Communications in Theoretical Physics

PAPER

Classification of Dark Modified KdV Equation

To cite this article: Na Xiong *et al* 2017 *Commun. Theor. Phys.* **68** 13

Related content

Commun. Theor. Phys. **68** (2017) 13–20 Vol. 68, No. 1, July 1, 2017

Classification of Dark Modified KdV Equation*

Na Xiong (熊娜),[1] Sen-Yue Lou (楼森岳),[1,2,†] Biao Li (李彪),[2] and Yong Chen (陈勇)[1,‡]

[1]Shanghai Key Laboratory of Trustworthy Computing, East China Normal University, Shanghai 200062, China

[2]Ningbo Collabrative Innovation Center of Nonlinear Harzard System of Ocean and Atmosphere, and Faculty of Science, Ningbo University, Ningbo 315211, China

(Received March 27, 2017)

Abstract *The dark Korteweg-de Vries (KdV) systems are defined and classified by Kupershmidt sixteen years ago. However, there is no other classifications for other kinds of nonlinear systems. In this paper, a complete scalar classification for dark modified KdV (MKdV) systems is obtained by requiring the existence of higher order differential polynomial symmetries. Different to the nine classes of the dark KdV case, there exist twelve independent classes of the dark MKdV equations. Furthermore, for the every class of dark MKdV system, there is a free parameter. Only for a fixed parameter, the dark MKdV can be related to dark KdV via suitable Miura transformation. The recursion operators of two classes of dark MKdV systems are also given.*

PACS numbers: 02.30.Ik, 02.30.Jr, 05.45.Yv, 11.10.Lm **DOI:** 10.1088/0253-6102/68/1/13

Key words: dark equations, the modified KdV equation, recursion operator

1 Introduction

In the during past decades, the mystery of dark matter[1−2] has attracted much attention, which is only a canonical result, emanating from nucleosynthesis considerations in the first minutes of the Big Bang. At the same time, the elegant discovery that our universe is in accelerating expansion, which has inspired a large number of studies on dark energy[3−5] introduced by cosmologists. The similar concept of dark equations is first proposed by Kupershmidt.[6] The basic idea of Ref. [6] is as follows:

Given two commuting dynamical systems X and Y:

$$(X) \quad u_t = X(u)\,, \tag{1a}$$

$$(Y) \quad u_t = Y(u)\,, \tag{1b}$$

find/classify all their linear extensions $X^{\rm ext}$ and $Y^{\rm ext}$ of the form:

$$(X^{\rm ext}) \quad u_t = X(u), \quad v_t = \hat{A}_1(u)v\,, \tag{2a}$$

$$(Y^{\rm ext}) \quad u_t = Y(u), \quad v_t = \hat{A}_2(u)v\,, \tag{2b}$$

which still commute, here v is a vector, and $\hat{A}_j(u)$ $(j = 1,2)$ are matrix linear differential operators only depend upon u, not v.

According to commutativity

$$[X,Y] = 0, \quad [X^{\rm ext}, Y^{\rm ext}] = 0\,, \tag{3}$$

dark equations (i.e. linear extension equation) are obtained. These equations are similar in spirit to what one gets when linearizing a given system, or studies how an external linear wave interacts with a particular solution of a given system.

Actually, the linear extensions $X^{\rm ext}$ and $Y^{\rm ext}$ can be regarded as some kinds of integrable coupling.[7] Integrable coupling with different dynamical systems have been explored by many researchers, such as Ablowitz–Kaup–Newell–Segur (AKNS) spectral problem,[8−9] Kaup–Newell (KN) spectral problem,[10−11] Wadati–Konno–Ichikawa (WKI) spectral problem[12] and Kadomtsev–Petviashvili (KP) equation.[13]

According to the definition of the dark equation (2), it is interesting to give a complete classification for a given integrable model (1). For the KdV equation, a complete scalar classification is given by Kupershmidt.[6] It is also interesting most of the dark equations are closely related to the physically meaningful models such as the symmetry equation of the KdV equation, the time part of the Lax pair, the bosonization of super-KdV,[14] the bosonization of supersymmetric bosonic superfield, the bosonization of Manin-Radul supersymmetric fermionic superfield,[15−16] and their dual systems. In addition to the models related to known physically meaningful models, there still exists a mysterious one and its dual.

Though the dark KdV equations have been classified sixteen years ago, there is no classifications for other kind of integrable systems. In this paper, we try to give out a complete classification for the dark MKdV equations in one component extensions by using the definition of high-order symmetry. It is found that the classifications of dark

*Supported by the Global Change Research Program of China under Grant No. 2015Cb953904, National Natural Science Foundation of China under Grant Nos. 11675054, 11435005, 11175092, and 11205092 and Shanghai Knowledge Service Platform for Trustworthy Internet of Things (No. ZF1213) and K. C. Wong Magna Fund in Ningbo University

†Corresponding author, E-mail: lousenyue@nbu.edu.cn

‡Corresponding author, E-mail: ychen@sei.ecnu.edu.cn

MKdV systems are quite different from those of the dark KdV equations, there will be not only three more classes of the dark MKdV systems but also more freedoms for every class of the dark MKdV system.

The dark equation may be integrable not only because its linearity but also the existence of higher order symmetries which usually implies the possible existence of infinitely many higher order symmetries. To guarantee the existence of infinitely many higher order symmetries, it is enough to find a recursion operator or a master symmetry. In this paper, the recursion operators of two classes of the dark MKdV equations have been arrived at.

The outline of this paper is as follows. In Sec. 2, twelve classes of the dark MKdV equations are derived. Section 3 is devoted to construct the recursion operators of two classes dark equations. Some conclusions and discussions are given in the last section.

2 Dark Equations of the MKdV Equation

In fact, the Kupershmidt's definition on dark equations means that both the original model (1a) and its extension (2a) are integrable under the meaning that they possess higher order symmetries (1b) and (2b) respectively.

For the MKdV equation

$$u_t = (2u^3 + u_{xx})_x\,, \tag{4}$$

its symmetry equation reads

$$\sigma_t^u = 6(u^2\sigma^u)_x + \sigma_{xxx}^u\,. \tag{5}$$

It is known that

$$\sigma^u = u_{5x} + 10u^2u_{xxx} + 40uu_xu_{xx} + 10u_x^3 + 30u^4u_x \tag{6}$$

is a fifth order symmetry of the MKdV (4).

Inspired by the main idea in Ref. [6] and based on the theory of high order symmetry, we look for the linear scalar homogeneous extensions of the MKdV equation (4) with the form

$$\begin{aligned} v_t = {}& a_1v_{xxx} + a_2uv_{xx} + (a_3u_x + a_4u^2)v_x \\ & + (a_5u_{xx} + a_6uu_x + a_7u^3)v\,, \end{aligned} \tag{7}$$

and its fifth order symmetry

$$\begin{aligned} \sigma^v = {}& b_1v_{5x}+b_2uv_{4x}+\big(b_3u_x+b_4u^2\big)v_{xxx}+\big(b_5u_{xx}+b_6uu_x+b_7u^3\big)v_{xx}+\big(b_8u_{xxx}+b_9uu_{xx}+b_{10}u_x^2 \\ & +b_{11}u^2u_x+b_{12}u^4\big)v_x+\big(b_{13}u_{4x}+b_{14}uu_{xxx}+b_{15}u_xu_{xx}+b_{16}u^2u_{xx}+b_{17}uu_x^2 + b_{18}u^5\big)v\,, \end{aligned} \tag{8}$$

under the condition that the field u possesses unchanged symmetry (6).

The combination of Eq. (4) and Eq. (7) is called the dark MKdV system (the one component integrable coupling system).

The only thing we need to do is to select out all independent versions of Eq. (7) by fixing a_i $(i = 1, \ldots, 7)$, b_j $(j = 1, \ldots, 19)$ with requiring σ^v satisfies the following symmetry equation

$$\begin{aligned} \sigma_t^v = {}& a_1\sigma_{xxx}^v + a_2\big(u\sigma_{xx}^v + v_{xx}\sigma^u\big) + a_3u\big(u\sigma_x^v + 2v_x\sigma^u\big) + a_4\big(u_x\sigma_x^v + v_x\sigma_x^u\big) \\ & + a_5\big(u_{xx}\sigma^v + v\sigma_{xx}^u\big) + a_6\big[uu_x\sigma^v + v\big(u\sigma^u\big)_x\big] + a_7u^2\big(u\sigma^v + 3v\sigma^u\big)\,, \end{aligned} \tag{9}$$

while σ^u is given by Eq. (6).

Substituting (8) and (6) with (4) and (7) into (9) yields the following 58 determining equations for 26 constants, a_i $(i = 1, \ldots, 7)$ and b_j $(j = 1, \ldots, 19)$,

$$\begin{aligned}
& 3a_1b_2 - 5a_2b_1 = 0, \quad 6a_1b_3 - 2a_2b_2 - 10a_3b_1 = 0, \quad a_1b_{17} - a_5b_1 + a_5 - b_{17} = 0\,, \\
& 5a_3b_{13} - 3a_7b_8 + 90a_7 - 30b_{13} = 0, \quad 3a_1b_2 + 3a_1b_4 - 10a_2b_1 - 5a_4b_1 = 0\,, \\
& 10a_2b_{13} + 2a_3b_8 - 6a_7b_5 + 60a_3 - 24b_8 = 0, \quad a_1b_{16} + 3a_1b_{19} - a_5b_{11} - 10a_6b_1 - b_{16} = 0\,, \\
& 3a_1b_{16} + 4a_1b_{19} - a_5b_7 - 15a_6b_1 - b_{19} = 0, \quad 9a_1b_5 + a_2b_3 - 7a_3b_2 - 15a_7b_1 - 6b_2 = 0\,, \\
& 12a_1b_8 + 4a_2b_5 - 4a_3b_3 - 12a_7b_2 - 12b_3 = 0, \quad a_1b_{11} + 3a_1b_{17} - a_4b_1 - 5a_5b_1 + a_4 - b_{11} = 0\,, \\
& 15a_1b_{13} + 7a_2b_8 - a_3b_5 - 9a_7b_3 + 30a_2 - 18b_5 = 0\,, \\
& a_1b_{16} + a_2b_{17} - a_5b_2 - a_6b_1 + a_6 - b_{16} = 0\,, \\
& 6a_1b_3 + 3a_1b_6 - 3a_2b_4 - 20a_3b_1 + a_4b_2 - 5a_6b_1 = 0\,, \\
& 3a_1b_{11} + a_1b_4 + 3a_1b_7 - 5a_2b_1 - 10a_4b_1 - 10a_5b_1 - b_4 = 0\,, \\
& a_1b_2 + 3a_1b_4 + 3a_1b_7 - 10a_2b_1 - 10a_4b_1 - 5a_5b_1 - b_2 = 0\,, \\
& 3a_1b_{16} + a_1b_{19} + a_4b_{17} - a_5b_4 - 6a_6b_1 + a_6 - b_{19} = 0\,, \\
& 6a_1b_3 + 3a_1b_6 - 5a_2b_2 + 2a_2b_4 - 20a_3b_1 - 4a_4b_2 - 5a_6b_1 = 0\,, \\
& 6a_1b_{14} + a_4b_{18} - a_6b_{12} - 6a_7b_4 + 10a_6 - 12b_{19} = 0\,, \\
& 3a_1b_{11} + 3a_1b_{17} + a_1b_7 - a_2b_1 - 5a_4b_1 - 10a_5b_1 + a_2 - b_7 = 0\,, \\
& 5a_2b_{13} + a_3b_{14} - a_6b_8 - 3a_7b_5 + 30a_6 - 6b_{14} = 0\,, \\
& 3a_1b_{10} + 6a_1b_{12} + 3a_1b_{19} + 3a_1b_6 - a_2b_7 - 30a_3b_1 - 2a_4b_7 - 30a_6b_1 = 0\,,
\end{aligned}$$

$$6a_1b_{15}+12a_1b_{18}+a_4b_{19}-3a_6b_4-3a_6b_7-90a_7b_1+180a_5-180b_{17}=0\,,$$
$$18a_1b_5+6a_1b_9+2a_2b_3-12a_3b_2-6a_3b_4+2a_4b_3-4a_6b_2-60a_7b_1-12b_4=0\,,$$
$$3a_1b_{10}+6a_1b_{12}+6a_1b_3+9a_1b_6-3a_2b_4-2a_2b_7-60a_3b_1-2a_4b_4-30a_6b_1=0\,,$$
$$20a_2b_{13}+3a_3b_{14}+5a_4b_{13}-a_6b_8-6a_7b_5-3a_7b_9+150a_6-30b_{14}=0\,,$$
$$a_1b_{10}+3a_1b_{16}+a_2b_{11}+2a_2b_{17}-2a_3b_1-a_4b_2-4a_5b_2-5a_6b_1+2a_3-b_{10}=0\,,$$
$$a_1b_{15}+a_2b_{16}+a_3b_{17}-a_5b_3-a_6b_2-3a_7b_1+10a_5+3a_7-b_{15}-6b_{17}=0\,,$$
$$3a_1b_{10}+2a_1b_{12}+6a_1b_{16}+3a_1b_{19}-10a_3b_1+a_4b_{11}-a_4b_4-3a_5b_4-25a_6b_1-2b_{12}=0\,,$$
$$4a_1b_{10}+6a_1b_{12}+3a_1b_{16}+9a_1b_{19}-20a_3b_1-a_4b_{11}-a_4b_7-2a_5b_7-50a_6b_1-b_{10}=0\,,$$
$$9a_1b_5+3a_1b_9-a_2b_3+2a_2b_6-12a_3b_2+a_3b_4-3a_4b_3-4a_6b_2-30a_7b_1-6b_4=0\,,$$
$$3a_1b_{18}+6a_1b_5+6a_1b_9-a_2b_{12}-6a_3b_4+a_4b_6-3a_6b_4-60a_7b_1+10a_2-12b_7=0\,,$$
$$3a_1b_{10}+3a_1b_{16}+a_1b_6+2a_2b_{11}-a_2b_2+a_2b_7-10a_3b_1-4a_4b_2-6a_5b_2-10a_6b_1-b_6=0\,,$$
$$a_1(3b_{10}+2b_3+3b_6)+a_2(b_4-4b_2+2b_7)-20a_3b_1-6a_4b_2-4a_5b_2-10a_6b_1-2b_3=0\,,$$
$$15a_1b_{13}+2a_2b_{14}+4a_2b_8-2a_3b_5+a_3b_9-a_4b_8-2a_6b_5-9a_7b_3+30a_4-6b_9=0\,,$$
$$5a_1b_{13}+a_2b_{14}+a_3b_{15}-a_5b_8-a_6b_5-3a_7b_3+30a_5+30a_7-5b_{13}-6b_{15}=0\,,$$
$$a_1b_{14}+a_2b_{15}+a_3b_{16}-a_5b_5-a_6b_3-3a_7b_2+10a_6-b_{14}-6b_{16}=0\,,$$
$$60a_1b_{13}+6a_2b_{14}+12a_2b_8-2a_3b_5+4a_4b_8-2a_6b_5-18a_7b_3-6a_7b_6+120a_4-24b_9=0\,,$$
$$3a_1b_{14}+12a_1b_8+2a_2b_5+2a_2b_9-6a_3b_3+a_3b_6-2a_4b_5-3a_6b_3-18a_7b_2-6b_6=0\,,$$
$$6a_1b_{15}+7a_1b_{18}+a_4b_{16}+a_4b_{19}-a_5b_{12}-a_6b_{11}-4a_6b_4-60a_7b_1+130a_5-120b_{17}-b_{18}=0\,,$$
$$a_1(6b_{15}+15b_{18}+12b_9)-2a_3(2b_4+b_7)+a_4(70+b_{10}+b_{12})-9a_6b_4-2a_6b_7-180a_7b_1-72b_{11}=0\,,$$
$$9a_1b_{14}+36a_1b_8+6a_2b_5+3a_2b_9-6a_3b_3-3a_3b_6+3a_4b_5-3a_6b_3-36a_7b_2-9a_7b_4-18b_6=0\,,$$
$$9a_1b_{14}+2a_2b_{15}+2a_2b_{18}+a_3b_{19}-a_6b_{10}-3a_6b_3-18a_7b_2-3a_7b_7+40a_6-36b_{16}-6b_{19}=0\,,$$
$$6a_1b_{15}+2a_1b_{18}+2a_2b_{16}+a_2b_{19}+a_4b_{16}-a_5b_6-5a_6b_2-a_6b_4-30a_7b_1+80a_5-60b_{17}-2b_{18}=0\,,$$
$$a_1(8b_{15}+6b_{18})+a_2(b_{16}+3b_{19})-a_5b_{10}-a_6(b_{11}+10b_2+b_7)-60a_7b_1+140a_5-2b_{15}-120b_{17}=0\,,$$
$$a_1(3b_{15}+b_9)+a_2(b_{10}+2b_{16})+a_3(b_{11}-2b_2)+a_4(10-b_3)-3a_5b_3-4a_6b_2-15a_7b_1-6b_{11}-b_9=0\,,$$
$$a_1(6b_{10}+6b_{12}+3b_{16}+3b_{19}+4b_6)-a_2b_{11}-b_4(a_2+3a_4+3a_5)-40a_3b_1+a_4b_7-40a_6b_1-b_6=0\,,$$
$$a_1(3b_{14}+4b_8)+a_2(2b_{15}+b_9)+a_3(20+b_{10}-2b_3)-(a_4+2a_5)b_5-3a_6b_3-12a_7b_2-6b_{10}-4b_8=0\,,$$
$$60a_1b_{13}+6a_2b_{14}+a_3b_{18}+3a_4b_{14}-a_6b_9+3a_7(10-b_{12}-2b_3-2b_6)+360a_5-12b_{15}-30b_{18}=0\,,$$
$$6a_1(b_{15}+b_{18}+b_9)+a_2\alpha_1-2a_3(b_7+3b_2)+a_4(40-b_{10})-2a_6(b_7+6b_2)-90a_7b_1-36b_{11}=0\,,$$
$$6a_1(3b_{14}+4b_8)+2a_2(b_{18}+b_9)+2a_3\alpha_2+2a_4b_9-2a_6b_6-6a_7(4b_2+3b_4)-12b_{10}-24b_{12}=0\,,$$
$$3a_1(b_{15}+b_5+b_9)+a_2\alpha_3+a_3(b_7-8b_2)-3b_3(a_4+a_5)-6a_6b_2-30a_7b_1-3b_5-6b_7=0\,,$$
$$36a_1b_{14}+2(a_2+a_4)b_{15}+(5a_2+2a_4)b_{18}+a_6\alpha_4-6a_7(6b_2+3b_4+b_7)-72b_{16}-48b_{19}=0\,,$$
$$60a_1b_{13}+(9a_2+a_4)b_{14}+2(a_3-24)b_{15}+2(a_3-6)b_{18}-3a_6b_5-a_6b_9+3a_7\alpha_5+360a_5=0\,,$$
$$6a_1\alpha_6+a_2(40+b_{10}+4b_{12}+2b_6)-a_3(24b_2+6b_4-4b_7)-a_4b_6-3a_6(6b_2+b_4)-180a_7b_1-36b_7=0\,,$$
$$9a_1(3b_{14}+4b_8)+a_2(4b_{15}+4b_{18}+6b_9)+a_3\alpha_7-a_6(9b_3+2b_6)-a_7\alpha_8-42b_{10}-12b_{12}=0\,,$$
$$3(4a_1-1)b_{14}+2a_2(2b_{15}+b_{18})+(a_3-54)b_{16}+(a_3-6)b_{19}+a_4b_{15}-a_5b_9+a_6\alpha_9-3a_7\alpha_{10}=0\,,$$
$$2a_1\alpha_{11}+2a_2\alpha_{12}-2a_3(b_{11}+4b_2)-2b_4+a_4\alpha_{13}-2a_5b_6-a_6(16b_2+3b_4)-120a_7b_1-48b_{11}-2b_9=0\,,$$

where $\alpha_1=b_{10}+2b_{12}+2b_{19}$, $\alpha_2=10-b_{12}-b_6$, $\alpha_3=10+2b_{10}-b_3+b_6$, $\alpha_4=110-b_{10}-b_{12}-3b_6$, $\alpha_5=40-b_{10}-6b_3-b_6$, $\alpha_6=b_{15}+b_{18}+3b_5+3b_9$, $\alpha_7=80-b_{10}+2b_{12}-6b_3-2b_6$, $\alpha_8=72b_2+9b_4+6b_7$, $\alpha_9=70-4b_3-b_6$, $\alpha_{10}=b_{11}+8b_2+b_4$, $\alpha_{11}=6b_{15}+3b_{18}+4b_9$, $\alpha_{12}=b_{10}+b_{12}+b_{16}+b_{19}$, and $\alpha_{13}=60+b_{10}-b_6$.

After the tedious calculations, we can find 32 sets of solutions and only 12 of them are independent. Thus twelve classes of independent dark MKdV models are obtained. Here, we just list the extended equations for the scalar field v and its related fifth order symmetry σ^v.

Class 1

$$v_t=a\big[v_{xxx}+3b(uv_x)_x+3b^2u^2v_x\big]+b\big[(ab^2-2)u^3+(a-1)u_{xx}+3abuu_x\big]v\,, \tag{10a}$$
$$\sigma^v=cv_{5x}+5bcuv_{4x}+10bc(u_x+bu^2)v_{xxx}+10bc(u_{xx}+3buu_x+b^2u^3)v_{xx}$$

$$+ 5bc(u_{xxx} + 4buu_{xx} + 3bu_x^2 + 6b^2u^2u_x + b^3u^4)v_x + b\big[(c-1)u_{4x} + 5bcuu_{xxx} + 10\big[bcu_x + (b^2c-1)u^2\big]u_{xx} + 5(3b^2c-2)uu_x^2 + 10b^3cu^3u_x + (b^4c-6)u^5\big]v\,, \tag{10b}$$

where a, b, and c are arbitrary constants.

Different from the dark KdV cases, for the scalar dark KdV extensions, there is no nontrivial free parameters. When $b = 0$, the first class of the dark MKdV system is reduced to

$$u_t = \partial(2u^3 + u_{xx}), \quad v_t = a\partial^3(v)\,, \tag{11}$$

which is just related to the Miura transformation of the completely decomposed case of the dark KdV equation given by Kupershmidt.

Obviously, such decomposed system is available for any system of the vector fields anywhere. For the MKdV hierarchy, we then have

$$u_t = \partial(2u^3 + u_{xx}), \quad v_t = a_n\partial^{2n-1}(v), \quad n \in \mathbb{N}, \quad a_n = \text{const.}, \tag{12}$$

and all these are self-dual extensions.

Class 2

$$v_t = 4(v_{xx} + 3auv_x)_x + 6\big[(2a^2+1)u^2 - \delta u_x\big]v_x + \big[4a(a^2+1)u^3 + 6(2a^2 - a\delta + 1)uu_x + 3(a-\delta)u_{xx}\big]v\,, \tag{13a}$$

$$\sigma^v = 16v_{5x} + 80auv_{4x} + 40\big[(4a-\delta)u_x + (4a^2+1)u^2\big]v_{xxx} + 20\big[(8a-3\delta)u_{xx} + 6(4a^2-a\delta+1)uu_x + 2a(4a^2+3)u^3\big]v_{xx} + 10\big[(8a-5\delta)u_{xxx} + 2(16a^2-6a\delta+5)uu_{xx} + (24a^2-12a\delta+7)u_x^2 + 6(8a^3-2a^2\delta+6a-\delta)u^2u_x + (8a^4+12a^2+3)u^4\big]v_x + \big[15(a-\delta)u_{4x} + 10(8a^2-5a\delta+3)uu_{xxx} + 10[2(8a^2-5a\delta+3)u_x + (16a^3-6a^2\delta+13a-3\delta)u^2]u_{xx} + 60(4a^3-2a^2\delta+3a-\delta)uu_x^2 + 20(8a^4-2a^3\delta+12a^2-3a\delta+3)u^3u_x + 8a(2a^2+3)(a^2+1)u^5\big]v\,, \tag{13b}$$

where a is an arbitrary constant and $\delta^2 = -1$.

It is interesting that there is no dark KdV correspondence for nonzero a. When $a = 0$, Eq. (13a) is reduced to

$$v_t = 4v_{xxx} + 6u^2v_x + 6uu_xv - 6\delta u_xv_x - 3\delta u_{xx}v\,, \tag{14}$$

which is related to the Miura transformation of one of the known dark KdV system.

Class 3

$$v_t = (v_{xx} + 3auv_x)_x + 3(a^2+2)u^2v_x + (a^2+4)(au^2 + 3u_x)uv\,, \tag{15a}$$

$$\sigma^v = v_{5x} + 5auv_{4x} + 10[au_x + (a^2+1)u^2]v_{xxx} + 10[au_{xx} + (3a^2+4)uu_x + a(a^2+3)u^3]v_{xx} + 5\big[au_{xxx} + 4(a^2+2)uu_{xx} + 3(a^2+2)u_x^2 + 2a(3a^2+11)u^2u_x + (a^4+6a^2+6)u^4\big]v_x + (a^2+4)\big[5uu_{xxx} + 10(u_x + au^2)u_{xx} + 15auu_x^2 + 10(a^2+3)u^3u_x + a(a^2+6)u^5\big]v\,, \tag{15b}$$

with a free parameter a.

Clearly, there is no similar known results for the dark KdV equation for arbitrary a. When $a = 0$, Eq. (15a) can be reduced to

$$v_t = v_{xxx} + 6(u^2v)_x\,, \tag{16}$$

which is just the symmetry equation of the MKdV. The symmetry coupling is valid dark extension for arbitrary integrable systems.

Class 4

$$v_t = (v_{xx} + 3auv_x)_x + 3(a^2+2)u^2v_x + a\big[(a^2+4)u^2 + 3au_x\big]uv\,, \tag{17a}$$

$$\sigma^v = v_{5x} + 5auv_{4x} + 10\big[au_x + (a^2+1)u^2\big]v_{xxx} + 10\big[au_{xx} + (3a^2+2)uu_x + a(a^2+3)u^3\big]v_{xx} + 5\big[au_{xxx} + 4(a^2+1)uu_{xx} + (3a^2+2)u_x^2 + 2a(3a^2+7)u^2u_x + (a^4+6a^2+6)u^4\big]v_x + a\big[5auu_{xxx} + 10[au_x + (a^2+2)u^2]u_{xx} + 5(3a^2+4)uu_x^2 + 10a(a^2+5)u^3u_x + (a^2+4)(a^2+6)u^5\big]v\,, \tag{17b}$$

where a is a free constant. There is also known correspondence for the dark KdV equation except for the special $a = 0$ case.

When $a = 0$, Eq. (17a) can be reduced to

$$v_t = v_{xxx} + 6u^2v_x\,, \tag{18}$$

which is just the dual equation of the symmetry equation of the MKdV. The dual symmetry coupling is also valid dark extension for all integrable systems.

Class 5

$$v_t = (v_{xx} + 3auv_x)_x + 3\big[(a^2+1)u^2 + \delta u_x\big]v_x + \big[a(a^2+1)u^3 + 3(a+2\delta)(a-\delta)uu_x + 3\delta u_{xx}\big]v\,, \tag{19a}$$

$$\begin{aligned}\sigma^v &= v_{5x}+5auv_{4x}+5\big[(2a+\delta)u_x+(2a^2+1)u^2\big]v_{xxx}+5\big[2(a+\delta)u_{xx}+(6a^2+3a\delta+4)uu_x+a(2a^2+3)u^3\big]v_{xx}\\ &\quad+5\big[(a+2\delta)u_{xxx}+4(a^2+a\delta+1)uu_{xx}+(3a^2+3a\delta+2)u_x^2+(6a^3+3a^2\delta+11a+4\delta)u^2u_x+(a^2+2)(a^2+1)u^4\big]v_x\\ &\quad+\big[5\delta u_{4x}+5(a^2+2a\delta+2)uu_{xxx}+5[(2a^2+3a\delta+2)u_x+(2a^3+2a^2\delta+3a+4\delta)u^2]u_{xx}\\ &\quad+5(3a^3+3a^2\delta+4a+8\delta)uu_x^2+5(2a^4+a^3\delta+7a^2+4a\delta+8)u^3u_x+a(a^2+4)(a^2+1)u^5\big]v\,,\end{aligned} \tag{19b}$$

with free constant a and $\delta^2 = -1$.

Except for $a = 0$ case, one has not yet found the similar results for the dark KdV system.

When $a = 0$, Eq. (19a) is reduced to

$$v_t = (v_{xx} + 3u^2v + 3\delta u_x v)_x\,, \tag{20}$$

which is the result of taking the Miura transformation of one known dark KdV system. The related dark KdV system can be considered as the bosonisation of the fermionic supersymmetric KdV equation.

Class 6

$$v_t = (v_{xx} + 3auv_x)_x + 3\big[(a^2+1)u^2 + \delta u_x\big]v_x + a\big[(a^2+1)u^2 + 3(a+\delta)u_x\big]uv\,, \tag{21a}$$

$$\begin{aligned}\sigma^v &= v_{5x}+5auv_{4x}+5\big[(2a+\delta)u_x+(2a^2+1)u^2\big]v_{xxx}+5\big[(2a+\delta)u_{xx}+(6a^2+3a\delta+2)uu_x+5a(2a^2+3)u^3\big]v_{xx}\\ &\quad+5\big[(a+\delta)u_{xxx}+2(2a^2+a\delta+1)uu_{xx}+3a(a+\delta)u_x^2+(6a^3+3a^2\delta+7a+4\delta)u^2u_x+(a^2+2)(a^2+1)u^4\big]v_x\\ &\quad+a\big[5(a+\delta)uu_{xxx}+5[2(a+\delta)u_x+(2a^2+a\delta+1)u^2]u_{xx}+15a(a+\delta)uu_x^2+5(2a^3+a^2\delta+5a+4\delta)u^3u_x\\ &\quad+(a^2+4)(a^2+1)u^5\big]v\,,\end{aligned} \tag{21b}$$

with arbitrary constant a and $\delta^2 = -1$.

Only for the special $a = 0$ case, one can find a correspondence of the dark KdV equation.

When $a = 0$, Eq. (21a) can be reduced to

$$v_t = v_{xxx} + 3u^2v_x + 3\delta u_x v_x\,, \tag{22}$$

which is just the dual equation of Eq. (20). The related dark KdV system can also be considered as the bosonisation of a trivial fermionic supersymmetric KdV model.

Class 7

$$v_t = (v_{xx} + 3auv_x)_x + 3\big[(a^2+2)u^2 + 2\delta u_x\big]v_x + \big[a(a^2+4)u^3 + 3(a^2+2a\delta+4)uu_x + 6\delta u_{xx}\big]v\,, \tag{23a}$$

$$\begin{aligned}\sigma^v &= v_{5x}+5auv_{4x}+10\big[(a+\delta)u_x+(a^2+1)u^2\big]v_{xxx}+10\big[(a+2\delta)u_{xx}+(3a^2+3a\delta+4)uu_x+a(a^2+3)u^3\big]v_{xx}\\ &\quad+5\big[(a+4\delta)u_{xxx}+4(a^2+2a\delta+2)uu_{xx}+(3a^2+6a\delta+2)u_x^2+2(3a^3+3a^2\delta+11a+6\delta)u^2u_x+(a^4+6a^2+6)u^4\big]v_x\\ &\quad+\big[10\delta u_{4x}+5(a^2+4a\delta+4)uu_{xxx}+10[a(a+3\delta)u_x+(a^3+2a^2\delta+4a+6\delta)u^2]u_{xx}\\ &\quad+5(3a^3+6a^2\delta+8a+24\delta)uu_x^2+10(a^4+a^3\delta+7a^2+6a\delta+12)u^3u_x+a(a^2+4)(a^2+6)u^5\big]v\,,\end{aligned} \tag{23b}$$

where a is an arbitrary constant and $\delta^2 = -1$.

Only for the special $a = 0$ case, we can find a correspondence of the dark KdV equation.

When $a = 0$, Eq. (23a) can be reduced to

$$v_t = (v_{xx} + 6u^2v + 6\delta u_x v)_x\,, \tag{24}$$

which is the results of taking the Miura transformation of the KdV symmetry equation.

Class 8

$$v_t = (v_{xx} + 3auv_x)_x + 3\big[(a^2+2)u^2 + 2\delta u_x\big]v_x + a\big[(a^2+4)u^2 + 3(a+2\delta)u_x\big]uv\,, \tag{25a}$$

$$\begin{aligned}\sigma^v &= v_{5x}+5auv_{4x}+5\big[(a+\delta)u_x+(a^2+1)u^2\big]v_{xxx}+10\big[(a+\delta)u_{xx}+(3a^2+3a\delta+2)uu_x+a(a^2+3)u^3\big]v_{xx}\\ &\quad+5\big[(a+2\delta)u_{xxx}+4(a^2+a\delta+1)uu_{xx}+(3a^2+6a\delta-2)u_x^2+2(3a^3+3a^2\delta+7a+6\delta)u^2u_x+(a^4+6a^2+6)u^4\big]v_x\\ &\quad+a\big[5(a+2\delta)uu_{xxx}+10[(a+2\delta)u_x+(a^2+a\delta+2)u^2]u_{xx}+15a(a+2\delta)uu_x^2+10(a^3+a^2\delta+5a+6\delta)u^3u_x\\ &\quad+(a^2+4)(a^2+6)u^5\big]v\,,\end{aligned} \tag{25b}$$

where a is an arbitrary constant and $\delta^2 = -1$.

Same as the class 8, for the special $a = 0$ situation, we can find a related model of the dark KdV system related to the dual of the symmetry equation.

When $a = 0$, Eq. (25a) can be reduced to

$$v_t = v_{xxx} + 6u^2 v_x + 6\delta u_x v_x \,, \tag{26}$$

which is just the dual equation of Eq. (24).

Class 9

$$v_t = -2(v_{xx} + 3auv_x)_x - 6\big[(a^2+1)u^2 + \delta u_x\big]v_x - \big[2a(a^2+4)u^3 + 6(a^2+a\delta+2)uu_x + 3(a+2\delta)u_{xx}\big]v \,, \tag{27a}$$

$$\begin{aligned}\sigma^v = {} & -4v_{5x} - 20auv_{4x} - 20\big[(2a+\delta)u_x + (2a^2+1)u^2\big]v_{xxx} - 20\big[2(a+\delta)u_{xx} + (6a^2+3a\delta+4)uu_x + a(2a^2+3)u^3\big]v_{xx} \\ & -10\big[(2a+3\delta)u_{xxx} + 2(4a^2+4a\delta+3)uu_{xx} + (6a^2+6a\delta+5)u_x^2 + 2(6a^3+3a^2\delta+11a+\delta)u^2u_x + (2a^4+6a^2+1)u^4\big]v_x \\ & -\big[5(a+2\delta)u_{4x} + 10(2a^2+3a\delta+2)uu_{xxx} + 10[2(2a^2+3a\delta+2)u_x + (4a^3+4a^2\delta+9a+2\delta)u^2]u_{xx} \\ & +20(3a^3+3a^2\delta+7a+2\delta)uu_x^2 + 20(2a^4+a^3\delta+7a^2+a\delta+2)u^3u_x + 4a(a^2+4)(a^2+1)u^5\big]v \,,\end{aligned} \tag{27b}$$

where a is an arbitrary constant and $\delta^2 = -1$.

Only for the $a = 0$ case, we can find a dark KdV correspondence, the socalled mysterious dark KdV equation.

Thus we can call the class 9 the general mysterious dark MKdV system and call the special case for $a = 0$ as the mysterious dark MKdV equation which possesses the form (4) and

$$v_t = -2(v_{xx} + 3u^2 v + 3\delta u_x v)_x \,, \tag{28}$$

which is the mysterious equation of MKdV (4).

Class 10

$$v_t = -2(v_{xx} + 3auv_x)_x - 6\big[(a^2+1)u^2 + \delta u_x\big]v_x - a\big[2(a^2+4)u^3 + 6(a+\delta)uu_x + 3u_{xx}\big]v \,, \tag{29a}$$

$$\begin{aligned}\sigma^v = {} & -4v_{5x} - 20auv_{4x} - 20\big[(2a+\delta)u_x + (2a^2+1)u^2\big]v_{xxx} - 20\big[(2a+\delta)u_{xx} + (6a^2+3a\delta+2)uu_x + a(2a^2+3)u^3\big]v_{xx} \\ & -10\big[(2a+\delta)u_{xxx} + 2(4a^2+2a\delta+1)uu_{xx} + (6a^2+6a\delta+1)u_x^2 + 2(6a^3+3a^2\delta+7a+\delta)u^2u_x + (2a^4+6a^2+1)u^4\big]v_x \\ & -a\big[5u_{4x} + 10(2a+\delta)uu_{xxx} + 10[4(a+\delta)u_x + (4a^2+2a\delta+5)u^2]u_{xx} \\ & +60(a^2+a\delta+1)uu_x^2 + 20(2a^3+a^2\delta+5a+\delta)u^3u_x + 4(a^2+4)(a^2+1)u^5\big]v \,,\end{aligned} \tag{29b}$$

where a is an arbitrary constant and $\delta^2 = -1$.

The special $a = 0$ case

$$v_t = -2(v_{xxx} + 3u^2 v_x + 3\delta u_x v_x) \,, \tag{30}$$

is the dual form of the mysterious dark MKdV equation.

Class 11

$$v_t = (v_{xx} + 3auv_x)_x + 3(a^2+1)u^2 v_x + (a^2+1)(au^2 + 3u_x)uv \,, \tag{31a}$$

$$\begin{aligned}\sigma^v = {} & v_{5x} + 5auv_{4x} + 5\big[2au_x + (2a^2+1)u^2\big]v_{xxx} + 5\big[2au_{xx} + 3(2a^2+1)uu_x + a(2a^2+3)u^3\big]v_{xx} \\ & + 5\big[au_{xxx} + (4a^2+3)uu_{xx} + (3a^2+2)u_x^2 + 3a(2a^2+3)u^2u_x + (a^2+2)(a^2+1)u^4\big]v_x \\ & + (a^2+1)\big[5uu_{xxx} + 10(u_x + au^2)u_{xx} + 15auu_x^2 + 10(a^2+2)u^3u_x + a(a^2+4)u^5\big]v \,,\end{aligned} \tag{31b}$$

where a is an arbitrary constant. It should be mentioned that there is no known correspondence of dark KdV case even for any fixed constant.

When $a = 0$, Eq. (31a) is reduced to

$$v_t = v_{xxx} + 3u^2 v_x + 3uu_x v \,, \tag{32}$$

which is the bosonization of fermion field[17] and self-dual.

Class 12

$$v_t = 2(u^2 + \delta u_x)v_x + a(u^2 + \delta u_x)_x v \,, \tag{33a}$$

$$\sigma^v = 2\big[\delta u_{xxx} + 2uu_{xx} - u_x^2 + 6\delta u^2 u_x + 3u^4\big]v_x + a\big[\delta u_{4x} + 2uu_{xxx} + 6\delta u^2 u_{xx} + 12uu_x^2 + 12u^3u_x\big]v \,, \tag{33b}$$

where a is free and $\delta^2 = -1$.

The class is just the Miura transformation of the first order dark KdV extension.

3 Recursion Operators of the Dark MKdV Equations

Usually, the existence of higher order symmetries may imply the possibility of the infinitely many symmetries. However, to guarantee the existence of infinitely many higher order symmetries, it is necessary to find a recursion operator or a master-symmetry such that infinitely many higher order symmetries can be obtained systematically. In this section, we write down the recursion operators of two class of dark MKdV.

The construction of the recursion operator of a given integrable system is not an easy task. A lot of works are devoted to this subject. Based on the Lax representation, some of the authors utilized eigenvalue equation for the squared eigenfunctions of the Lax operator to construct recursion operator,[18−22] others[23−26] used Hamiltonian operators to realize it. Moreover, a simple, effective method[27] for constructing the recursion operator has been proposed when the Lax representation is given. However, for the dark equations, the corresponding Lax representations are unknown. Here, on the basis of the known symmetries, the recursion operators of the first two classes of dark equations are directly constructed.

Take Class 1 as an example, the first order symmetry is

$$\sigma_1^u = u_x, \quad \sigma_1^v = v_x\,, \tag{34}$$

the third order symmetry is

$$\sigma_3^u = u_{xxx} + 6u^2 u_x, \quad \sigma_3^v = c\big[v_{xxx} + 3b(uv_x)_x + 3b^2u^2v_x\big] + b\big[(a-1)u_{xx} + 3bcuu_x + (b^2c-2)u^3\big]v\,, \tag{35}$$

and the fifth order symmetry is

$$\sigma_5^u = u_{5x} + 10u^2u_{xxx} + 40uu_xu_{xx} + 10u_x^3 + 30u^4u_x\,, \tag{36a}$$

$$\begin{aligned}\sigma_5^v = {} & dv_{5x} + 5bduv_{4x} + 10bd(u_x + bu^2)v_{xxx} + 10bd(u_{xx} + 3buu_x + b^2u^3)v_{xx} \\ & + 5bd(u_{xxx} + 4buu_{xx} + 3bu_x^2 + 6b^2u^2u_x + b^3u^4)v_x + b\big[(d-1)u_{4x} + 5bduu_{xxx} \\ & + 10(bdu_x + (b^2d-1)u^2)u_{xx} + 5(3b^2d-2)uu_x^2 + 10b^3du^3u_x + (b^4d-6)u^5\big]v\,. \end{aligned} \tag{36b}$$

By virtue of these symmetries and the properties of MKdV, we assume that the strong symmetry $\mathcal{R}$ is of the form:

$$\mathcal{R} = \begin{pmatrix} \mathcal{R}_{11} & 0 \\ \mathcal{R}_{21} & \mathcal{R}_{22} \end{pmatrix}, \tag{37}$$

where

$$\mathcal{R}_{11} = \partial_x^2 + 4u^2 + 4u_x\partial_x^{-1}u\,, \tag{38}$$

is just the recursion operator of the MKdV equation, and

$$\begin{aligned}\mathcal{R}_{21} = {} & c_1\partial_x^2 + (c_2u + c_3v)\partial_x + (c_4u_x + c_5v_x + c_6u^2 + c_7v^2 + c_8uv) + \big[c_9u^3 + c_{10}u^2v \\ & + c_{11}uv^2 + c_{12}v^3 + (c_{13}u + c_{14}v)u_x + (c_{15}u + c_{16}v)v_x + c_{17}u_{xx} + c_{18}v_{xx}\big]\partial_x^{-1} \\ & + \big[c_{19}u^2 + c_{20}uv + c_{21}v^2 + c_{22}u_x + c_{23}v_x\big]\partial_x^{-1}u + \big[c_{24}u^2 + c_{25}uv + c_{26}v^2 + c_{27}u_x \\ & + c_{28}v_x\big]\partial_x^{-1}v + (c_{29}u + c_{30}v)\partial_x^{-1}u^2 + (c_{31}u + c_{32}v)\partial_x^{-1}uv + (c_{33}u + c_{34}v)\partial_x^{-1}v^2 \\ & + c_{35}\partial_x^{-1}u^3 + c_{36}\partial_x^{-1}u^2v + c_{37}\partial_x^{-1}uv^2 + c_{38}\partial_x^{-1}v^3 + c_{39}\partial_x^{-1}u_xv + c_{40}\partial_x^{-1}uv_x\,. \end{aligned} \tag{39}$$

Meanwhile, $\mathcal{R}_{22}$ is same as $\mathcal{R}_{21}$ by replacing c_j with d_j, where c_j and d_j $(j = 1, 2, \ldots, 40)$ are undetermined constants. According to the definition of strong symmetry, we have

$$\mathcal{R}\begin{pmatrix} \sigma_1^u \\ \sigma_1^v \end{pmatrix} = \begin{pmatrix} \sigma_3^u \\ \sigma_3^v \end{pmatrix}, \quad \mathcal{R}\begin{pmatrix} \sigma_3^u \\ \sigma_3^v \end{pmatrix} = \begin{pmatrix} \sigma_5^u \\ \sigma_5^v \end{pmatrix}. \tag{40}$$

After the tedious calculations with *Maple*, we get

$$\mathcal{R}_{21} = b\big[(c-1)v\partial_x + 2c(v_x + buv) + c\big[v_{xx} + b(uv)_x + bu(v_x + buv)\big]\partial_x^{-1} - 4uv\partial_x^{-1}u\big]\,, \tag{41}$$

$$\mathcal{R}_{22} = c\big[\partial_x^2 + b^2u^2 + b(u\partial_x + \partial_xu)\big]\,, \tag{42}$$

with the constraint $d = c^2$.

Next, it is need to verify that strong symmetry $\mathcal{R}$ is just the recursion operator of the dark equation. That is to say, we need to prove that $\mathcal{R}$ has the property of heredity. The only thing we need to do is to prove that $\mathcal{R}$ is also the strong symmetry of the corresponding higher order flows. One of the higher order flows of dark equation (4) and (10a) is

$$u_t = u_{5x} + 10u^2u_{xxx} + 40uu_xu_{xx} + 10u_x^3 + 30u^4u_x\,, \tag{43}$$

$$\begin{aligned} v_t = {} & dv_{5x}+5bduv_{4x}+10bd(u_x+bu^2)v_{xxx}+10bd(u_{xx}+3buu_x+b^2u^3)v_{xx} \\ & +5bd(u_{xxx}+4buu_{xx}+3bu_x^2+6b^2u^2u_x+b^3u^4)v_x+b\big[(d-1)u_{4x}+5bduu_{xxx} \\ & +10(bdu_x+(b^2d-1)u^2)u_{xx}+5(3b^2d-2)uu_x^2+10b^3du^3u_x+(b^4d-6)u^5\big]v\,. \end{aligned} \tag{44}$$

Proceeding as before, it is not difficult to verify that $\mathcal{R}$ with Eqs. (38), (41), and (42) is also the strong symmetry of Eqs. (43) and (44). Thereby, the recursion operator $\mathcal{R}$ of Class 1 is given. The exist of $\mathcal{R}$ indicates that the dark equation of Class 1 has an infinite set of symmetries and conserved quantities.

In the same way, the recursion operator of Class 2 is

$$\mathcal{R}_{21}=3(a-\delta)v\partial_x+2\big[(4a-\delta)v_x+(4a^2-a\delta+3)uv\big]+4\big[av_{xx}+2a^2uv_x+a(a-\delta)vu_x+a(a^2+1)u^2v\big]\partial_x^{-1}+4v_x\partial_x^{-1}u\,, \quad (45)$$

$$\mathcal{R}_{22}=4\big[\partial_x^2+2au\partial_x+(a-\delta)u_x+(a^2+1)u^2\big]\,. \quad (46)$$

This method of constructing recursion operator can also be applied to the rest ten dark equations, we do not consider them here.

4 Summary and Discussion

In summary, a complete classification of the dark MKdV systems with scalar extension has been given. The results show that there are twelve classes of dark MKdV equations, which are all integrable coupling systems under meaning that they possess higher order symmetries. Furthermore, the recursion operators of the former two classes of dark MKdV equations are explicitly given.

It is quite remarkable that the dark MKdV systems are quite different from the Miura transformations of the dark KdV systems. Firstly, there are only nine classes for the dark KdV systems while there twelve sets of dark MKdV equations. For the Classes 3, 4 and 11, there is no known dark KdV partner.

Secondly, every class of the dark MKdV equation possesses a further free parameter. Except for the last first order system, eight classes of dark MKdV systems linked with the Miura transformations of the dark KdV systems only for the special cases by vanishing theses free parameters.

From the discovery of the dark MKdV and dark KdV systems, one can explore various open problems to further investigate. Especially, the close relations dark equations, super (Kuper) systems, supersymmetric systems and other non-decoupled systems are worthy of investigation.

The linear homogeneous extensions have been discussed in this paper, the linear inhomogeneous or the nonlinear extensions will be investigated in our further work. In addition, the explicit solutions of the generalized dark equations will be produced through the famous methods, such as symmetry reduction method, the Hirota bilinear method, Darboux transformation, Painlevé analysis and so on.

References

[1] G. Jungmana, M. Kamionkowskib, and K. Griestd, Phys. Rep. **267** (1996) 195.

[2] G. Bertone and D. Hopper, arXiv:1605.04909v2[astro-ph.CO].

[3] Planck Collaboration: P. A. R. Ade, N. Aghanim, *et al.*, arXiv:1303.5062v2[astro-ph.CO].

[4] B. C. Paul, Bibechana **11** (2014) 8; M. S. Turner, Physics in Collision **19** (1999) 203.

[5] S. Perlmutter, M. S. Turner, and M. White, Phys. Rev. Lett. **83** (1999) 670.

[6] B. A. Kupershmidt, J. Nonlinear Math. Phys. **8** (2001) 363.

[7] W. X. Ma and B. Fuchssteiner, Phys. Lett. A **213** (1996) 49; W. X. Ma, J. Math. Phys. **46** (2005) 033507.

[8] D. Levi and O. Ragnisco, IL Nuovo Cimento **83** (1984) 34.

[9] Z. Y. Yan and H. Q. Zhang, J. Math. Phys. **42** (2001) 330.

[10] X. G. Geng and W. X. Ma, IL Nuovo Cimento **108** (1995) 477.

[11] E. G. Fan, Physica A **301** (2001) 105.

[12] X. X. Xu, Phys. Lett. A **301** (2002) 250.

[13] L. Peng, S. Y. Lou, and X. D. Yang, Commun. Theor. Phys. **58** (2012) 1.

[14] B. A. Kupershmidt, Phys. Lett. A **102** (1984) 213.

[15] Yu. I. Maniu and A.O. Radul, Commun. Math. Phys. **98** (1985) 65.

[16] X. N. Gao and S. Y. Lou, Phys. Lett. B **707** (2012) 209.

[17] I. Yamanaka and R. Sasaki, Prog. Theor. Phys. **79** (1988) 1167.

[18] A. P. Fordy and J. Gibbons, J. Math. Phys. **22** (1980) 2508.

[19] A. S. Fokas and R. L. Anderson, J. Math. Phys. **23** (1982) 1066.

[20] A. S. Fokas, Stud. Appl. Math. **77** (1987) 253.

[21] P. M. Santini and A. S. Fokas, Commun. Math. Phys. **115** (1988) 375.

[22] A. S. Fokas and P. M. Santini, Commun. Math. Phys. **116** (1988) 449.

[23] W. Symes, J. Math. Phys. **20** (1979) 721.

[24] M. Adler, Invent. Math. **50** (1979) 219.

[25] A. S. Fokas, I. M. Gelfand, *Important Developments in Soliton Theory*, *Springer Series in Nonlinear Dynamics*, Springer-Verlag, Berlin (1993).

[26] M. Antonowicz and A. P. Fordy, *Nonlinear Evolution Equations and Dynamical Systems*, World Scientific, Singapore (1988).

[27] M. Gürses, A. Karasu, and V. V. Sokolov, J. Math. Phys. **40** (1999) 6473.