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Conservation laws and self-consistent sources for a super integrable equation hierarchy

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ABSTRACT

In this paper, a super integrable equation hierarchy is considered based on a Lie superalgebra and supertrace identity. Then, a super integrable equation hierarchy with self-consistent sources is established. Furthermore, we introduce two variables F and G to construct conservation laws of the super integrable equation hierarchy and the first two conserved densities and fluxes are listed. It would be specially mentioned that the Fermi variables play an important role in super integrable systems which is different from the ordinary integrable systems.

1. Introduction

As is well known, soliton theory is being applied to mathematics, physics, biology, astrophysics and other potential field, and considerable progress has been made in the study of soliton theory [1–17]. The diversity and complexity of soliton theory enables investigators to do research from different views, such as Hamiltonian structure, conservation laws, self-consistent sources and various solutions of soliton equations. Conservation law plays an important role in mathematical physics, such as it describes the conservation of fundamental physical quantities, provides a method to study quantitative and qualitative properties of equations and their solutions, verifies complete integrability of nonlinear partial differential equations and is used to test numerical integrators. Generally, the infinitely many conservation laws or conserved quantities for both continuous system and discrete system can be obtained from the scattering problem [18,19], from the formal solutions of eigenfunctions [20], from the Bäcklund transformation [18], from the couple of Ricatti equations [18], from the quasi-diffierential operator based on the Sato theory [21], from the trace identity [22] or from Lax pair [23].

The trace identity [24] provides a powerful tool for constructing Hamiltonian structures of soliton equations. It is based on the killing form on a semisimple Lie algebra. Various integrable equation hierarchies, such as AKNS hierarchy, BPT hierarchy, TB hierarchy and Jaulent–Miodek hierarchy, along with their Hamiltonian structures are obtained [24–38]. Recently, Ma and Chen [39] developed this method to nonsemisimple Lie algebras and proposed the variational identity – a generalized trace identity. In Ref. [40], Ma further gave the supertrace identity on Lie superalgebras and its application to super-AKNS hierarchy and super-Dirac hierarchy to get their super Hamiltonian structures. Then, super C-KdV hierarchy [41] and super Boussinesq hierarchy [42] and super NLS-mKdV hierarchy [43] hierarchies as well as their super Hamiltonian structures are presented. The binary nonlinearization of the super-AKNS system [44] and an implicit symmetry constraint, the Bargmann symmetry constraint [45] and binary nonlinearization of the super-Dirac systems are obtained by He et al.

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Very recently, a super-CKdV equation hierarchy [46] with self-consistent sources is presented. Soliton equations with self-consistent sources can provide variety of dynamics of physical models due to the nonconstant velocities of solitary waves resulting from sources. They are usually used to describe interactions between different solitary waves. Therefore such systems have attracted considerable attention in recent research of soliton theory [47].

Our letter would like to derive a super integrable equation hierarchy with self-consistent sources and give the conservation laws of the super integrable equation hierarchy.

The paper is organized as follows. In Section 2, we give a brief introduction about supertrace identity and super integrable equation hierarchy with self-consistent sources. In Section 3, we give a specific super integrable equation hierarchy with self-consistent sources. In Section 4, the conservation laws for the super integrable equation hierarchy are obtained. Finally,-some conclusions are given in Section 5.

2. The supertrace identity and super integrable equation hierarchy with self-consistent sources

We consider the following loop superalgebras [48] $\left\{\widetilde{G}=e_i(n)|\ i=1,\ldots,5\right\}$,

$$e_1(n)=\begin{pmatrix}0&\lambda^n&0\\ \lambda^n&0&0\\ 0&0&0\end{pmatrix},\quad e_2(n)=\begin{pmatrix}0&\lambda^n&0\\ -\lambda^n&0&0\\ 0&0&0\end{pmatrix},\quad e_3(n)=\begin{pmatrix}\lambda^n&0&0\\ 0&-\lambda^n&0\\ 0&0&0\end{pmatrix},$$
$$e_4(n)=\begin{pmatrix}0&0&\lambda^n\\ 0&0&0\\ 0&-\lambda^n&0\end{pmatrix},\quad e_5(n)=\begin{pmatrix}0&0&0\\ 0&0&\lambda^n\\ \lambda^n&0&0\end{pmatrix}, \tag{1}$$

along with the communicative operation

$$\begin{aligned}&[e_1(m),e_2(n)]=-2e_3(m+n),\quad [e_1(m),e_3(n)]=-2e_2(m+n),\quad [e_1(m),e_4(n)]=e_5(m+n),\quad [e_1(m),e_5(n)]=e_4(m+n),\\ &[e_2(m),e_3(n)]=-2e_1(m+n),\quad [e_2(m),e_4(n)]=-e_5(m+n),\quad [e_2(m),e_5(n)]=e_4(m+n),\quad [e_3(m),e_4(n)]=e_4(m+n),\\ &[e_3(m),e_5(n)]=-e_5(m+n),\quad [e_4(m),e_4(n)]_+=-(e_1(m+n)+e_2(m+n)),\quad [e_4(m),e_5(n)]_+=e_3(m+n),\\ &[e_5(m),e_5(n)]_+=e_1(m+n)-e_2(m+n).\end{aligned} \tag{2}$$

where $\widetilde{G}_1=\{e_1(n),e_2(n),e_3(n)\}$ are even and $\widetilde{G}_2=\{e_4(n),e_5(n)\}$ are odd, $[\cdot,\cdot]$ and $[\cdot,\cdot]_+$ denote the commutator and the anticommutator.

Considering an auxiliary linear spectral problem

$$\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}_x=U(u,\lambda)\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix},\qquad \begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}_t=V(u,\lambda)\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}, \tag{3}$$

where $U(u,\lambda)=e_1(1)+\sum_{j=1}^{4}u_je_{j+1}(0),\ \{e_{j+1}(n),1\leqslant j\leqslant 4\}\subset\widetilde{G},\ u=(u_1,\ldots,u_4)^T$ is a vector function, $u_i=u_i(x,t)$, $\phi_i=\phi_i(x,t)$, ϕ_{ix} and ϕ_{it} denote the partial derivatives with respect to x and t, λ is a spectral parameter.

From the spectral problem (3), the compatibility condition gives to the zero curvature equation

$$U_t-V_x+[U,V]=0,\quad \lambda_t=0. \tag{4}$$

Solve zero curvature Eq. (4), we could get

$$u_t=K\left(u,u_x,\ldots,\frac{\partial^p u}{\partial x^p}\right), \tag{5}$$

which is called super-evolution equation.

Considering the supertrace identity

$$\frac{\delta}{\delta u}\int str(ad_v ad_{\partial U/\partial\lambda})dx=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}str(ad_v ad_{\partial U/\partial\lambda}). \tag{6}$$

With the help of supertrace identity (6), if we could get a super Hamiltonian operator J and a super Hamiltonian function H such that

$$u_t=K(u)=J\frac{\delta H_{n+1}}{\delta u},\quad n=1,2,\ldots, \tag{7}$$

where

$$\frac{\delta H_n}{\delta u} = L\frac{\delta H_{n-1}}{\delta u} = \cdots = L^n\frac{\delta H_0}{\delta u}, \quad \frac{\delta}{\delta u} = \left(\frac{\delta}{\delta u_1}, \ldots, \frac{\delta}{\delta u_5}\right), \quad n = 1, 2, \ldots. \tag{8}$$

Then Eq. (5) is posses a super-Hamiltonian equation. Thus, we called Eq. (5) has a super-Hamiltonian structure.

According to the Eq. (3), we consider a new auxiliary linear problem which is constructed with the help of loop superalgebra (1)

$$\begin{aligned}
\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\end{pmatrix}_x &= U(u,\lambda_j)\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\end{pmatrix} = \sum_{i=1}^{5} u_i e_i(\lambda)\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\end{pmatrix}, \quad j = 1, \ldots, N,\\
\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\end{pmatrix}_{t_n} &= V_n(u,\lambda_j)\begin{pmatrix}\phi_{1}\\ \phi_{2}\\ \phi_{3}\end{pmatrix} = \left[\sum_{m=0}^{n} V_m(u)\lambda_j^{n-m}\right]\begin{pmatrix}\phi_{1}\\ \phi_{2}\\ \phi_{3}\end{pmatrix}, \quad j = 1, \ldots, N.
\end{aligned} \tag{9}$$

Based on the result in [49], we show that the following equation

$$\frac{\delta H_k}{\delta u} + \sum_{j=1}^{N} \alpha_j \frac{\delta \lambda_j}{\delta u} = 0, \tag{10}$$

where α_j are constants. Eq. (10) determines a finite-dimensional invariant set for the flows (8).

For (9), it is known that

$$\frac{\delta \lambda_j}{u_i} = \frac{1}{3} Str\left(\Psi_j \frac{\partial U(u,\lambda_j)}{\delta u_i}\right) = \frac{1}{3} Str(\Psi_j e_i \lambda_j), \quad i = 1, \ldots 5, \tag{11}$$

where *Str* denotes the supertrace of a matrix and

$$\Psi_j = \begin{pmatrix}\phi_{1j}\phi_{2j} & -\phi_{1j}^2 & \phi_{1j}\phi_{3j}\\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j} & \phi_{2j}\phi_{3j}\\ \phi_{2j}\phi_{3j} & -\phi_{1j}\phi_{3j} & 0\end{pmatrix}, \quad j = 1, \ldots, N. \tag{12}$$

From the Eqs. (10) and (11), a kind of super Hamiltonian integrable equation hierarchy with self-consistent sources is presented as follows

$$u_{it} = J\frac{\delta H_{n+1}}{\delta u_i} + J\sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u_i}, \quad n = 1, 2\cdots. \tag{13}$$

3. A super integrable equation hierarchy with self-consistent sources

Considering the following spectral problem based on loop superalgebra (1)

$$\varphi_x = U\varphi, \quad \varphi_t = V\varphi, \tag{14}$$

where

$$U = \begin{pmatrix}u_2 & \lambda + u_1 & u_3\\ \lambda - u_1 & -u_2 & u_4\\ u_4 & -u_3 & 0\end{pmatrix}, \quad V = \begin{pmatrix}c & a+b & d\\ a-b & -c & e\\ e & -d & 0\end{pmatrix}, \tag{15}$$

and

$$a = \sum_{m\geqslant 0} a_m\lambda^{-m}, \quad b = \sum_{m\geqslant 0} b_m\lambda^{-m}, \quad c = \sum_{m\geqslant 0} c_m\lambda^{-m}, \quad d = \sum_{m\geqslant 0} d_m\lambda^{-m}, \quad e = \sum_{m\geqslant 0} e_m\lambda^{-m}. \tag{16}$$

It must be pointed out that u_3 and u_4 are fermi variables, in other words, $u_3u_4 = -u_4u_3, u_3^2 = u_4^2 = 0$.

Starting from the stationary zero curvature equation

$$V_x = [V, U], \tag{17}$$

we have

$$\begin{cases} a_{m,x} = 2u_2 b_m - 2u_1 c_m - u_3 d_m + u_4 e_m, \\ b_{mx} = 2u_2 a_m - 2c_{m+1} - u_3 d_m - u_4 e_m, \\ c_{mx} = 2u_1 a_m - 2b_{m+1} + u_4 d_m + u_3 e_m, \\ d_{mx} = e_{m+1} - u_4 a_m - u_4 b_m - u_3 c_m + u_2 d_m + u_1 e_m, \\ e_{mx} = d_{m+1} - u_3 a_m + u_3 b_m + u_4 c_m - u_1 d_m - u_2 e_m \\ c_0 = d_0 = e_0 = b_0 = 0, \quad a_0 = m_0, \quad c_1 = u_2 m_0, \quad d_1 = u_3 m_0, \\ e_1 = u_4 m_0, \quad b_1 = u_1 m_0, \quad a_1 = m_1, \quad c_2 = u_2 m_1 - \frac{1}{2} u_{1x} m_0, \quad d_2 = u_3 m_1 + u_{4x} m_0, \\ e_2 = u_4 m_1 + u_{3x} m_0, \quad b_2 = u_1 m_1 - \frac{1}{2} u_{2x} m_0, \quad a_2 = -\frac{1}{2} u_2^2 m_0 + \frac{1}{2} u_1^2 m_0 + u_4 u_3 m_0. \end{cases} \tag{18}$$

Note

$$-V_+^{(n)} = \sum_{m=0}^{n} (a_m e_1(n-m) + b_m e_2(n-m) + c_m e_3(n-m) + d_m e_4(n-m) + e_m e_5(n-m)) = \lambda^n V - V_-^{(n)}. \tag{19}$$

A direct calculation reads

$$-\left(V_{+x}^{(n)} + V_{+t}^{(n)}\right) + \left[V_+^{(n)}, U\right] = 2c_{n+1} e_2(0) + 2b_{n+1} e_3(0) - e_{n+1} e_4(0) - d_{n+1} e_5(0). \tag{20}$$

Substituting it into the zero curvature equation

$$U_t - V^{(n)} + \left[U, V^{(n)}\right] = 0, \tag{21}$$

we get a super integrable equation hierarchy

$$u_t = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t = \begin{pmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -b_{n+1} \\ c_{n+1} \\ -e_{n+1} \\ d_{n+1} \end{pmatrix} = JP_{n+1} = JLP_n, \tag{22}$$

where

$$L = \begin{pmatrix} 2u_1\partial^{-1}u_2 & -\frac{1}{2}\partial + 2u_1\partial^{-1}u_1 & -\frac{1}{2}u_3 + u_1\partial^{-1}u_4 & -\frac{1}{2}u_4 + u_1\partial^{-1}u_3 \\ \frac{1}{2}\partial - 2u_2\partial^{-1}u_2 & -2u_2\partial^{-1}u_1 & \frac{1}{2}u_4 - u_2\partial^{-1}u_4 & -\frac{1}{2}u_3 - u_2\partial^{-1}u_3 \\ u_4 + 2u_4\partial^{-1}u_2 & 2u_4\partial^{-1}u_1 - u_3 & u_4\partial u_4 - u_1 & u_4\partial^{-1}u_3 + u_2 - \partial \\ u_3 - 2u_3\partial^{-1}u_2 & -u_4 - 2u_3\partial^{-1}u_1 & -\partial - u_2 - u_3\partial^{-1}u_4 & u_1 - u_3\partial^{-1}u_3 \end{pmatrix}. \tag{23}$$

When we take $n = 2$, the hierarchy (22) can be reduced to the super equations:

$$\begin{cases} u_{1t_2} = u_{1x} m_1 - \frac{1}{2} u_{2xx} m_0 + u_2^3 m_0 - u_2 u_1^2 m_0 - 2u_2 u_4 u_3 m_0 + u_3 u_{4x} m_0 + u_4 u_{3x} m_0, \\ u_{2t_2} = u_{2x} m_1 - \frac{1}{2} u_{1xx} m_0 - u_1^3 m_0 + u_1 u_2^2 m_0 - 2u_1 u_4 u_3 m_0 - u_4 u_{4x} m_0 - u_3 u_{3x} m_0, \\ u_{3t_2} = u_{4xx} m_0 + u_{3x} m_1 - \frac{1}{2} u_4 u_2^2 m_0 - \frac{1}{2} u_4 u_1^2 m_0 - \frac{1}{2} u_4 u_{2x} m_0 - \frac{1}{2} u_3 u_{1x} m_0 - u_{4x} u_2 m_0 - u_1 u_{3x} m_0, \\ u_{4t_2} = u_{3xx} m_0 + u_{4x} m_1 - \frac{1}{2} u_3 u_2^2 m_0 - \frac{1}{2} u_3 u_1^2 m_0 + \frac{1}{2} u_3 u_{2x} m_0 + \frac{1}{2} u_4 u_{1x} m_0 + u_1 u_{4x} m_0 + u_2 u_{3x} m_0. \end{cases} \tag{24}$$

According to supertrace identity on Lie superalgebras, a direct calculation reads

$$Str(ad_V ad_{U_\lambda}) = 6a, \quad Str\left(ad_V ad_{\frac{\partial U}{\partial u_1}}\right) = -6b, \quad Str\left(ad_V ad_{\frac{\partial U}{\partial u_2}}\right) = 6c, \quad Str\left(ad_V ad_{\frac{\partial U}{\partial u_3}}\right) = -6e, \quad Str\left(ad_V ad_{\frac{\partial U}{\partial u_4}}\right) = 6d. \tag{25}$$

Substituting the above formulae into the supertrace identity yields

$$\frac{\delta}{\delta u}\left(\int (6a) dx\right) = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} (-6b, 6c, -6e, 6d)^T. \tag{26}$$

Comparing the coefficient of λ^{-n-1} yields

$$\frac{\delta}{\delta u}\left(\int (6a_{n+1}\right) = (\gamma - n)(-6b_n, 6c_n, -6e_n, 6d_n)^T. \tag{27}$$

Therefore, we conclude that

$$P_{n+1} = \frac{\delta H_n}{\delta u}, \quad H_n = -\int \frac{a_{n+2}}{n+1} dx. \tag{28}$$

This super integrable equation hierarchy (22) has the following super bi-Hamiltonian structure

$$u_t = JP_{n+1} = J\frac{\delta H_n}{\delta u} = M\frac{\delta H_{n-1}}{\delta u}, \quad n \geqslant 0, \tag{29}$$

where

$$M = JL = \begin{pmatrix} -\partial + 4u_2\partial^{-1}u_2 & 4u_2\partial^{-1}u_1 & -u_4 + 2u_2\partial^{-1}u_4 & u_3 + 2u_2\partial^{-1}u_3 \\ 4u_1\partial^{-1}u_2 & \partial + 4u_1\partial^{-1}u_1 & u_3 + 2u_1\partial^{-1}u_4 & -u_4 + 2u_1\partial^{-1}u_3 \\ -u_4 - 2u_4\partial^{-1}u_2 & -2u_4\partial^{-1}u_1 + u_3 & -u_4\partial^{-1}u_4 + u_1 & -u_4\partial^{-1}u_3 - u_2 + \partial \\ u_3 - 2u_3\partial^{-1}u_2 & -u_4 - 2u_3\partial^{-1}u_1 & -\partial - u_2 - u_3\partial^{-1}u_4 & u_1 - u_3\partial^{-1}u_3 \end{pmatrix}. \tag{30}$$

Next, we will construct the super integrable equation hierarchy with self-consistent sources. Considering the linear system

$$\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_x = U\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}, \quad \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_t = V\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}. \tag{31}$$

From Eq. (31), we set

$$\frac{\delta H_n}{\delta u} = \sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u}, \tag{32}$$

and obtain the following $\frac{\delta\lambda_j}{\delta u}$

$$\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u} = \sum_{j=1}^{N}\begin{pmatrix} Str\left(\Psi_j\frac{\delta U}{\delta u_1}\right) \\ Str\left(\Psi_j\frac{\delta U}{\delta u_2}\right) \\ Str\left(\Psi_j\frac{\delta U}{\delta u_3}\right) \\ Str\left(\Psi_j\frac{\delta U}{\delta u_4}\right) \end{pmatrix} = \begin{pmatrix} <\Psi_1,\Psi_1> + <\Psi_2,\Psi_2> \\ 2<\Psi_1,\Psi_2> \\ -2<\Psi_2,\Psi_3> \\ 2<\Psi_1,\Psi_3> \end{pmatrix}, \tag{33}$$

where $\Psi_1 = (\phi_{i1},\ldots\phi_{N1})^T$, $(i = 1,2,3)$.

According to (13), the super integrable equation hierarchy with self-consistent sources is proposed

$$u_t = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t = J\begin{pmatrix} -b_{n+1} \\ c_{n+1} \\ -e_{n+1} \\ d_{n+1} \end{pmatrix} + J\begin{pmatrix} <\Psi_1,\Psi_1> + <\Psi_2,\Psi_2> \\ 2<\Psi_1,\Psi_2> \\ -2<\Psi_2,\Psi_3> \\ 2<\Psi_1,\Psi_3> \end{pmatrix}. \tag{34}$$

For $n = 2$, we obtain the super integrable equation with self-consistent sources

$$\begin{cases} u_{1t_2} = u_{1x}m_1 - \frac{1}{2}u_{2xx}m_0 + u_2^3m_0 - u_2u_1^2m_0 - 2u_2u_4u_3m_0 + u_3u_{4x}m_0 + u_4u_{3x}m_0 - 4\sum_{j=1}^{N}\Phi_{1j}\Phi_{2j}, \\ u_{2t_2} = u_{2x}m_1 - \frac{1}{2}u_{1xx}m_0 - u_1^3m_0 + u_1u_2^2m_0 - 2u_1u_4u_3m_0 - u_4u_{4x}m_0 - u_3u_{3x}m_0 + 2\sum_{j=1}^{N}\Phi_{1j}^2 + 2\sum_{j=1}^{N}\Phi_{2j}^2, \\ u_{3t_2} = u_{4xx}m_0 + u_{3x}m_1 - \frac{1}{2}u_4u_2^2m_0 - \frac{1}{2}u_4u_1^2m_0 - \frac{1}{2}u_4u_{2x}m_0 - \frac{1}{2}u_3u_{1x}m_0 - u_{4x}u_2m_0 - u_1u_{3x}m_0,\ 2\sum_{j=1}^{N}\Phi_{2j}\Phi_{3j}, \\ u_{4t_2} = u_{3xx}m_0 + u_{4x}m_1 - \frac{1}{2}u_3u_2^2m_0 - \frac{1}{2}u_3u_1^2m_0 + \frac{1}{2}u_3u_{2x}m_0 + \frac{1}{2}u_4u_{1x}m_0 + u_1u_{4x}m_0 + u_2u_{3x}m_0\ 2\sum_{j=1}^{N}\Phi_{1j}\Phi_{3j}. \end{cases} \tag{35}$$

4. Conservation laws for the super integrable equation hierarchy

In the following, we will construct conservation laws of the super integrable equation hierarchy. Introducing two variables

$$F = \frac{\Psi_2}{\Psi_1}, \quad G = \frac{\Psi_3}{\Psi_1}. \tag{36}$$

From Eqs. (9) and (15), we have

$$F_x = \lambda - u_1 - 2u_2F + u_4G - u_3FG - (\lambda + u_1)F^2, \quad G_x = u_4 - u_3F - u_2G - (\lambda + u_1)FG - u_3G^2. \tag{37}$$

Expanding F and G in the power of λ^{-1}

$$F = \sum_{j=0}^{\infty} f_j \lambda^{-j}, \quad G = \sum_{j=0}^{\infty} g_j \lambda^{-j}. \tag{38}$$

Substituting Eq. (38) into Eq. (37) and comparing the coefficients of the same power of λ, we obtain

$$\begin{aligned} &f_0 = 1, \quad g_0 = 0, \quad f_1 = -u_1 - u_2, \quad g_1 = u_4 - u_3, \quad f_2 = \frac{1}{2}(u_1^2 + u_2^2 + u_{1x} + u_{2x}) + u_1 u_2, \\ &g_2 = u_1 u_3 + u_2 u_3 + u_{3x} - u_{4x}, \end{aligned} \tag{39}$$

and the recursion formulas for f_n and g_n are given

$$\begin{aligned} &f_0 = 1, \quad f_{n+1} = \tfrac{1}{2}\left(-f_{n,x} + 2u_2 f_n + u_4 g_n - u_3 \sum_{i=0}^{n} f_i g_{n-i} - u_1 \sum_{i=0}^{n} f_i f_{n-i} - \sum_{i=1}^{n} f_i f_{n+1-i}\right), \quad n = 0,1,2,\ldots, \\ &g_0 = 0, \quad g_{n+1} = -g_{n,x} - u_3 f_n - u_2 g_n - u_3 \sum_{i=0}^{n} g_i g_{n-i} - u_1 \sum_{i=0}^{n} f_i g_{n-i} - \sum_{i=1}^{n} f_i g_{n+1-i}, \quad n = 0,1,2,\ldots \end{aligned} \tag{40}$$

It is easy to calculate that

$$\frac{\partial}{\partial t}(u_2 + (\lambda + u_1)F + u_3 G) = \frac{\partial}{\partial x}(c + (a+b)F + dG), \tag{41}$$

which is derived from

$$\frac{\partial}{\partial t}\frac{\phi_{1,x}}{\phi_1} = \frac{\partial}{\partial x}\frac{\phi_{1,t}}{\phi_1}, \tag{42}$$

where

$$\begin{aligned} &a = \lambda^2 m_0 + \lambda m_1 + \left(-\tfrac{1}{2}u_2^2 + \tfrac{1}{2}u_1^2 + u_4 u_3\right)m_0, \quad b = \lambda u_1 m_0 - \tfrac{1}{2}u_{2x} m_0 + u_1 m_1, \\ &c = \lambda u_2 m_0 - \tfrac{1}{2}u_{1x} m_0 + u_2 m_1, \quad d = \lambda u_3 m_0 + u_{4x} m_0 + u_3 m_1. \end{aligned} \tag{43}$$

In order to obtain the conservation laws for super integrable hierarchy, we difine

$$\sigma = u_2 + (\lambda + u_1)F + u_3 G, \theta = c + (a+b)F + dG. \tag{44}$$

Then the Eq. (41) can be rewritten as $\sigma_t = \theta_x$, which is just the formal definition of conservation laws. We expand σ and θ as series in powers of λ with the coefficients, which are called conserved densities and fluxes respectively

$$\sigma = \lambda + \sum_{j=0}^{\infty} \sigma_j \lambda^{-j}, \quad \theta = m_0 \lambda^2 + m_1 \lambda + \sum_{j=0}^{\infty} \theta_j \lambda^{-j}, \tag{45}$$

where m_0, m_1 are constants of integration.

With the help of Eqs. (41), (43), (45), the recursion relation for σ_n and θ_n are given

$$\begin{aligned} &\sigma_n = f_{n+1} + u_1 f_n + u_3 g_n, \quad n = 0,1,2,\ldots, \\ &\theta_n = m_0\left[\left(-\tfrac{1}{2}u_2^2 + \tfrac{1}{2}u_1^2 + u_4 u_3 - \tfrac{1}{2}u_{2x}\right)f_n + u_1 f_{n+1} + f_{n+2} + u_{4x} g_n + u_3 g_{n+1}\right] + m_1(u_1 f_n + f_{n+1} + u_3 g_n), \quad n = 0,1,2,\ldots, \end{aligned} \tag{46}$$

where f_n and g_n can be calculated from Eq. (40).

The first two conserved densities and fluxes are read

$$\begin{aligned} &\sigma_0 = 0, \\ &\theta_0 = m_0(u_4 u_3 - u_1 u_3 - u_2 u_3 + u_{4x}) + m_1 u_3, \\ &\sigma_1 = \tfrac{1}{2}(u_2^2 - u_1^2) + \tfrac{1}{2}(u_{1x} + u_{2x}) + u_3 u_4, \\ &\theta_1 = m_0\left(-\tfrac{1}{2}u_1^3 - u_1 u_4 u_3 + u_1 u_{2x} + \tfrac{1}{2}u_2^3 - \tfrac{1}{2}u_1^3 u_2 - u_2 u_4 u_3 + \tfrac{1}{2}u_1 u_2^2 - \tfrac{1}{2}u_2 u_{1x} + \tfrac{1}{2}u_4 u_{3x} - \tfrac{3}{2}u_4 u_{4x} - \tfrac{1}{4}u_{1xx} - \tfrac{1}{2}u_{2xx} \right. \\ &\qquad \left. + \tfrac{1}{2}u_3 u_{3x} + \tfrac{1}{2}u_3 u_{4x}\right) + m_1\left(-\tfrac{1}{2}u_1^2 + \tfrac{1}{2}u_2^2 + \tfrac{1}{2}u_{1x} + \tfrac{1}{2}u_{2x} + u_3 u_4\right). \end{aligned} \tag{47}$$

The infinitely many conservation laws of Eq. (22) can be easily obtained into Eqs. (36)–(47), respectively.

5. Conclusions

Finding the integrable couplings of integrable systems is always an important part in soliton theory. With the help of proper Lie superalgebra and supertrace identity, we can derive some significative super integrable equation hierarchies as well as their Hamiltonian structures and bi-Hamiltonian structures. Based on these super integrable equation hierarchies, the integrable couplings of the self-consistent sources can be obtained. In our work, a super integrable equation hierarchy with self-consistent sources is obtained based on this idea. The conservation laws of the super integrable equation hierarchy are also obtained. It is important to note that the coupling terms of super integrable hierarchies involve fermi variables. In other words, the parameters in the coupling terms are fermi variables which is different from the ordinary one.

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