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Bäcklund Transformations and Solutions of a Generalized Kadomtsev–Petviashvili Equation*

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Abstract *In this paper, the bilinear form of a generalized Kadomtsev–Petviashvili equation is obtained by appling the binary Bell polynomials. The N-soliton solution and one periodic wave solution are presented by use of the Hirota direct method and the Riemann theta function, respectively. And then the asymptotic analysis demonstrates one periodic wave solution can be reduced to one soliton solution. In the end, the bilinear Bäcklund transformations are derived.*

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1 Introduction

As is well known, seeking for solutions of the nonlinear evolution equations (NLEEs) has always been an interesting problem in soliton theory. At present, a variety of powerful methods, such as the inverse scattering transformation,[1−2] Bäcklund transformation,[3−4] Painlevé analysis,[5−6] Darboux transformation,[3,7−8] Hirota direct method[9−11] and rational expansion method[12] have been proposed. The Hirota direct method was originally proposed by R. Hirota, which has been successful applied to construct soliton solution, Bäcklund transformation and Lax pair for some NLEEs. Furthermore, the Hirota direct method has also been developed to obtain explicit periodic wave solutions based on the Riemann theta functions.[13−18] Hence, how to transform NLEEs into the bilinear forms is of the utmost importance. Lately, Lambert *et al.*[19−20] proposed a lucid approach to construct the bilinear form, bilinear Bäcklund transformation and Lax pair for the constant coefficients NLEEs. Afterwards, Fan extended this method to variable coefficients NLEEs and supersymmetric equation.[21−22]

It is well known that (2+1)-dimensional Kadomtsev–Petviashvili equation can be used to model water waves of long wavelength with weakly nonlinear restoring forces and frequency dispersion. Various of generalization of Kadomtsev–Petviashvili equation are proposed,[22−24] in present paper, with the help of symbolic computation, we will focus on the following generalized Kadomtsev–Petviashvili equation

$$u_{tx} + u_{4x} + 6uu_{2x} + 6u_x^2 + 3u_{2y} + 3u_{xy} = 0\,, \qquad (1)$$

which can be transformed to the standard Kadomtsev–Petviashvili equation under transformation

$$x \to x, \quad y \to y - 3t, \quad t \to t\,. \qquad (2)$$

The organization of this paper is as follows. In Sec. 2, we give a brief introduction about the binary Bell polynomial. In Sec. 3, first of all, we obtain the bilinear form of Eq. (1) by appling binary Bell polynomial. Then the N-soliton solution and periodic wave solution are presented by use of the Hirota direct method and the Riemann theta function, respectively. The asymptotic analysis demonstrates one periodic wave solution can be reduced to one soliton solution. In Sec. 4, Bilinear Bäcklund transformation are obtained in a quick and natural manner. Finally, some conclusions are given in Sec. 5.

2 Binary Bell Polynomial

Seeking for the bilinear form of a given nonlinear equation is generally recognized as the first step by appling the Hirota direct method. Lambert *et al.* linked the Bell polynomials to Hirota D operator, proposed a lucid approach to construct the bilinear form and bilinear Bäcklund transformation for the NNEEs.[19−22,25−28] In the following, we will briefly introduce the necessary notations of the Bell polynomials.

The multi-dimensional binary Bell polynomials are defined as the following

$$\begin{aligned} &Y_{n_1x_1,\ldots,n_lx_l}(f) \equiv Y_{n_1,\ldots,n_l}(f_{r_1x_1,\ldots,r_lx_l}) = \mathrm{e}^{-f}\partial_{x_1}^{n_1}\ldots\partial_{x_l}^{n_l}\,\mathrm{e}^{f}\,,\\ &\mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(v,w) \equiv Y_{n_1,\ldots,n_l}(f)\Big|_{f_{r_1x_1,\ldots,r_lx_l}=\begin{cases} v_{r_1x_1,\ldots,r_lx_l}\,, & r_1+\cdots+r_l \text{ is odd}\,,\\ w_{r_1x_1,\ldots,r_lx_l}\,, & r_1+\cdots+r_l \text{ is even}\,,\end{cases}} \end{aligned} \qquad (3)$$

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with $f_{r_1x_1,\ldots,r_lx_l}=\partial_{x_1}^{n_1}\ldots\partial_{x_l}^{n_l}f$, $r_1=0,\ldots,n_1;\ldots,r_l=0,\ldots,n_l$.

The first few lowest order binary Bell polynomials are

$$\mathcal{Y}_x(v)=v_x\,,\quad \mathcal{Y}_{2x}(v,w)=w_{2x}+v_x^2\,,$$
$$\mathcal{Y}_{x,y}(v,w)=w_{x,y}+v_xv_y\,,$$
$$\mathcal{Y}_{3x}(v,w)=v_{3x}+3v_xw_{2x}+v_x^3\,,\ldots \tag{4}$$

The link between $\mathcal{Y}$-polynomials and the standard Hirota expressions can be given by the identity

$$\mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(v=\ln F/G,w=\ln FG)$$
$$=(FG)^{-1}D_{x_1}^{n_1}\ldots D_{x_l}^{n_l}F\cdot G\,. \tag{5}$$

When $F=G$, the formula (5) becomes

$$F^{-2}D_{x_1}^{n_1}\ldots D_{x_l}^{n_l}F\cdot F=\mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(0,q=2\ln F)$$
$$=\begin{cases}0\,, & n_1+\cdots+n_l \text{ is odd}\,,\\ P_{n_1x_1,\ldots,n_lx_l}(q)\,, & n_1+\cdots+n_l \text{ is even}\,,\end{cases} \tag{6}$$

in which the P-polynomials can be characterized by an equally recognizable even part partitional structure

$$P_{2x}(q)=q_{2x},\quad P_{x,t}(q)=q_{xt},\quad P_{4x}(q)=q_{4x}+3q_{2x}^2,$$
$$P_{3x,y}(q)=q_{3xy}+3q_{xy}q_{2x},\ldots \tag{7}$$

3 Bilinear Form and Solutions

Equation (1) is invariant under the scale transformation

$$x\to\lambda x\,,\quad y\to\lambda^3y\,,\quad t\to\lambda^3t\,,\quad u\to\lambda^{-2}u\,. \tag{8}$$

Thus, a dimensionless field q can be introduced by setting

$$u=q_{2x}. \tag{9}$$

Substituting (9) into (1) and integrating it twice with respect to x yields

$$q_{x,t}+q_{4x}+3cq_{2x}^2+3q_{2y}+3q_{x,y}-\varsigma=0\,, \tag{10}$$

where ς is an integral constant. Then, in terms of P-polynomials (7), Eq. (10) can be written as

$$E(q)=P_{x,t}(q)+P_{4x}(q)+3P_{2y}(q)+3P_{x,y}(q)-\varsigma=0\,. \tag{11}$$

Introducing a change of dependent variable

$$q=2\ln F\Leftrightarrow u=q_{2x}=2(\ln F)_{2x}, \tag{12}$$

we get the bilinear representation of Eq. (1) as follows

$$G(D_x,D_y,D_t)\equiv(D_xD_t+D_x^4+3D_y^2+3D_xD_y)F\cdot F$$
$$-\varsigma F^2=0, \tag{13}$$

where Hirota D operator is defined as

$$D_x^mD_y^sD_t^nF(x,y,t)\cdot G(x',y',t')$$
$$=(\partial_x-\partial_{x'})^m(\partial_y-\partial_{y'})^s(\partial_t-\partial_{t'})^nF(x,y,t)$$
$$\times\,G(x,y,t)|_{x'=x,y'=y,t'=t}. \tag{14}$$

The D-operators have good property when acting on exponential functions,[29–32] which is used for seeking for the bilinear Bäcklund transformation of NLEEs

$$D_x^mD_y^sD_t^n\,\mathrm{e}^{\xi_1}\cdot\mathrm{e}^{\xi_2}$$
$$=(\kappa_1-\kappa_2)^m(\iota_1-\iota_2)^s(\omega_1-\omega_2)^n\,\mathrm{e}^{\xi_1+\xi_2}\,, \tag{15}$$

where $\xi_j=\kappa_jx+\iota_jy+\omega_jt+\xi_j^{(0)}$, $j=1,2$.

More generally, we have

$$G(D_x,D_y,D_t)\,\mathrm{e}^{\xi_1}\cdot\mathrm{e}^{\xi_2}$$
$$=G(\kappa_1-\kappa_2,\iota_1-\iota_2,\omega_1-\omega_2)\,\mathrm{e}^{\xi_1+\xi_2}. \tag{16}$$

3.1 *N-Soliton Solutions*

In the following, we will apply the Hirota direct method to construct the N-soliton solutions of Eq. (1) based on bilinear form (13). Assuming $\varsigma=0$ and expanding the function F in powers of a parameter ϵ as

$$F=1+\epsilon f_1+\epsilon^2f_2+\cdots, \tag{17}$$

where $F=F(x,y,t), f_i=f_i(x,t,y), i=1,2,\ldots$

Substituting Eq. (17) into Eq. (13) and setting the coefficients of each power of ϵ to zero. According to Hirota direct method, the one-soliton solution, two-soliton solution, and three-soliton solution are obtained as follows.

(i) One-Soliton Solution

$$F=1+f_1,\quad f_1=\mathrm{e}^{\xi_1},\quad \xi_1=\kappa_1x+\iota_1y+\omega_1t+\xi_1^0,$$
$$\omega_1=-\kappa_1^3-\frac{3\iota_1^2}{\kappa_1}-3\iota_1,\quad f_i=0\,,\quad i=2,3,\ldots \tag{18}$$

Thus we get corresponding one-soliton solution

$$u=2(\ln F)_{2x}=\frac{\kappa_1^2}{2}\mathrm{sech}^2\frac{\xi_1}{2}. \tag{19}$$

(ii) Two-Soliton Solution

$$F=1+f_1+f_2,\quad f_1=\mathrm{e}^{\xi_1}+\mathrm{e}^{\xi_2},\quad f_2=\mathrm{e}^{\xi_1+\xi_2+A_{12}},$$
$$\mathrm{e}^{A_{12}}=\frac{(\kappa_2\iota_1+\kappa_1^2\kappa_2-\kappa_1\kappa_2^2-\kappa_1\iota_2)(-\kappa_2\iota_1+\kappa_1^2\kappa_2-\kappa_1\kappa_2^2+\kappa_1\iota_2)}{(-\kappa_2\iota_1+\kappa_1^2\kappa_2+\kappa_1\kappa_2^2+\kappa_1\iota_2)(\kappa_2\iota_1+\kappa_1^2\kappa_2+\kappa_1\kappa_2^2-\kappa_1\iota_2)},$$
$$\xi_i=\kappa_ix+\iota_iy+\omega_it+\xi_i^0,\quad \omega_i=-\kappa_i^3-\frac{3\iota_i^2}{\kappa_i}-3\iota_i,\quad f_i=0\,,\quad i=3,4,\ldots \tag{20}$$

Corresponding two-soliton solution is

$$u=2(\ln F)_{2x}=2[\ln(1+f_1+f_2)]_{2x}. \tag{21}$$

(iii) Three-Soliton Solution

$$F=1+f_1+f_2+f_3,\quad f_1=\mathrm{e}^{\xi_1}+\mathrm{e}^{\xi_2}+\mathrm{e}^{\xi_3},\quad f_2=\mathrm{e}^{\xi_1+\xi_2+A_{12}}+\mathrm{e}^{\xi_1+\xi_3+A_{13}}+\mathrm{e}^{\xi_2+\xi_3+A_{23}},$$
$$f_3=\mathrm{e}^{\xi_1+\xi_2+\xi_3+A_{12}+A_{13}+A_{23}},\quad \mathrm{e}^{A_{ij}}=\frac{(\kappa_j\iota_i+\kappa_i^2\kappa_j-\kappa_i\kappa_j^2-\kappa_i\iota_j)(-\kappa_j\iota_i+\kappa_i^2\kappa_j-\kappa_i\kappa_j^2+\kappa_i\iota_j)}{(-\kappa_j\iota_i+\kappa_i^2\kappa_j+\kappa_i\kappa_j^2+\kappa_i\iota_j)(\kappa_j\iota_i+\kappa_i^2\kappa_j+\kappa_i\kappa_j^2-\kappa_i\iota_j)},$$
$$\xi_i=\kappa_ix+\iota_iy+\omega_it+\xi_i^0,\quad \omega_i=-\kappa_i^3-\frac{3\iota_i^2}{\kappa_i}-3\iota_i,\quad f_i=0\,,\quad i=4,5,\ldots \tag{22}$$

Corresponding three-soliton solution is

$$u = 2(\ln F)_{2x} = 2[\ln(1 + f_1 + f_2 + f_3)]_{2x}\,. \tag{23}$$

Continuing this process, the N-soliton solution formular are given,[33–34]

$$\mathrm{e}^{A_{ij}} = \frac{(\kappa_j\iota_i + \kappa_i^2\kappa_j - \kappa_i\kappa_j^2 - \kappa_i\iota_j)(-\kappa_j\iota_i + \kappa_i^2\kappa_j - \kappa_i\kappa_j^2 + \kappa_i\iota_j)}{(-\kappa_j\iota_i + \kappa_i^2\kappa_j + \kappa_i\kappa_j^2 + \kappa_i\iota_j)(\kappa_j\iota_i + \kappa_i^2\kappa_j + \kappa_i\kappa_j^2 - \kappa_i\iota_j)}\,, \tag{24}$$

where $\sum_{\mu=0,1}$ indicates the summation over all possible combination of $\mu_j = 0, 1, j = 1, 2, \ldots$

3.2 *Periodic Wave Solutions*

Nakamura presented a direct method to multiperiodic wave solutions for NLEEs by using Riemann theta function.[13,35] Thus, we consider the Riemann theta function solution of Eq. (1)

$$F = \sum_{n=-\infty}^{\infty} \mathrm{e}^{2\pi\mathrm{i}n\xi + \pi\mathrm{i}n^2\tau}\,, \tag{25}$$

where $n \in \mathcal{Z}, \tau \in \mathcal{C}, \mathrm{Im}\tau > 0$, and $\xi = \kappa x + \iota y + \omega t + \delta_0$, with κ, ι, and ω are constants to be determined, δ_0 is a constant.

Inserting (25) into (13), we have

$$\begin{aligned}
GF\cdot F &= G(D_x, D_y, D_t)\sum_{n=-\infty}^{\infty}\mathrm{e}^{2\pi\mathrm{i}n\xi+\pi\mathrm{i}n^2\tau}\sum_{m=-\infty}^{\infty}\mathrm{e}^{2\pi\mathrm{i}m\xi+\pi\mathrm{i}m^2\tau}\\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} G(D_x, D_y, D_t)\,\mathrm{e}^{2\pi\mathrm{i}n\xi+\pi\mathrm{i}n^2\tau}\cdot\mathrm{e}^{2\pi\mathrm{i}m\xi+\pi\mathrm{i}m^2\tau}\\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} G(2\pi\mathrm{i}(n-m)\kappa, 2\pi\mathrm{i}(n-m)\iota, 2\pi\mathrm{i}(n-m)\omega)\,\mathrm{e}^{2\pi\mathrm{i}(n+m)\xi+\pi\mathrm{i}(n^2+m^2)\tau}\\
&= \sum_{p=-\infty}^{\infty}\Big\{\sum_{n=-\infty}^{\infty} G[2\pi\mathrm{i}(2n-p)\kappa, 2\pi\mathrm{i}(2n-p)\iota, 2\pi\mathrm{i}(2n-p)\omega]\,\mathrm{e}^{\pi\mathrm{i}(n^2+(p-n)^2)\tau}\Big\}\,\mathrm{e}^{2\pi\mathrm{i}p\xi} = \sum_{p=-\infty}^{\infty}\bar{G}(p)\,\mathrm{e}^{2\pi\mathrm{i}p\xi}\,.
\end{aligned} \tag{26}$$

Noting that

$$\begin{aligned}
\bar{G}(p) &= \sum_{n=-\infty}^{\infty} G[2\pi\mathrm{i}(2n-p)\kappa, 2\pi\mathrm{i}(2n-p)\iota, 2\pi\mathrm{i}(2n-p)\omega]\,\mathrm{e}^{\pi\mathrm{i}(n^2+(p-n)^2)\tau}\\
&= \sum_{h=-\infty}^{\infty} G[A\kappa, A\iota, A\omega]\,\mathrm{e}^{\pi\mathrm{i}(h^2+(p-h-2)^2)\tau}\cdot\mathrm{e}^{2\pi\mathrm{i}(p-1)\tau} = \bar{G}(p-2)\,\mathrm{e}^{2\pi\mathrm{i}(p-1)\tau}\,,
\end{aligned} \tag{27}$$

where $A = 2\pi\mathrm{i}(2h - (p-2))$, $p = m + n$.

In view of (26) and by induction method, we can get

$$\bar{G}(p) = \begin{cases} \bar{G}(0)\,\mathrm{e}^{\pi\mathrm{i}np\tau}\,, & p = 2n\,,\\ \bar{G}(1)\,\mathrm{e}^{\pi\mathrm{i}(2n+2n^2)(p+1)\tau}\,, & p = 2n+1\,. \end{cases} \tag{28}$$

In this way, we may let

$$\begin{aligned}
\bar{G}(0) &= \sum_{n=-\infty}^{\infty}(-16n^2\pi^2\kappa\omega + 256n^4\pi^4\kappa^4\\
&\quad - 48n^2\pi^2\iota^2 - 48n^2\pi^2\kappa\iota - \varsigma)\,\mathrm{e}^{2\pi\mathrm{i}n^2\tau} = 0,\\
\bar{G}(1) &= \sum_{n=-\infty}^{\infty}(-4(2n-1)^2\pi^2\kappa\omega + 16(2n-1)^4\pi^4\kappa^4\\
&\quad - 12(2n-1)^2\pi^2\iota^2 - 12(2n-1)^2\pi^2\kappa\iota - \varsigma)\\
&\quad\times\,\mathrm{e}^{\pi\mathrm{i}(2n^2-2n+1)\tau} = 0\,.
\end{aligned} \tag{29}$$

For the sake of convenience, if we denote that

$$q_1(n) = \mathrm{e}^{2\pi\mathrm{i}n^2\tau}, \quad q_2(n) = \mathrm{e}^{\pi\mathrm{i}(2n^2-2n+1)\tau},$$

$$a_{11} = -16n^2\pi^2\kappa q_1(n), \quad a_{12} = \sum_{n=-\infty}^{\infty} q_1(n),$$

$$a_{21} = -\sum_{n=-\infty}^{\infty} 4(2n-1)^2\pi^2\kappa q_2(n), \quad a_{22} = \sum_{n=-\infty}^{\infty} q_2(n),$$

$$b_1 = \sum_{n=-\infty}^{\infty}(256n^4\pi^4\kappa^4 - 48n^2\pi^2\iota^2 - 48n^2\pi^2\kappa\iota)q_1(n),$$

$$\begin{aligned}
b_2 &= \sum_{n=-\infty}^{\infty}(16(2n-1)^4\pi^4\kappa^4 - 12(2n-1)^2\pi^2\iota^2\\
&\quad - 12(2n-1)^2\pi^2\kappa\iota)q_2(n)\,,
\end{aligned} \tag{30}$$

then (29) can be written as

$$a_{11}\omega + b_1 - \varsigma a_{12} = 0\,, \quad a_{21}\omega + b_2 - \varsigma a_{22} = 0\,. \tag{31}$$

Solving this system, we obtain

$$\omega = \frac{a_{12}b_2 - b_1a_{22}}{a_{11}a_{22} - a_{21}a_{12}}, \quad \varsigma = \frac{a_{11}b_2 - b_1a_{21}}{a_{11}a_{22} - a_{21}a_{12}}\,. \tag{32}$$

Thus, we obtain the one periodic wave solution

$$u = 2(\ln F)_{2x}, \tag{33}$$

where F and ω are given by (25) and (32), respectively. Figure 1 shows the periodic wave solution (32) for one choice of the parameters.

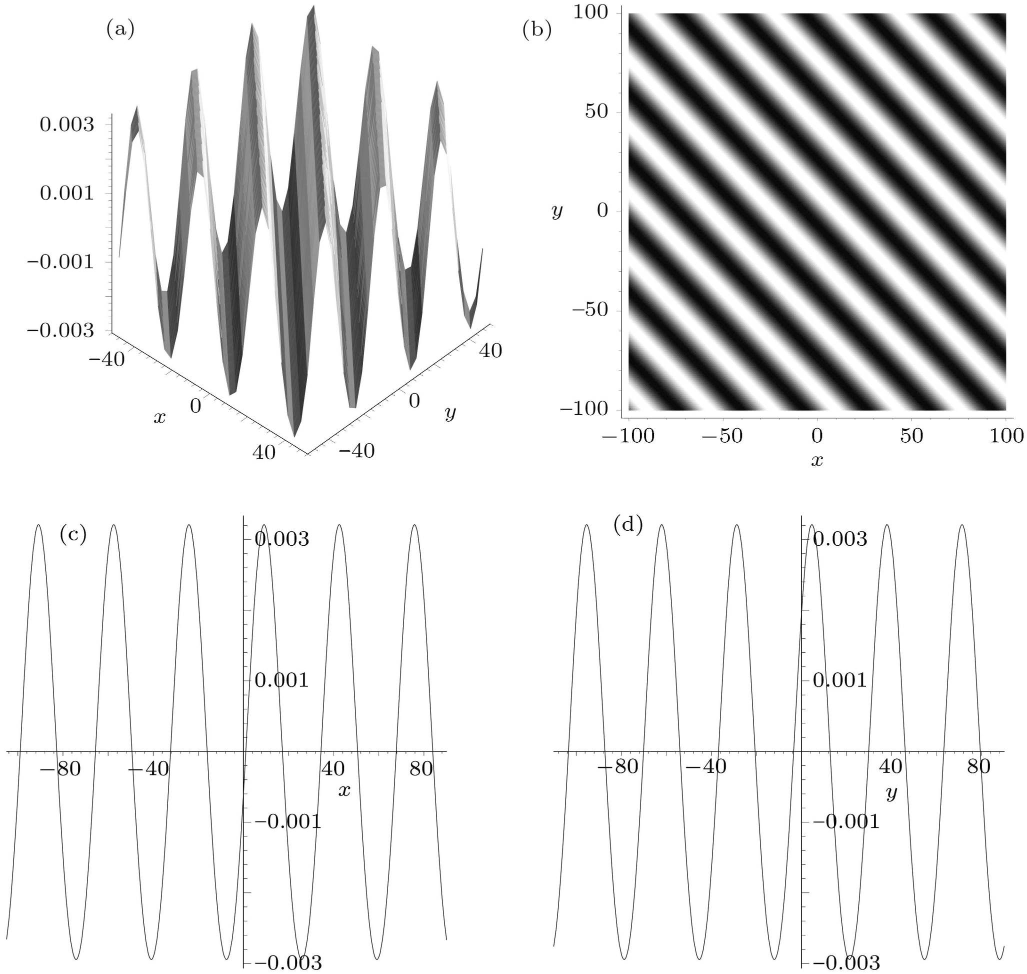

Fig. 1 One periodic wave-(33) of Eq. (1) with parameters: $\delta_0 = 0$, $\kappa = 0.03$, $\tau = i$, $\iota = 0.03$. (a) Perspective view of the wave; (b) Overhead view of the wave, with contour plot shown. The bright lines are crests and the dark lines are troughs; (c) Wave propagation pattern of the wave along the x axis; (d) Wave propagation pattern of the wave along the y axis.

3.3 *Asymptotic Property of Periodic Wave Solution*

We are interested in the asymptotic properties of the periodic wave solutions of Eq. (1). The relation between the periodic wave solution and soliton solution can be established as follows.

System (31) can be rewritten as matrix form

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \omega \\ -\varsigma \end{pmatrix} = \begin{pmatrix} -b_1 \\ -b_2 \end{pmatrix}. \tag{34}$$

Expand the system (34) into power series of γ

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = A_0 + A_1\gamma + A_2\gamma^2 + \cdots,$$
$$\begin{pmatrix} -b_1 \\ -b_2 \end{pmatrix} = B_0 + B_1\gamma + B_2\gamma^2 + \cdots,$$
$$\begin{pmatrix} \omega \\ -\varsigma \end{pmatrix} = X_0 + X_1\gamma + X_2\gamma^2 + \cdots, \tag{35}$$

where $\gamma = \mathrm{e}^{\pi \mathrm{i} \tau}$.

Theorem 1 Suppose that the vector $(\omega, -\varsigma)^{\mathrm{T}}$ is a solution of the system (34), and for the periodic solution wave solution, we let

$$\kappa = \frac{\kappa_1}{2\pi \mathrm{i}}, \quad \iota = \frac{\iota_1}{2\pi \mathrm{i}}, \quad \delta_0 = \frac{\xi_1^0 - \pi \mathrm{i} \tau}{2\pi \mathrm{i}},$$

where κ_1, ι_1 are the same as those in Eq. (19).

Then we have the following asympotic properties:

$$\varsigma \to 0, \quad \xi \to \frac{\xi_1 - \pi \mathrm{i} \tau}{2\pi \mathrm{i}},$$
$$F \to 1 + \mathrm{e}^{\xi_1}, \quad \text{as} \quad \gamma \to 0. \tag{37}$$

In other words, the one periodic wave solution (33) tends to the one-soliton solution (19) as $\gamma \to 0$.

Proof By using Eq. (30), we write functions a_{ij}, b_j, $i, j = 1, 2$ as the series about γ

$$a_{11} = -32\pi^2\kappa(\gamma^2 + 4\gamma^4 + \cdots),$$
$$a_{12} = 1 + 2\gamma^2 + 2\gamma^8 + \cdots,$$
$$a_{21} = -8\pi^2\kappa(\gamma + 9\gamma^5 + \cdots),$$
$$a_{22} = 2\gamma + 2\gamma^5 + \cdots,$$
$$b_1 = (256\pi^4\kappa^4 - 48\pi^2\iota^2 - 48\pi^2\kappa\iota)\gamma^2 + \cdots,$$
$$b_2 = 2(16\pi^4\kappa^4 - 12\pi^2\iota^2 - 12\pi^2\kappa\iota)\gamma + 2(16 \cdot 3^4\pi^4\kappa^4 - 12 \cdot 3^2\pi^2\iota^2 - 12 \cdot 3^2\pi^2\kappa\iota)\gamma^5 + \cdots \tag{38}$$

Thus, using Eq. (35), we have

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ -8\pi^2\kappa & 2 \end{pmatrix},$$
$$A_2 = \begin{pmatrix} -32\pi^2\kappa & 2 \\ 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
$$A_4 = \begin{pmatrix} -128\pi^2\kappa & 0 \\ 0 & 0 \end{pmatrix}, \ldots, \tag{39}$$
$$B_1 = \begin{pmatrix} 0 \\ -32\pi^4\kappa^4 + 24\pi^2\iota^2 + 24\pi^2\kappa\iota \end{pmatrix},$$
$$B_0 = B_3 = B_4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$B_2 = \begin{pmatrix} -256\pi^4\kappa^4 + 48\pi^2\iota^2 + 48\pi^2\kappa\iota \\ 0 \end{pmatrix}, \ldots \tag{40}$$

Then we obtain

$$X_0 = \begin{pmatrix} 4\kappa^3\pi^2 - 3\iota^2/\kappa - 3\iota \\ 0 \end{pmatrix},$$
$$X_2 = \begin{pmatrix} 32\kappa^3\pi^2 - 24\iota^2/\kappa - 24\iota \\ 128\pi^4\kappa^4 - 96\pi^2\iota^2 - 96\pi^2\kappa\iota \end{pmatrix},$$
$$X_1 = X_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots, \tag{41}$$

and thus

$$\varsigma = (-128\pi^4\kappa^4 + 96\pi^2\iota^2 + 96\pi^2\kappa\iota)\gamma^2 + o(\gamma^2),$$
$$\omega = 4\kappa^3\pi^2 - \frac{3\iota^2}{\kappa} - 3\iota + \Big(32\kappa^3\pi^2 - \frac{24\iota^2}{\kappa} - 24\iota\Big)\gamma^2 + o(\gamma^2), \tag{42}$$

which exactly implies by using relation (34) that

$$\varsigma \to 0,$$
$$2\pi \mathrm{i}\omega = \omega_1 \to 8\pi^3\kappa^3\mathrm{i} - \frac{6\pi\iota^2\mathrm{i}}{\kappa} - 6\pi\iota\mathrm{i} = -\kappa_1^3 - \frac{3l_1^2}{\kappa_1} - 3l_1,$$
$$\text{as} \quad \gamma \to 0. \tag{43}$$

It remains to show that the periodic wave (33) possesses the same form with the soliton solution (19) under the limit $\gamma \to 0$. For this purpose, we expand the periodic functions F as

$$F = 1 + \gamma(\mathrm{e}^{2\pi\mathrm{i}\xi} + \mathrm{e}^{-2\pi\mathrm{i}\xi}) + \gamma^4(\mathrm{e}^{4\pi\mathrm{i}\xi} + \mathrm{e}^{-4\pi\mathrm{i}\xi}) + \cdots$$
$$= 1 + \mathrm{e}^{\xi_1} + \gamma^2(\mathrm{e}^{-\xi_1} + \mathrm{e}^{2\xi_1}) + \gamma^6(\mathrm{e}^{-2\xi_1} + \mathrm{e}^{3\xi_1}) + \cdots$$
$$\to 1 + \mathrm{e}^{\xi_1}, \quad \text{as} \quad \gamma \to 0, \tag{44}$$

where $\xi_1 = 2\pi\mathrm{i}\xi + \pi\mathrm{i}\tau = \kappa_1 x + \iota_1 y + \omega_1 t + \xi_1^0$.

Therefore we conclude that the periodic solution just be reduced to the one-soliton solution as $\gamma \to 0$.

4 Two Kinds of Bäcklund Transformation

Bäcklund transformation can be used to derive new solutions from known solutions, especially for the bilinear Bäcklund transformation, it can be transformed into a new bilinear Bäcklund transformation. Binary Bell polynomials provide us a new approach to obtain the bilinear BT for the NNEE. In order to obtain the bilinear Bäcklund transformation of the Eq. (1), taking q and q' as two different solutions of the Eq. (10) and introduce two new variables.

$$v = \frac{q' - q}{2}, \quad w = \frac{q' + q}{2}. \tag{45}$$

We get

$$E(q') - E(q) = E(w + v) - E(w - v)$$
$$= v_{xt} + v_{4x} + 6v2xw_{2x} + 3v_{xy} + 3v_{2y}$$
$$= 2\partial_x[\mathcal{Y}_t(v) + \mathcal{Y}_{3x}(v, w)] + \mathcal{R}(v, w), \tag{46}$$

where

$$\mathcal{R}(v, w) = 3v_{2x}w_{2x} - 3v_x w_{3x} - 3v_x^2 v_{2x} + 3v_{xy} + 3v_{2y}. \tag{47}$$

If we could search for such a constraint that $\mathcal{R}(v, w)$ as the x-derivative of a combination of $\mathcal{Y}$-polynomials, the two-field condition (46) can be decomposed into a pair of constraints. The simplest possible choice of such constraint may be

$$\mathcal{Y}_y(v, w) + \mathcal{Y}_{2x}(v, w) = \lambda, \tag{48}$$

where λ is a constant.

Then, we deduce a coupled system of $\mathcal{Y}$-polynomials

$$\mathcal{Y}_y(v, w) + \mathcal{Y}_{2x}(v, w) = \lambda,$$
$$\partial_x[\mathcal{Y}_t(v) + \mathcal{Y}_{3x}(v, w) + 3\mathcal{Y}_y(v, w) + 3\lambda\mathcal{Y}_x(v, w) - 3\mathcal{Y}_{x,y}(v, w)] = 0. \tag{49}$$

Naturally, Eq. (49) leads to the bilinear Bäcklund transformation

$$(D_y + D_x^2)F \cdot G = \lambda F \cdot G,$$
$$(D_t + D_x^3 + 3D_y - 3D_xD_y + 3\lambda D_x)F \cdot G = 0. \tag{50}$$

In the following, a new Bäcklund transformation are obtained based on Eq. (50). If we take $\lambda = 0$, Eq. (50) reduce to

$$(D_y + D_x^2)F \cdot G = 0,$$
$$(D_t + D_x^3 + 3D_y - 3D_xD_y)F \cdot G = 0. \tag{51}$$

To derive a new Bäcklund transformation, we make

$$F \to \mathrm{e}^{\xi}F, \quad G \to \mathrm{e}^{\eta}G, \tag{52}$$

where $\xi = w_1x + k_1y + l_1t + \xi_1^0, \eta = w_2x + k_2y + l_2t + \xi_2^0$.

Using the bilinear operator identity

$$D_t^m D_x^n D_y^s \mathrm{e}^{\xi}F \cdot \mathrm{e}^{\eta}G = \mathrm{e}^{\xi+\eta}(D_t + l_1 - l_2)^m(D_x + w_1 - w_2)^n \times (D_y + k_1 - k_2)^s F \cdot G. \tag{53}$$

Equations (51) is transformed into

$$[(D_y + k_1 - k_2) + (D_x + w_1 - w_2)^2]F \cdot G = 0,$$
$$[(D_t + l_1 - l_2) + (D_x + w_1 - w_2)^3 + 3(D_y + k_1 - k_2) - 3(D_x + w_1 - w_2) \times (D_y + k_1 - k_2)]F \cdot G = 0. \tag{54}$$

Equating the constant terms zero, we obtain

$$w_1 - w_2 = \gamma, \quad k_2 - k_1 = (w_1 - w_2)^2,$$
$$l_1 - l_2 = 3\gamma^2 - 4\gamma^3. \tag{55}$$

Then, a new Bäcklund transformation is yielded as follows

$$(D_y + D_x^2 + 2\gamma D_x)F \cdot G = 0,$$
$$[D_t + D_x^3 + 3(1 - \gamma)D_y - 3D_xD_y + 3\lambda D_x^2 + 6\gamma^2 D_x]F \cdot G = 0, \tag{56}$$

where γ is an arbitrary parameter.

5 Conclusions

In this paper, we investigate the generalized Kadomtsev–Petviashvili equation (1) and obtain its bilinear form, N-soliton solutions, one periodic solution, and bilinear Bäcklund transformation. As mentioned above, Eq. (1) can be transformed to the standard Kadomtsev–Petviashvili equation under transformation (2), this means its bilinear form, soliton solutions, periodic solutions, and bilinear Bäcklund transformation should be related to those of the standard Kadomtsev–Petviashvili equation. In fact, it can be observed that if we taking $\omega_1 = -\kappa_1^3 - 3\iota_1^2/\kappa_1$, the soliton solutions (19), (21), (23), and (24) are just the soliton solutions of the standard Kadomtsev–Petviashvili equation.[34] In Refs. [32,36], based on the bilinear Bäcklund transformation of the standard Kadomtsev–Petviashvili equation, multi-soliton solutions and Wronskian solutions are obtained, respectively. This means that multi-soliton solutions and Wronskian solutions of the Eq. (1) can also be obtained based on the idea of Refs. [32,36].

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