1} = -iB\sigma_3 - iX(k)\mathrm{e}^{-iBx\sigma_3}E(k)\sigma_3(E(k))^{-1}\mathrm{e}^{iBx\sigma_3}, \tag{A5}$$

Then $\Psi^p(x;k)$ will be a solution of the x-equation if

$$-ik^2\sigma_3 + kQ^p + \frac{i}{2}|q^p|^2\sigma_3 = -iB\sigma_3 - iX(k)\mathrm{e}^{-iBx\sigma_3}E(k)\sigma_3(E(k))^{-1}\mathrm{e}^{iBx\sigma_3}, \tag{A6}$$

To solve the last equation we put

$$E(k) = \begin{pmatrix} a(k) & b(k) \\ b(k) & a(k) \end{pmatrix} \tag{A7}$$

where $a^2(k) - b^2(k) = 1$. Substituting (A7) into (A6), we have

$$-iX(k)(a^2+b^2) - iB = -ik^2 + \frac{i}{2}|A|^2, \qquad 2iX(k)ab\mathrm{e}^{-2iBx} = kA\mathrm{e}^{-2iBx}. \tag{A8}$$

If

$$a(k) = \frac{1}{2}\left(\varphi(k) + \frac{1}{\varphi(k)}\right), \quad b(k) = \frac{1}{2}\left(\varphi(k) - \frac{1}{\varphi(k)}\right),$$

then $a^2(k) - b^2(k) = 1$ is fulfilled and

$$\varphi^2 + \frac{1}{\varphi^2} = \frac{-ik^2 + iB + \frac{i}{2}|A|^2}{-\frac{i}{2}X(k)}, \qquad \varphi^2 - \frac{1}{\varphi^2} = \frac{kA}{\frac{i}{2}X(k)},$$

Hence,

$$X(k)^2 = \left(k^2 - B - \frac{1}{2}A^2\right)^2 + (kA)^2,$$

and

$$\varphi = \left(\frac{k^2 - B - \frac{A^2}{2} - ikA}{k^2 - B - \frac{A^2}{2} + ikA}\right)^{\frac{1}{4}}.$$

We look for the solution of the x-equation and t-equation in the form

$$\Psi^p(x,t;k) = \mathrm{e}^{i(\omega t - Bx)\sigma_3} E(k) \mathrm{e}^{-i(xX(k)+t\Omega(k))\sigma_3},$$

where matrix $E(k)$ is the same one, and function $\Omega(k)$ have to be determined. Consider now logarithmic derivative

$$\Psi_t^p(x,t;k)(\Psi^p(x,t;k))^{-1} = i\omega\sigma_3 - i\Omega(k)\mathrm{e}^{i(\omega t - Bx)\sigma_3} E(k)\sigma_3(E(k))^{-1}\mathrm{e}^{-i(\omega t - Bx)\sigma_3},$$

Then $\Psi^p(x,t;k)$ will be a compatible solution of the x- and t-equations if

$$-2ik^4\sigma_3 + 2k^3 Q^p + ik^2|q^p|^2\sigma_3 - ikQ_x^p\sigma_3 + \frac{i}{4}|q^p|^4\sigma_3 + \frac{1}{2}\left(q^p\bar{q}_x^p - \bar{q}^p q_x^p\right)\sigma_3 = i\omega\sigma_3 - i\Omega(k)\mathrm{e}^{i(\omega t - Bx)\sigma_3} E(k)\sigma_3(E(k))^{-1}\mathrm{e}^{-i(\omega t - Bx)\sigma_3},$$

This equation gives

$$\Omega(k)(a^2+b^2) = 2k^4 - k^2 A - 2B^2 - A^2 B, \quad i\Omega(k)ab = kA(k^2+B),$$

Both of which imply that $\Omega(k) = 2(k^2+B)X(k)$.

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