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Localized waves in three-component coupled nonlinear Schrödinger equation*

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We study the generalized Darboux transformation to the three-component coupled nonlinear Schrödinger equation. First- and second-order localized waves are obtained by this technique. In first-order localized wave, we get the interactional solutions between first-order rogue wave and one-dark, one-bright soliton respectively. Meanwhile, the interactional solutions between one-breather and first-order rogue wave are also given. In second-order localized wave, one-dark-one-bright soliton together with second-order rogue wave is presented in the first component, and two-bright soliton together with second-order rogue wave are gained respectively in the other two components. Besides, we observe second-order rogue wave together with one-breather in three components. Moreover, by increasing the absolute values of two free parameters, the nonlinear waves merge with each other distinctly. These results further reveal the interesting dynamic structures of localized waves in the three-component coupled system.

Keywords: localized waves, three-component coupled nonlinear Schrödinger equation, generalized Darboux transformation

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1. Introduction

In the past several years, localized waves including dark or bright soliton, breather and rogue wave have been of great interests in nonlinear science. The dark and bright soliton are special cases of soliton. The breather is localized in time or space, such as Ma breather (time-periodic breather solution)[1] and Akhmediev breather (space-periodic breather solution).[2] While the rogue wave (also called freak wave, monster wave, killer wave, rabid-dog wave, and other names) is localized in both time and space, and seems to appear from nowhere and disappear without a trace.[3–6] There have been many articles on rogue waves of single-component systems, such as the nonlinear Schrödinger (NLS) equation,[7–9] the derivative NLS equation,[10,11] the Kundu–Eckhaus equation,[12–14] the Sasa–Satsuma equqtion,[15] the higher-order dispersive NLS equation,[16] and so on.

However, a variety of complex systems,[17–19] such as Bose–Einstein condensates and nonlinear optical fibers, usually involve more than one component. So recent studies are extended to localized waves in multicomponent coupled systems, and many interesting and appealing results have been obtained. The bright–dark–rogue solution[20,21] and other higher-order localized waves[22] are all found in two-component coupled NLS equation. Some semi-rational, multi-parametric localized wave solutions are obtained in coupled Hirota equation.[23–25] A four-petaled flower structure rogue wave is exhibited in three-component coupled NLS equation.[26]

Motivated by the works of Baronio[27] and Guo,[9,28] we study the localized wave solutions of the three-component coupled NLS equation

$$\begin{aligned} &\mathrm{i}q_{1t}+q_{1xx}+2(|q_1|^2+|q_2|^2+|q_3|^2)q_1=0,\\ &\mathrm{i}q_{2t}+q_{2xx}+2(|q_1|^2+|q_2|^2+|q_3|^2)q_2=0,\\ &\mathrm{i}q_{3t}+q_{3xx}+2(|q_1|^2+|q_2|^2+|q_3|^2)q_3=0, \end{aligned} \tag{1}$$

where each non-numeric subscripted variable stands for partial differentiation. Besides, q_i $(i=1,2,3)$ is the complex function of x and t.

Here we are interested in the interactional solutions between rogue waves and some nonlinear wave solutions in the Eq. (1), for example, dark, bright soliton and breather. To the best of our knowledge, this is not reported in other articles. By using Darboux-dressing transformation, Baronio *et al.*[21] has obtained some semi-rational solutions in two-component coupled NLS equation, which include rogue wave, dark–bright–rogue wave and breather–rogue wave. But, Baronio's method is very complicated and can not obtain higher-order localized waves. In order to overcome this difficulty, we construct a specifical vector solution of Lax pair for the vector NLS equation, which is firstly put forward.[22,24] Combining the generalized Darboux transformation (DT) with the special vector solution, we have conveniently obtained several interesting higher-order localized waves.

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The localized waves, such as second-order dark–bright–rogue wave and second-order breather–rogue wave, have been discussed in detail.[22,24] N-component NLS equation has been solved to get multi-dark soliton.[29] Meanwhile, performing Hirota bilinear method to Eq. (1), one can obtain two-bright-one-dark soliton and one-bright-two-dark soliton.[30] Four-petaled flower structure rogue wave has also been found.[26] However, three-component and two-component NLS equation are not exactly the same. Using our method, some meaningful results can be obtained. From the special seed solutions of Lax pair, we can get the basic solutions of Eq. (1) with several free parameters by generalized DT. Then, choosing the appropriate values of these free parameters, some interesting interactional solutions are exhibited.

2. Generalized Darboux transformation

In this section, we construct the generalized DT of Eq. (1). The system (1) admits the following Lax pair

$$\phi_x = \boldsymbol{U}\phi = (\mathrm{i}\lambda\boldsymbol{\Lambda} + \boldsymbol{Q})\phi, \tag{2}$$

$$\phi_t = \boldsymbol{V}\phi = [3\mathrm{i}\lambda^2\boldsymbol{\Lambda} + 3\lambda\boldsymbol{Q} + \mathrm{i}\boldsymbol{\sigma}_3(\boldsymbol{Q}_x - \boldsymbol{Q}^2)]\phi, \tag{3}$$

where

$$\boldsymbol{\Lambda} = \mathrm{diag}(-2,1,1,1), \quad \boldsymbol{\sigma}_3 = \mathrm{diag}(1,-1,-1,-1),$$

$$\boldsymbol{Q} = \begin{bmatrix} 0 & q_1 & q_2 & q_3 \\ -q_1^* & 0 & 0 & 0 \\ -q_2^* & 0 & 0 & 0 \\ -q_3^* & 0 & 0 & 0 \end{bmatrix}.$$

Here $\phi = (\phi,\varphi,\chi,\psi)^{\mathrm{T}}$, q_i $(i=1,2,3)$ is potential function, λ is spectral parameter, and q_i^* $(i=1,2,3)$ denotes the complex conjugate of q_i. In fact, a direct calculation shows that the zero-curvature equation, $\boldsymbol{U}_t - \boldsymbol{V}_x + [\boldsymbol{U},\boldsymbol{V}] = 0$, is implied in Eq. (1).

Based on the DT of the Ablowitz–Kaup–Newell–Segur (AKNS) spectral problem,[28] the generalized DT of Eq. (1) can be also constructed. In Eq. (2), $\boldsymbol{U}$ and $\boldsymbol{V}$ are 4×4 matrixs, so it is more complicated than two-component NLS equation for getting a special vector solution of Lax pair. Let $\phi_1 = (\phi_1,\varphi_1,\chi_1,\psi_1)^{\mathrm{T}} = \phi_1(\lambda_1+\delta)$ be a solution of Eqs. (2) and (3) with $q_1 = q_1[0]$, $q_2 = q_2[0]$, $q_3 = q_3[0]$ and $\lambda = \lambda_1 + \delta$, then ϕ_1 can be expanded as the Taylor series at $\delta = 0$,

$$\phi_1 = \phi_1^{[0]} + \phi_1^{[1]}\delta + \phi_1^{[2]}\delta^2 + \cdots + \phi_1^{[N]}\delta^N + \cdots, \tag{4}$$

where

$$\phi_1^{[l]} = (\phi_1^{[l]},\varphi_1^{[l]},\chi_1^{[l]},\psi_1^{[l]})^{\mathrm{T}},$$

$$\phi_1^{[l]} = \frac{1}{l!}\frac{\partial^l\phi_1}{\partial\delta^l}|_{\delta=0} \ (l=0,1,2,3\cdots).$$

Thus the generalized DT of Eq. (1) can be defined as the following form:

$$\phi_1[N-1] = \phi_1^{[0]} + \sum_{l=1}^{N-1}\boldsymbol{T}_1[l]\phi_1^{[l]} + \sum_{l=1}^{N-1}\sum_{k=1}^{l-1}\boldsymbol{T}_1[l]\boldsymbol{T}_1[k]\phi_1^{[2]} + \cdots + \boldsymbol{T}_1[N-1]\boldsymbol{T}_1[N-2]\cdots\boldsymbol{T}_1[1]\phi_1^{[N-1]}, \tag{5}$$

$$\phi[N] = \boldsymbol{T}[N]\boldsymbol{T}[N-1]\cdots\boldsymbol{T}[1]\phi, \quad \boldsymbol{T}[l] = \lambda\boldsymbol{I} - \boldsymbol{H}[l-1]\boldsymbol{\Lambda}_l\boldsymbol{H}[l-1]^{-1}, \tag{6}$$

$$q_1[N] = q_1[N-1] + \frac{3\mathrm{i}(\lambda_1^*-\lambda_1)\phi_1[N-1]\varphi_1[N-1]^*}{|\phi_1[N-1]|^2+|\varphi_1[N-1]|^2+|\chi_1[N-1]|^2+|\psi_1[N-1]|^2}, \tag{7}$$

$$q_2[N] = q_2[N-1] + \frac{3\mathrm{i}(\lambda_1^*-\lambda_1)\phi_1[N-1]\chi_1[N-1]^*}{|\phi_1[N-1]|^2+|\varphi_1[N-1]|^2+|\chi_1[N-1]|^2+|\psi_1[N-1]|^2}, \tag{8}$$

$$q_3[N] = q_3[N-1] + \frac{3\mathrm{i}(\lambda_1^*-\lambda_1)\phi_1[N-1]\psi_1[N-1]^*}{|\phi_1[N-1]|^2+|\varphi_1[N-1]|^2+|\chi_1[N-1]|^2+|\psi_1[N-1]|^2}. \tag{9}$$

Here

$$(\phi_1[N-1],\varphi_1[N-1],\chi_1[N-1],\psi_1[N-1])^{\mathrm{T}} = \phi_1[N-1], \quad \boldsymbol{T}_k[l] = \lambda_k\boldsymbol{I} - \boldsymbol{H}[l-1]\boldsymbol{\Lambda}_l\boldsymbol{H}[l-1]^{-1},$$

$$\boldsymbol{H}[l-1] = \begin{bmatrix} \phi_1[l-1] & \varphi_1[l-1]^* & \psi_1[l-1]^* & 0 \\ \varphi_1[l-1] & -\phi_1[l-1]^* & 0 & 0 \\ \chi_1[l-1] & 0 & 0 & \psi_1[l-1]^* \\ \psi_1[l-1] & 0 & -\phi_1[l-1]^* & -\chi_1[l-1]^* \end{bmatrix}, \quad \boldsymbol{\Lambda}_l = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1^* & 0 & 0 \\ 0 & 0 & \lambda_1^* & 0 \\ 0 & 0 & 0 & \lambda_1^* \end{bmatrix}, 1 \leqslant l \leqslant N.$$

Here, $\boldsymbol{I}$ is the 4×4 identity matrix. We can see that Eqs. (7)–(9) give rise to the N-order localized waves solutions of Eq. (1). If we iterative above procedures, some higher-order localized waves solutions may be obtained. Certainly, the determinant representation of the high-order localized wave solutions can be derived by Crum theorem.[31] In order to avoid cumbersome calculation of determinant of high order matrix, we prefer to iterative the DT of degree one. Besides, it is very convenient to figure out these expressions through some computer softwares.[32] By choosing appropriate eigenfunction ϕ_1, we can get some interesting localized waves solutions of Eq. (1) by the above formulas.

3. Localized waves solutions

We begin with the nontrivial seed solution of Eq. (1)

$$q_1[0]=d_1\,\mathrm{e}^{\mathrm{i}\theta},\ q_2[0]=d_2\,\mathrm{e}^{\mathrm{i}\theta},\ q_3[0]=d_3\,\mathrm{e}^{\mathrm{i}\theta}. \tag{10}$$

Here, $\theta=2(d_1^2+d_2^2+d_3^2)t$, and d_1,d_2,d_3 are three arbitrary real constants, which denote the backgrounds where nonlinear localized waves emerge. For convenience, we choose the seed solutions as periodic plane waves without independent variable x. Then the special vector solution of Lax pair of Eq. (1) with λ at $q_1[0]$, $q_2[0]$, and $q_3[0]$ can be written as

$$\phi_1 = \begin{pmatrix} (c_1\,\mathrm{e}^{M_1+M_2}-c_2\,\mathrm{e}^{M_1-M_2})\,\mathrm{e}^{\frac{\mathrm{i}\theta}{2}} \\ \rho_1(c_1\,\mathrm{e}^{M_1-M_2}-c_2\,\mathrm{e}^{M_1+M_2})\,\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}-(\alpha d_2+\beta d_3)\,\mathrm{e}^{M_3} \\ \rho_2(c_1\,\mathrm{e}^{M_1-M_2}-c_2\,\mathrm{e}^{M_1+M_2})\,\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}+\alpha d_1\,\mathrm{e}^{M_3} \\ \rho_3(c_1\,\mathrm{e}^{M_1-M_2}-c_2\,\mathrm{e}^{M_1+M_2})\,\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}+\beta d_1\,\mathrm{e}^{M_3} \end{pmatrix}, \tag{11}$$

where

$$\begin{aligned}
c_1 &= \frac{\left(3\lambda-\sqrt{9\lambda^2+4(d_1^2+d_2^2+d_3^2)}\right)^{\frac{1}{2}}}{\sqrt{9\lambda^2+4(d_1^2+d_2^2+d_3^2)}},\\
c_2 &= \frac{\left(3\lambda+\sqrt{9\lambda^2+4(d_1^2+d_2^2+d_3^2)}\right)^{\frac{1}{2}}}{\sqrt{9\lambda^2+4(d_1^2+d_2^2+d_3^2)}},\\
\rho_1 &= \frac{d_1}{\sqrt{d_1^2+d_2^2+d_3^2}},\ \rho_2=\frac{d_2}{\sqrt{d_1^2+d_2^2+d_3^2}},\\
\rho_3 &= \frac{d_3}{\sqrt{d_1^2+d_2^2+d_3^2}},\ M_1=-\frac{\mathrm{i}}{2}\lambda(x+3tx),\\
M_2 &= \frac{\mathrm{i}}{2}\sqrt{9\lambda^2+4(d_1^2+d_2^2+d_3^2)}\left(x+3tx+\sum_{k=1}^{N}s_kf^{2k}\right),\\
M_3 &= \mathrm{i}\lambda(x+3\lambda t).
\end{aligned}$$

Here $s_k=m_k+\mathrm{i}n_k$, and $\alpha,\beta,m_k,n_k (1\leqslant k\leqslant N)$ are real free parameters. Let $\tau=d_1^2+d_2^2+d_3^2$ and $\lambda=2\sqrt{\tau}\mathrm{i}(1+f^2)/3$ with a small parameter f. So we can expand the vector function $\phi_1(f)$ at $f=0$ as[8]

$$\phi_1(f)=\phi_1^{[0]}+\phi_1^{[1]}f^2+\phi_1^{[2]}f^4+\phi_1^{[3]}f^6+\cdots, \tag{12}$$

where

$$\begin{aligned}
\phi_1^{[0]} &= \frac{1}{2}\frac{(-1+\mathrm{i})(4\mathrm{i}\tau t+2\sqrt{\tau}x+1)}{\tau^{\frac{1}{4}}}\,\mathrm{e}^{\xi_1}, && (13)\\
\varphi_1^{[0]} &= -(\alpha d_2+\beta d_3)\,\mathrm{e}^{\xi_2}\\
&\quad -\frac{1}{2}\frac{(1+\mathrm{i})(2\mathrm{i}\sqrt{\tau}x-4\tau t-\mathrm{i})d_1}{\tau^{\frac{3}{4}}}\,\mathrm{e}^{\xi_3}, && (14)\\
\chi_1^{[0]} &= -\frac{1}{2}\frac{(1+\mathrm{i})(2\mathrm{i}\sqrt{\tau}x-4\tau t-\mathrm{i})d_2}{\tau^{\frac{3}{4}}}\,\mathrm{e}^{\xi_3}+\alpha d_1\,\mathrm{e}^{\xi_2}, && (15)\\
\psi_1^{[0]} &= -\frac{1}{2}\frac{(1+\mathrm{i})(2\mathrm{i}\sqrt{\tau}x-4\tau t-\mathrm{i})d_3}{\tau^{\frac{3}{4}}}\,\mathrm{e}^{\xi_3}+\beta d_1\,\mathrm{e}^{\xi_2}, && (16)\\
\phi_1^{[1]} &= \tau^{-\frac{1}{4}}\Bigg[\frac{1}{24}(1-\mathrm{i})\Big(96\tau^{\frac{5}{2}}t^2x+64\mathrm{i}\tau^3t^3-96\mathrm{i}\tau^{\frac{3}{2}}tx\\
&\quad -8\tau^{\frac{3}{2}}x^3-48\mathrm{i}\tau^2tx^2+112\tau^2t^2-24\mathrm{i}\sqrt{\tau}n_1-76\mathrm{i}\tau t\\
&\quad -20\tau x^2-10\sqrt{\tau}x-24\sqrt{\tau}m_1+3\Big)\Bigg]\,\mathrm{e}^{\xi_1}, && (17)\\
\varphi_1^{[1]} &= \frac{1}{3}(8\mathrm{i}\alpha t\tau d_2+8\mathrm{i}\beta t\tau d_3+2\alpha x\sqrt{\tau}d_2\\
&\quad +2\beta x\sqrt{\tau}d_3)\,\mathrm{e}^{\xi_2}+\Omega d_1\,\mathrm{e}^{\xi_3}, && (18)\\
\chi_1^{[1]} &= -\frac{2}{3}\alpha d_1(4\mathrm{i}\tau t+\sqrt{\tau}x)\,\mathrm{e}^{\xi_2}+\Omega d_2\,\mathrm{e}^{\xi_3}, && (19)\\
\psi_1^{[1]} &= -\frac{2}{3}\beta d_1(4\mathrm{i}\tau t+\sqrt{\tau}x)\,\mathrm{e}^{\xi_2}+\Omega d_3\,\mathrm{e}^{\xi_3}, && (20)\\
&\quad \cdots\cdots
\end{aligned}$$

with

$$\begin{aligned}
\xi_1 &= \frac{5}{3}\mathrm{i}\tau t+\frac{1}{3}\sqrt{\tau}x,\\
\xi_2 &= -\frac{2}{3}\sqrt{\tau}(2\mathrm{i}\sqrt{\tau}t+x),\\
\xi_3 &= -\frac{1}{3}\mathrm{i}\tau t+\frac{1}{3}\sqrt{\tau}x,\\
\Omega &= \frac{1}{24}\frac{1}{\tau^{\frac{3}{4}}}(1+\mathrm{i})(96\mathrm{i}\tau^{\frac{5}{2}}t^2x-8\mathrm{i}\tau^{\frac{3}{2}}x^3-64\tau^3t^3+16\mathrm{i}\tau^2t^2\\
&\quad +48\tau^2tx^2+4\mathrm{i}\tau x^2-2\mathrm{i}\sqrt{\tau}x-24\mathrm{i}\sqrt{\tau}m_1\\
&\quad +24\sqrt{\tau}n_1+44\tau t-3\mathrm{i}).
\end{aligned}$$

Here $\phi_1^{[k]}=(\phi_1^{[k]},\varphi_1^{[k]},\chi_1^{[k]},\psi_1^{[k]})^{\mathrm{T}}$ $(1\leqslant k\leqslant N)$. It is straightforward to calculate that the vector function $\phi_1^{[0]}$ is a solution of the Lax pair Eq. (1) at $q_1=q_1[0],q_2=q_2[0],q_3=q_3[0]$, and $\lambda=\lambda_1=2\mathrm{i}\sqrt{\tau}/3$. Hence, by using Eqs. (7)–(9), we can arrive at

$$q_1[1]=d_1\,\mathrm{e}^{\mathrm{i}\theta}+\frac{4\tau F_1d_1\,\mathrm{e}^{\mathrm{i}\theta}+4\tau^{\frac{7}{4}}G_1\,\mathrm{e}^{3\mathrm{i}\tau t-\sqrt{\tau}x}}{D_1+2\tau^{\frac{3}{2}}D_2\,\mathrm{e}^{-2\sqrt{\tau}x}}, \tag{21}$$

$$q_2[1]=d_2\,\mathrm{e}^{\mathrm{i}\theta}+\frac{4\tau d_2F_1\,\mathrm{e}^{\mathrm{i}\theta}+4\tau^{\frac{7}{4}}\alpha G_2\,\mathrm{e}^{3\mathrm{i}\tau t-\sqrt{\tau}x}}{D_1+2\tau^{\frac{3}{2}}D_2\,\mathrm{e}^{-2\sqrt{\tau}x}}, \tag{22}$$

$$q_3[1]=d_3\,\mathrm{e}^{\mathrm{i}\theta}+\frac{4\tau d_3F_1\,\mathrm{e}^{\mathrm{i}\theta}+4\beta\tau^{\frac{7}{4}}G_2\,\mathrm{e}^{3\mathrm{i}\tau t-\sqrt{\tau}x}}{D_1+2\tau^{\frac{3}{2}}D_2\,\mathrm{e}^{-2\sqrt{\tau}x}}, \tag{23}$$

where

$$\begin{aligned}
F_1 &= -16t^2\tau^2+8\mathrm{i}t\tau-4\tau x^2+1,\\
G_1 &= (1-\mathrm{i})(4\mathrm{i}\tau t+2\sqrt{\tau}x+1)(\alpha d_2+\beta d_3),\\
D_1 &= 16\tau^3t^2+4(4t^2d_1^2+4t^2d_2^2+4t^2d_3^2+x^2)\tau^2\\
&\quad +4\tau^{\frac{3}{2}}x+(4x^2d_1^2+4x^2d_2^2+4x^2d_3^2+1)\tau
\end{aligned}$$

$$
\begin{aligned}
& -4x(d_1^2+d_2^2+d_3^2)\sqrt{\tau}+d_1^2+d_2^2+d_3^2,\\
D_2 &= \alpha^2(d_1^2+d_2^2)+\beta^2(d_1^2+d_3^2)+2\alpha\beta d_2 d_3,\\
G_2 &= -4td_1(1+\mathrm{i})\tau+2xd_1(\mathrm{i}-1)\sqrt{\tau}+d_1(\mathrm{i}-1).
\end{aligned}
$$

The validity of Eqs. (21)–(23) can be directly verified by putting them back into Eq. (1). At this point, we get first-order localized wave solutions of Eq. (1) with two free parameters α and β, which play important role in controlling the dynamics of these localized waves. Next, we discuss the dynamics of these solutions through three different cases.

(i) When $\alpha = 0$ and $\beta = 0$, q_1, q_2, and q_3 are proportional to each other, and they are first-order rogue waves. We can find that these solutions shown in Fig. 1 are similar to the standard NLS equation.

(ii) When $\alpha \neq 0$, $\beta \neq 0$, $d_1 \neq 0$, and $d_2 = d_3 = 0$, the first-order dark–bright–rogue wave solution can be gained. In Fig. 2, we can see the interaction between first-order rogue wave and one-dark, one-bright soliton respetively. Figure 2(a) reveals that q_1 component has the first-order one-dark-rogue wave solution. Meanwhile, q_2 and q_3 both have similar structure, i.e., first-order one-bright-rogue wave solution, which can be seen in Figs. 2(b) and 2(c). The maximum amplitude of first-order rogue wave in Figs. 2(b) and 2(c) are very small, because they appear at the zero-amplitude background crest and are very difficult to observe.

(iii) When $\alpha \neq 0$, $\beta \neq 0$, $d_1 \neq 0$, $d_2 \neq 0$, and $d_3 \neq 0$, the first-order one-breather–rogue wave solution can be obtained. In Fig. 3, we can observe that q_1, q_2, and q_3 have the similar solution structure.

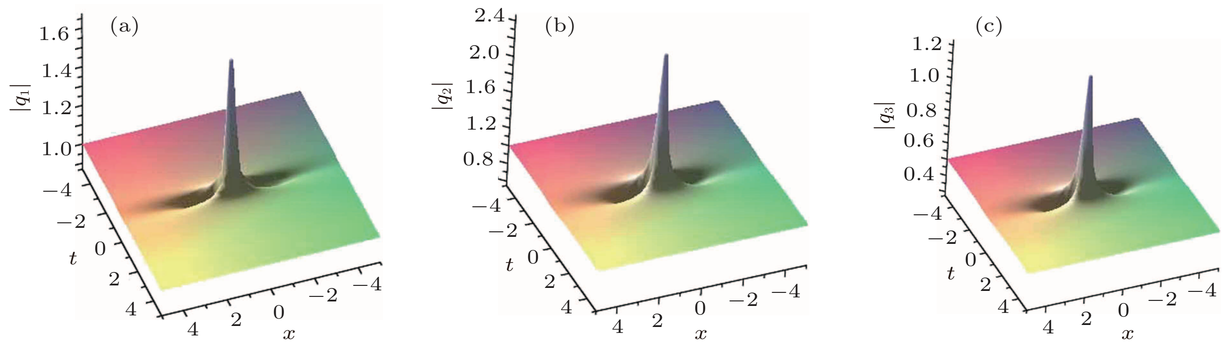

Fig. 1. (color online) Evolution of the first-order rogue wave of the three-component coupled NLS equation by choosing $d_1 = 1$, $d_2 = -1$, and $d_3 = -0.5$. Three component q_1 (a), q_2 (b) and q_3 (c) have the similar structure.

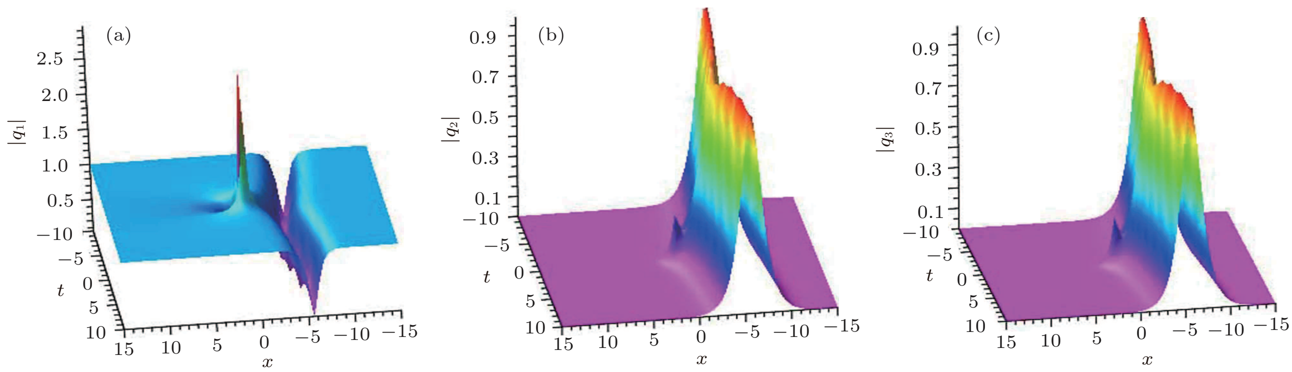

Fig. 2. (color online) Evolution of the first-order dark–bright–rogue wave of the three-component NLS equation by choosing $\alpha = 1/10, \beta = -1/10$, $d_1 = 1$, and $d_2 = d_3 = 0$.

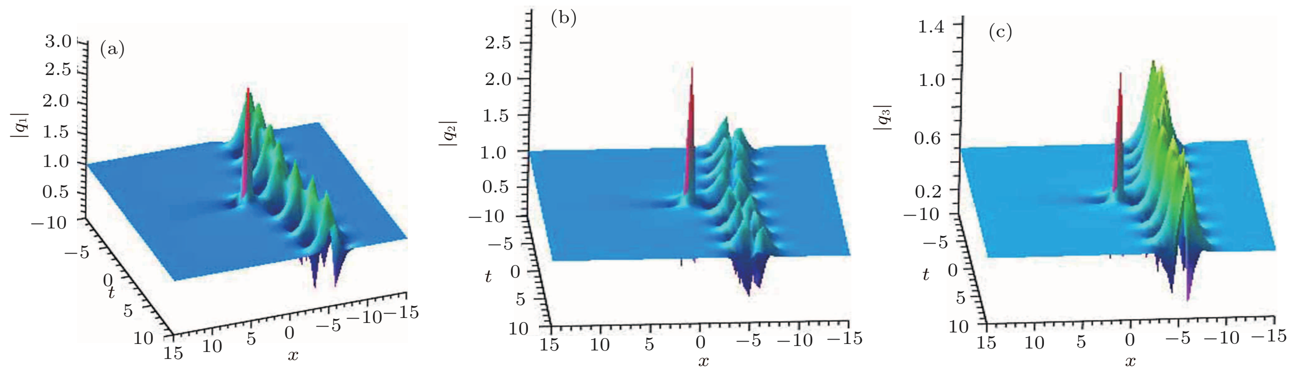

Fig. 3. (color online) Evolution of the first-order breather–rogue wave of the three-component NLS equation by choosing $\alpha = 1/100, \beta = -1/100$, $d_1 = 1$, $d_2 = 1$, and $d_3 = -0.5$.

Here, we give a classification about values of parameters α, β, and d_i $(i=1,2,3)$ corresponding to different types of first-order local wave solutions.

Case 1: When $\alpha \neq 0$ and $\beta \neq 0$, the solutions $q_i (i = 1,2,3)$ are all first-order rogue wave (RW).

Case 2: One of these two parameters α and β is 0, for convenience, we consider the case of $\alpha = 0, \beta \neq 0$. The classification is shown in Table 1.

Table 1. Classification of first-order local wave solutions generated by the first-step generalized DT.

d_i	q_1	q_2	q_3
$d_1 \neq 0, d_2 = d_3 = 0$	RW and one-dark soliton	0	RW and one-bright soliton
$d_1 = 0, d_2 \neq 0, d_3 = 0$	0	RW	0
$d_1 = 0, d_2 = 0, d_3 \neq 0$	RW and one-bright soliton	0	RW and one-dark soliton
$d_1 \neq 0, d_2 \neq 0, d_3 = 0$	RW and one-dark soliton	RW and one-dark soliton	RW and one-bright soliton
$d_1 \neq 0, d_2 = 0, d_3 \neq 0$	RW and one-breather	0	RW and one-breather
$d_1 = 0, d_2 \neq 0, d_3 \neq 0$	RW and one-bright soliton	RW and one-dark soliton	RW and one-dark soliton
$d_1 \neq 0, d_2 \neq 0, d_3 \neq 0$	RW and one-dark soliton	RW and one-breather	RW and one-breather

Case 3: When $\alpha \neq 0$ and $\beta \neq 0$, the classification is shown in Table 2.

Table 2. Classification of first-order local wave solutions generated by the first-step generalized DT.

d_i	q_1	q_2	q_3
$d_1 \neq 0, d_2 = d_3 = 0$	RW and one-dark soliton	RW and one-bright soliton	RW and one-bright soliton
$d_1 = 0, d_2 \neq 0, d_3 = 0$	RW and one-bright soliton	RW and one-dark soliton	RW and one-bright soliton
$d_1 = 0, d_2 = 0, d_3 \neq 0$	RW and one-bright soliton	0	RW and one-dark soliton
$d_1 \neq 0, d_2 \neq 0, d_3 = 0$	RW and one-breather	RW and one-breather	RW and one-bright soliton
$d_1 \neq 0, d_2 = 0, d_3 \neq 0$	RW and one-breather	RW and one-bright soliton	RW and one-breather
$d_1 = 0, d_2 \neq 0, d_3 \neq 0$	RW and one-bright soliton	RW and one-dark soliton	RW and one-dark soliton
$d_1 \neq 0, d_2 \neq 0, d_3 \neq 0$	RW and one-breather	RW and one-breather	RW and one-breather

Next, we consider the following limit:

$$\begin{aligned}
&\lim_{f\to 0} \frac{\boldsymbol{T}[1]|_{\lambda=\frac{2\mathrm{i}}{3}\sqrt{\tau}(1+f^2)}\phi_1}{f^2} \\
&= \lim_{f\to 0} \frac{(\frac{2\mathrm{i}}{3}\sqrt{\tau}f^2 + \boldsymbol{T}_1[1])\phi_1}{f^2} \\
&= \frac{2\mathrm{i}}{3}\sqrt{\tau}\phi_1^{[0]} + \boldsymbol{T}_1[1]\phi_1^{[1]} \equiv \phi_1[1],
\end{aligned} \tag{24}$$

$$\begin{aligned}
&\boldsymbol{T}_1[1] \\
&= \lambda_1 \boldsymbol{I} - \boldsymbol{H}[0]\boldsymbol{\Lambda}_1 H[0]^{-1} \\
&= (\lambda_1 - \lambda_1^*)\left(\boldsymbol{I} - \frac{\phi_1^{[0]}\phi_1^{[0]\dagger}}{\phi_1^{[0]\dagger}\phi_1^{[0]}}\right) \\
&= \frac{4\mathrm{i}}{3}\sqrt{\tau}\left(\boldsymbol{I} - \frac{\phi_1^{[0]}\phi_1^{[0]\dagger}}{\phi_1^{[0]\dagger}\phi_1^{[0]}}\right),
\end{aligned} \tag{25}$$

where † denotes the transposition and conjugation of a matrix (vector), and $\phi_1^{[1]} = \partial^2\phi_1/\partial f^2|_{f=0}$. We can obtain a special solution of the Lax pair (2) and (3) with $q_1[1]$, $q_2[1]$, $q_3[1]$, and $\lambda = \lambda_1 = 2\mathrm{i}\sqrt{\tau}/3$. Using Eqs. (7)–(9), the explicit expressions of second-order localized wave solutions can be figured out. Considering the complexity of the explicit expressions of $q_1[2], q_2[2]$, and $q_3[2]$, we only give their expressions in the simplest case of $\alpha = \beta = 0$. For the case of $\alpha \neq 0$ and $\beta \neq 0$, we omit writing down these expressions since they are rather cumbersome. Besides, it isn't difficult to verify the validity of these solutions $q_1[2], q_2[2]$, and $q_3[2]$ by putting them into Eq. (1) using Maple.

(i) $\alpha = \beta = 0$. Letting $d_1 = 1, d_2 = -1, d_3 = 2, m_1 = 100$, and $n_1 = 0$, we can get

$$q_1[2] = 3\,\mathrm{e}^{12\mathrm{i}t}\frac{32\mathrm{i}t - 192t^2 - 8x^2 + 1}{576t^2 + 24x^2 + 1} + \frac{\mathrm{i}p_1 + p_2}{r}\mathrm{e}^{12\mathrm{i}t}, \tag{26}$$

$$q_2[2] = -3\,\mathrm{e}^{12\mathrm{i}t}\frac{32\mathrm{i}t - 192t^2 - 8x^2 + 1}{576t^2 + 24x^2 + 1} - \frac{\mathrm{i}p_1 + p_2}{r}\mathrm{e}^{12\mathrm{i}t}, \tag{27}$$

$$q_3[2] = 6\,\mathrm{e}^{12\mathrm{i}t}\frac{32\mathrm{i}t - 192t^2 - 8x^2 + 1}{576t^2 + 24x^2 + 1} + 2\frac{\mathrm{i}p_1 + p_2}{r}\mathrm{e}^{12\mathrm{i}t}, \tag{28}$$

$$\begin{aligned}
r = {} & \sqrt{6}[10616832x^{10} + 1019215872x^8t^2 + 38220595200x^6t^4 + 697143656448x^4t^6 \\
& + 6164217593856x^2t^8 + 21134460321792t^{10} + 5308416\sqrt{6}x^9 + 509607936\sqrt{6}x^7t^2 \\
& + 15543042048\sqrt{6}x^5t^4 + 183458856960\sqrt{6}x^3t^6 + 733835427840\sqrt{6}xt^8 + 3096576x^8 \\
& + 1130692608x^6t^2 + 24524881920x^4t^4 + 165112971264x^2t^6 + 330225942528t^8 + (1061683200 \\
& - 884736\sqrt{6})x^7 + (98205696\sqrt{6} - 50960793600)x^5t^2 + (3163815936\sqrt{6} - 1681706188800)x^3t^4 \\
& + (20384317440\sqrt{6} - 11007531417600)xt^6 + (221184000\sqrt{6} - 511488)x^6 - (530841600\sqrt{6} \\
& + 14929920)x^4t^2 + (578617344 - 140142182400\sqrt{6})x^2t^4 + (12867600384 - 1223059046400\sqrt{6})t^6 \\
& - (154828800 + 11520\sqrt{6})x^5 - (6598656\sqrt{6} + 530841600)x^3t^2 + (7962624\sqrt{6} + 31850496000)xt^4
\end{aligned}$$

$$
\begin{aligned}
&+(26542106688-15667200\sqrt{6})x^4-(5990400+2211839552\sqrt{6})x^3-(26541778944\sqrt{6}\\
&+254361600)xt^2+(552962744-192000\sqrt{6})x^2+(13271120832-14745600\sqrt{6})t^2+(288000\\
&+46079916\sqrt{6})x+11520039-9600\sqrt{6}],\\
p_1 =\ & 192\sqrt{6}t[-36691771392t^8-7644119040t^6x^2-573308928t^4x^4-18579456t^2x^6-221184x^8\\
&+92160\sqrt{6}x^7+5750784\sqrt{6}t^2x^5+116785152\sqrt{6}t^4x^3+764411904\sqrt{6}t^6x-32256x^6\\
&+15261696x^4t^2+321159168x^2t^4+1528823808t^6+(44236800+768\sqrt{6})x^5+(2617344\sqrt{6}\\
&+530841600)x^3t^2+34062336\sqrt{6}t^4x+(5529600\sqrt{6}-2496)x^4+(594432-66355200\sqrt{6})t^2x^2\\
&+(12165120-2123366400\sqrt{6})t^4+2144\sqrt{6}x^3+(2211840-60672\sqrt{6})xt^2+(276480024\\
&-422400\sqrt{6})x^2+(331775078-5529600\sqrt{6})t^2-(38400+23039948\sqrt{6})x-3200\sqrt{6}-5759959],\\
p_2 =\ & -3522410053632t^8x-733835427840t^6x^3-55037657088t^4x^5-1783627776t^2x^7\\
&-21233664x^9-1769472\sqrt{6}x^8-488374272\sqrt{6}x^6t^2-24970788864\sqrt{6}x^4t^4-452531847168\sqrt{6}x^2t^6\\
&-2641807540224\sqrt{6}t^8+6635520x^7+424673280x^5t^2-183458856960t^6x-13504610304t^4x^3\\
&+(626688\sqrt{6}-530841600)x^6+(157483008\sqrt{6}-82811289600)x^4t^2+3078881280\sqrt{6}x^2t^4\\
&+11007531417600t^6+(165888+398131200\sqrt{6})x^5+(87146496+5308416000\sqrt{6})x^3t^2\\
&+(1571291136-50960793600\sqrt{6})t^4x-(44544\sqrt{6}+11059200)x^4+(2985984\sqrt{6}\\
&-1061683200)x^2t^2+(104398848\sqrt{6}-70071091200)t^4+(26542022400+9216000\sqrt{6})x^3\\
&+(76947200\sqrt{6}+318500204544)xt^2+(1105921728\sqrt{6}-16588800)x^2+(53083975680\sqrt{6}\\
&-254361600)t^2-(552961632+153600\sqrt{6})x-172800+72\sqrt{6}.
\end{aligned}
$$

Here, $q_2[2]$ and $q_1[2]$ are opposite, and $q_3[2]$ is the double of $q_1[2]$. Thus, they are the second-order rogue wave of Eq. (1). Besides,we can find that these three components q_1, q_2, and q_3 have the similar structure from Fig. 4.

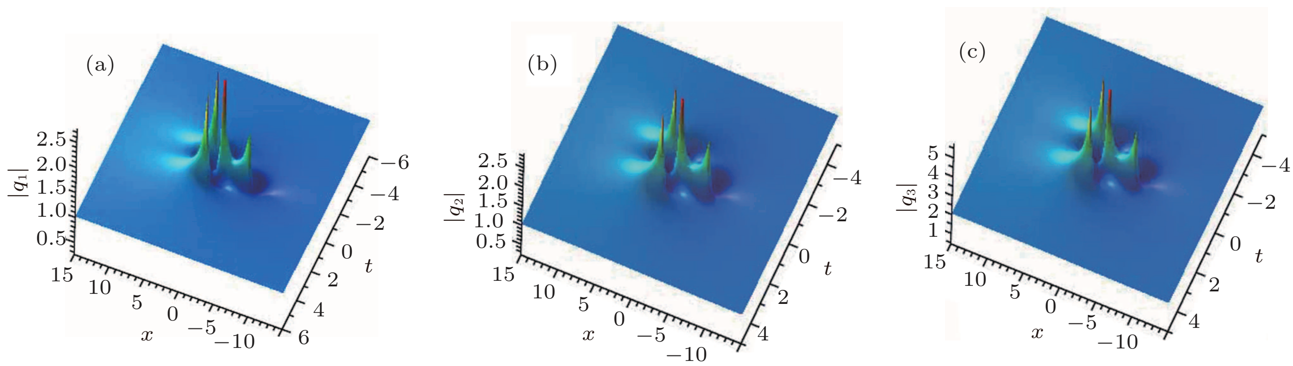

Fig. 4. (color online) Evolution of the second-order rogue wave of the three-component NLS equation by choosing $\alpha=0, \beta=0$, $d_1=1, d_2=-1, d_3=2$, $m_1=100$, and $n_1=0$.

(ii) $\alpha \neq 0, \beta \neq 0$, $d_1 \neq 0$, and $d_2=d_3=0$. We can get the interactional solution between second-order rogue wave and dark–bright soliton. Furthermore, we can observe that one-dark-one-bright soliton and two-bright soliton together with the second-order rogue wave of fundamental pattern and triangular pattern present in the second-order localized wave solution respectively in Figs. 5 and 6. Figures 5(a) and 6(a) show that q_1 component is an interactional solution between one-dark-one-bright soliton and second-order rogue wave.

The solution q_1 here is greatly distinct from that in Ref. [22], which is an interactional solution between two-dark soliton and second-order rogue wave. Owing to the zero-amplitude background crest, homoplastically, it is difficult to observe the rogue wave in q_2 and q_3 component. Besides, in the expressions of this solutions, α and β are greatly important free parameters. In Fig. 7, with increasing the absolute values of α and β, we can find that the dark–bright soliton merge with the rogue wave distinctly.

(iii) $\alpha \neq 0, \beta \neq 0$, $d_1 \neq 0, d_2 \neq 0$ and $d_3 \neq 0$. The interactional solutions[33] between one-breather and second-order rogue wave of triangular pattern can be obtained. q_1 component clearly shows that the triangular pattern rogue wave and breather coexist in Fig. 8(a). But, in Figs. 8(b) and 8(c), the triangular pattern is not obvious. Analogously, if we increase the absolute values of α and β, we can also see the breather solution and the rogue wave merge with each other distinctly. Here, we do not give the figures in this case.

In Eq. (11), there are two important parameters α and β. In Ref. [22], except for the parameters $d[1]$ and $d[2]$, there is only one parameter α. Thus these two parameters, α and β here, determine greatly different structures of localized waves in three-component NLS equation.

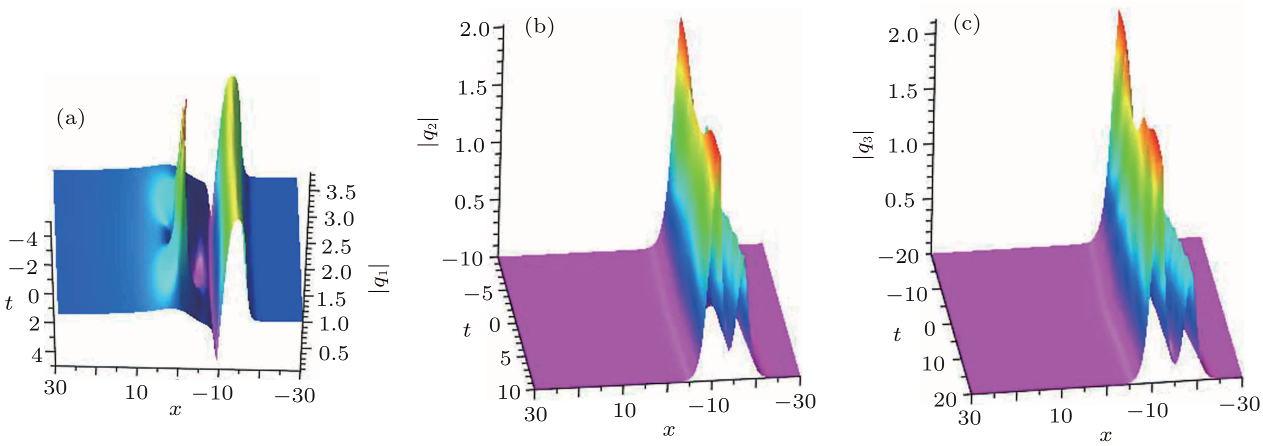

Fig. 5. (color online) Evolution of the second-order dark–bright–rogue wave of the three-component NLS equation by choosing $\alpha = 1/1000, \beta = -1/1000$, $d_1 = 1, d_2 = 0, d_3 = 0$, $m_1 = 0$, and $n_1 = 0$.

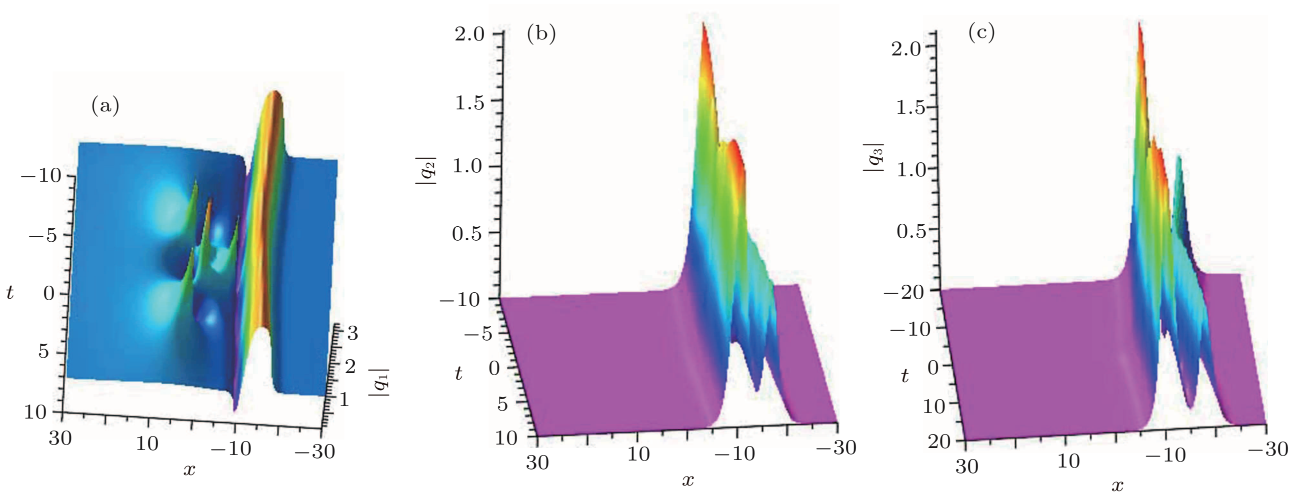

Fig. 6. (color online) Evolution of the second-order dark–bright–rogue wave of the three-component NLS equation by choosing $\alpha = 1/1000, \beta = -1/1000$, $d_1 = 1, d_2 = 0, d_3 = 0$, $m_1 = 100$, and $n_1 = 0$.

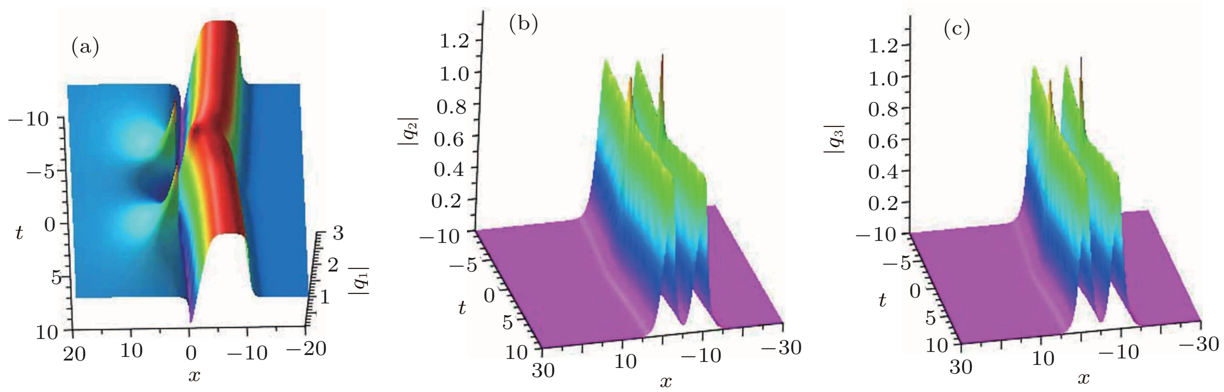

Fig. 7. (color online) Evolution of the second-order dark–bright–rogue wave of the three-component NLS equation by choosing $\alpha = 10, \beta = -10$, $d_1 = 1, d_2 = 0, d_3 = 0$, $m_1 = 100$, and $n_1 = 0$.

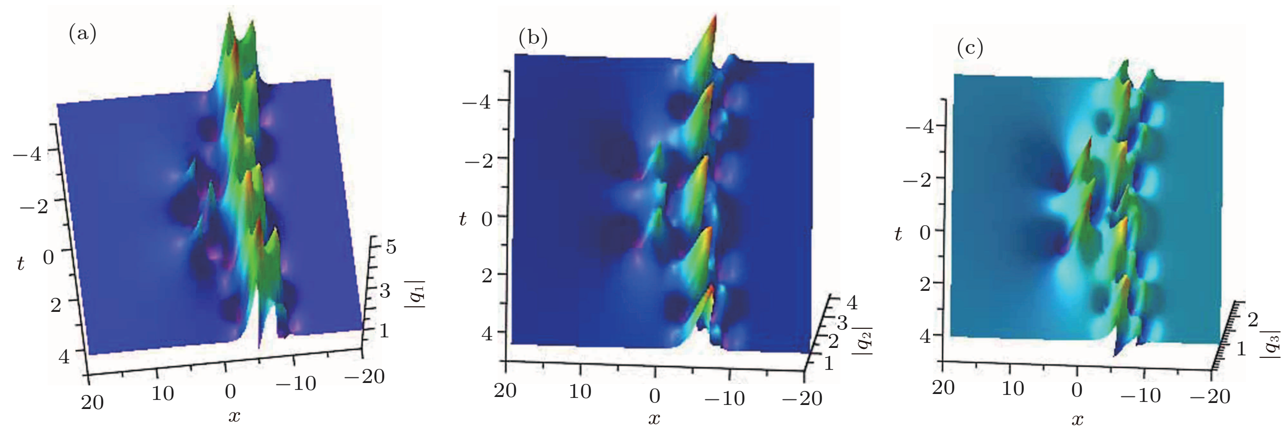

Fig. 8. (color online) Evolution of the second-order breather–rogue wave of the three-component NLS equation by choosing $\alpha = 1/100, \beta = -1/100$, $d_1 = 1, d_2 = -1, d_3 = -1$, $m_1 = 100$, and $n_1 = 0$.

4. Conclusion

We give some interesting localized waves of three-component NLS equation by the generalized Darboux transformation. With a fixed spectral parameter and a special vector solution of Lax pair of Eqs. (2) and (3), we apply the Taylor series expansion to Eq. (1), and give the generalized Darboux transformation.[34] Applying the formula (7)–(9),[35] the interactions between rogue wave and some nonlinear waves (dark, bright solitions and breather) are obtained. In the expressions of these solutions, some parameters play an important role in dynamic properties, such as α, β, $d_i (i = 1, 2, 3)$, and s_1.

We mainly discuss the dynamics of these solutions through three different cases. (i) When $\alpha = 0$ and $\beta = 0$, the first- and second-order rogue wave are given, which are similar to one-component and two-component NLS equation. (ii) When $\alpha \neq 0, \beta \neq 0$, $d_1 \neq 0$, $d_2 = 0$, and $d_3 = 0$, the first-order one-dark-rogue and one-bright-rogue wave can be gained. Meanwhile, the second-order one-dark-one-bright-rogue wave and two-bright-rogue wave are also presented. The parameter s_1 determines the shape of rogue wave, such as fundamental pattern and triangular pattern. (iii) When $\alpha \neq 0, \beta \neq 0$, and $d_i \neq 0$ $(i = 1, 2, 3)$, the first- and second-order one-breather–rogue wave are observed.[36] With increasing the absolute values of α and β, we can observe that rogue wave and those other nonlinear waves merge distinctly.

The localized waves of three-component coupled NLS equation are not absolutely identical with ones of two-component coupled NLS equation.[22] Second-order one-dark-one-bright-rogue wave can be obtained in $q[1]$ component, instead of second-order two-dark-rogue wave in the two-component case. Furthermore, we get second-order rogue wave which contains four fundamental ones and this type of rogue wave interacts with one-dark-one-bright soliton, which is different with the case of two-component. We can only get one-breather–rogue wave solution, which is not the two-breather–rogue wave ones in two-component NLS equation. Through considering both two-component and three-component NLS equation, we may well understand the localized waves of the multi-component NLS equation.[37]

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