Commun. Theor. Phys. **66** (2016) 189–195 Vol. 66, No. 2, August 1, 2016

Nonlocal Symmetry and Interaction Solutions of a Generalized Kadomtsev–Petviashvili Equation*

Li-Li Huang (黄丽丽),[1] Yong Chen (陈勇),[1,†] and Zheng-Yi Ma (马正义)[2]

[1]Shanghai Key Laboratory of Trustworthy Computing, East China Normal University, Shanghai 200062, China

[2]Department of Mathematics, Lishui University, Lishui 323000, China

(Received March 24, 2016; revised manuscript received May 11, 2016)

Abstract *A generalized Kadomtsev–Petviashvili equation is studied by nonlocal symmetry method and consistent Riccati expansion (CRE) method in this paper. Applying the truncated Painlevé analysis to the generalized Kadomtsev–Petviashvili equation, some Bäcklund transformations (BTs) including auto-BT and non-auto-BT are obtained. The auto-BT leads to a nonlocal symmetry which corresponds to the residual of the truncated Painlevé expansion. Then the nonlocal symmetry is localized to the corresponding nonlocal group by introducing two new variables. Further, by applying the Lie point symmetry method to the prolonged system, a new type of finite symmetry transformation is derived. In addition, the generalized Kadomtsev–Petviashvili equation is proved consistent Riccati expansion (CRE) solvable. As a result, the soliton-cnoidal wave interaction solutions of the equation are explicitly given, which are difficult to be found by other traditional methods. Moreover, figures are given out to show the properties of the explicit analytic interaction solutions.*

PACS numbers: 02.30.Ik, 05.45.Yv, 04.20.Jb
Key words: nonlocal symmetry, consistent riccati expansion, Painlevé expansion, soliton-cnoidal wave solution

1 Introduction

With the development of science and technology in modern society, nonlinear science plays a more and more important role both in the science advancement and in our life. As one of the main parts of nonlinear science, the theory of solitons has been applied to many areas of mathematics, fluid physics, micro-physics, solid state physics, condensed matter physics, hydrodynamics, fluid dynamics, cosmology, field theory. To find exact solutions of nonlinear systems is a difficult and tedious but very important and meaningful work. With the development of nonlinear science, many methods have been established by mathematicians and physicists to obtain exact solutions of soliton equations, such as the Inverse Scattering transformation (IST),[1] Bäcklund transformation (BT),[2] Darboux transformation (DT),[3–4] Hirota bilinear method,[5] Painlevé method,[6–7] Lie symmetry method[8–10] and so on.

It is known that Painlevé analysis is one of the best approaches to investigate the integrable property of a given nonlinear evolution equations, and the truncated Painlevé expansion is a straight way to provide some Bäcklund transformations (BTs) including auto-BT and non-auto-BT. Furthermore, it can also be used to obtain nonlocal symmetries and analytic solutions. As the nonlocal symmetries are connected with integrable models and they enlarge the class of symmetries, therefore, to search for nonlocal symmetries of the nonlinear systems is an interesting work. In 1969, Bluman[11] introduced the concept of potential symmetry for a differential system by writting the given system in a conserved form. In 1991, Akhatov and Gazizov[12] provided a method for constructing nonlocal symmetries of differential equations based on the Lie–Bäcklund theory. In 1992, Galas[13] obtained the nonlocal Lie–Bäcklund symmetries by introducing the pesudo-potentials as an auxiliary system. In 1993, Guthrie[14] got nonlocal symmetries with the help of a recursion operator. In 1997, Lou and Hu[15–17] have made some efforts to obtain infinite many nonlocal symmetries by inverse recursion operators, the conformal invariant form (Schwartz form) and Darboux transformation. More recently, in 2012, Lou, Hu, and Chen[18–20] obtained nonlocal symmetries that were related to the Darboux transformation with the Lax pair and Bäcklund transformation. In 2013, Xin and Chen[21] gave a systemic method to find the nonlocal symmetry of nonlin-

*Supported by the Global Change Research Program of China under Grant No. 2015CB953904, National Natural Science Foundation of under Grant Nos. 11275072 and 11435005, Doctoral Program of Higher Education of China under Grant No. 20120076110024, the Network Information Physics Calculation of Basic Research Innovation Research Group of China under Grant No. 61321064, and Shanghai Collaborative Innovation Center of Trustworthy Software for Internet of Things under Grant No. ZF1213, and Zhejiang Provincial Natural Science Foundation of China under Grant No. LY14A010005

†Corresponding author, E-mail: ychen@sei.ecnu.edu.cn

ear evolution equation and improved previous methods to avoid missing some important results such as integral terms or high order derivative terms of nonlocal variables in the symmetries. In 2014, Cheng etc.[22] demonstrated the nonlocal symmetries can be successfully used to discover some types of important interaction solutions. In recent years, it is found that Painlevé analysis can be used to obtain nonlocal symmetries. This type of nonlocal symmetries related to the truncated Painlevé expansion is just the residual of the expansion with respect to singular manifold, and is also called residual symmetry.[23−24] The localization of this type of residual symmetry seems easily performed than that coming from DT and BT. In order to develop some types of relatively simple and understandable methods to construct exact solutions, Lou proposed a consistent Riccati expansion (CRE) method to identify CRE solvable systems in Ref. [25]. A system is defined to be CRE solvable if it has a CRE. It is clear that various integrable systems are CRE solvable, and many systems have been verified having this property, such as the Korteweg de-Vries (KdV) equation, the mKdV equation, the Ablowitz–Kaup–Newell–Segur (AKNS) system, the Kadomtsev–Petviashvili equation, the Sawada–Kortera equation, the Kaup–Kupershmidt equation, the Boussinesq equation, the Sine-Gordon equation, the Burgers equation, the dispersive water wave equation, the modified asymmetric Veselov–Novikov equation. It has been revealed that many similar interaction solutions between a soliton and a cnoidal wave were found in various CRE solvable systems. By this method, recent studies[26−36] have found a lot of intercation solutions in many nonlinear equations.

In the present paper, we focus on nonlocal symmetry, prolonged system, Bäcklund transformation, CRE solvable and exact interaction solutions of a generalized Kadomtsev–Petviashvili equation[37] as follows:

$$u_{xt} + u_{xxxx} + 6uu_{xx} + 6u_x^2 + 3u_{yy} + 3u_{xy} = 0\,, \tag{1}$$

where subscript means a partial derivative such as $u_{xt} = \partial u/\partial t\partial x$.

It is well known that (2+1)-dimensional Kadomtsev–Petviashvili equation can be used to model water waves of long wavelength with weakly nonlinear restoring forces and frequency dispersion. Various of generalization of Kadomtsev–Petviashvili equation[38−40] are proposed. Equation (1) can be transformed to the standard Kadomtsev–Petviashvili equation under transformation

$$x \to x\,,\quad y \to y - 3t\,,\quad t \to t\,. \tag{2}$$

Therefore, Eq. (1) must have abundant physical phenomena, finding more types of solutions of Eq. (1) is interesting to understand the Kadomtsev–Petviashvili equation fully.

This paper is arranged as follows: In Sec. 2, the auto Bäcklund transformation, non-auto Bäcklund transformation and nonlocal symmetry of the generalized Kadomtsev–Petviashvili equation are obtained by the truncated Painlevé expansion approach, then the nonlocal symmetry is localized by introducing another three dependent variables and the corresponding nonlocal transformation group is found. In Sec. 3, some exact solutions are derived via the similarity reductions of the prolonged system. In Sec. 4, the generalized Kadomtsev–Petviashvili equation is verified CRE solvable and the soliton-cnoidal wave solutions are constructed. The last section contains a summary and discussion.

2 Nonlocal Symmetry from the Truncated Painlevé Expansion

For the generalized Kadomtsev–Petviashvili equation (1), there exists a truncated Painlevé expansion

$$u = \frac{u_2}{\phi^2} + \frac{u_1}{\phi} + u_0\,, \tag{3}$$

with u_0, u_1, u_2, ϕ being the functions of x, y and t, the function $\phi(x,y,t) = 0$ is the equation of singularity manifold.

Substituting Eq. (3) into Eq. (1) and balancing all the coefficients of different powers of ϕ, we can get

$$u_2 = -2\phi_x^2\,,\quad u_1 = 2\phi_{xx}\,,$$
$$u_0 = -\frac{\phi_t}{6\phi_x} - \Big(\frac{\phi_y}{2\phi_x}\Big)^2 - \frac{\phi_y}{2\phi_x} + \Big(\frac{\phi_{xx}}{2\phi_x}\Big)^2 - \frac{2\phi_{xxx}}{3\phi_x}\,, \tag{4}$$

and the generalized Kadomtsev–Petviashvili equation (1) is successfully satisfing the following Schwarzian form:

$$(P + S + 3C)_x + 3(CC_x + C_y) = 0\,. \tag{5}$$

Here, we denote

$$P = \frac{\phi_t}{\phi_x}\,,\ S = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\Big(\frac{\phi_{xx}}{\phi_x}\Big)^2\,,\ C = \frac{\phi_y}{\phi_x}\,, \tag{6}$$

where P, C are the usual Schwarzian variables, S is the Schwarzian derivative and both invariant under the Möbious transformation, i.e.,

$$\phi \to \frac{a + b\phi}{c + d\phi}\,,\quad (ad \neq bc)\,. \tag{7}$$

If we take a special case $a = 0$, $b = c = 1$, $d = \epsilon$, then Eq. (7) can be rewritten as:

$$\phi \to \phi - \epsilon\phi^2 + o(\epsilon^2)\,, \tag{8}$$

which means (5) possesses the point symmetry[23]

$$\sigma^\phi = -\phi^2\,. \tag{9}$$

From the standard truncated Painlevé expansion (3), we have the following auto-Bäcklund transformation and non-auto Bäcklund transformation theorem of Eq. (1).

Theorem 1a (auto-BT theorem) If the function ϕ satisfies Eq. (5), then:

$$u = -\frac{2\phi_x^2}{\phi^2} + \frac{2\phi_{xx}}{\phi} + u_0\,, \tag{10}$$

is an auto-BT between the solution u and u_0 of the generalized Kadomtsev–Petviashvili equation (1).

Theorem 1b (non-auto-BT theorem) If the function ϕ satisfies Eq. (5), then:

$$u=-\frac{\phi_t}{6\phi_x}-\Big(\frac{\phi_y}{2\phi_x}\Big)^2-\frac{\phi_y}{2\phi_x}+\Big(\frac{\phi_{xx}}{2\phi_x}\Big)^2-\frac{2\phi_{xxx}}{3\phi_x}\,,\quad(11)$$

is a non-auto-BT between ϕ and the solution u_0 of the generalized Kadomtsev–Petviashvili equation (1).

One knows that the symmetry equation for Eq. (1) reads:

$$\sigma^u_{xt}+\sigma^u_{4x}+6\sigma^u u_{xx}+6u\sigma^u_{xx}+12\sigma^u_x u_x + 3\sigma^u_{yy}+3\sigma^u_{xy}=0\,,\quad(12)$$

where σ^u denotes the symmetry of u, respectively. From the truncated Painlevé expansion (3) and the Theorem 1a and 1b, a new nonlocal symmetry of Eq. (1) is presented and studied as follows.

Theorem 2 Equation (1) has the nonlocal symmetry given by

$$\sigma^u=2\phi_{xx}\,,\quad(13)$$

where u and ϕ satisfy the non-auto BT (11).

Proof The nonlocal symmetry (13) is residual of the singularity manifold ϕ. The nonlocal symmetry (13) will also be obtained with substituting the Möbious transformation symmetry σ^ϕ into the linearized equation (4).

To find out the group of the nonlocal symmetry (13)

$$u\to\bar u(\epsilon)=u+\epsilon\sigma^u\,,\quad(14)$$

we have to solve the following initial value problem

$$\frac{\mathrm{d}\bar u(\epsilon)}{\mathrm{d}\epsilon}=2\bar\phi_{xx}\,,\quad \bar u(\epsilon)\,|_{\epsilon=0}=u\,,\quad(15)$$

with ϵ being the infinitesimal parameter.

However, since it is difficult to solve Eqs. (15) for $\bar u(\epsilon)$ due to the intrusion of the function $\bar\phi(\epsilon)$ and its differentiations, we introduce new variables to eliminate the space derivatives of $\bar\phi(\epsilon)$

$$f=\phi_x\,,\quad g=f_x\,.\quad(16)$$

Now the nonlocal symmetry (13) of the original equation (1) becomes a Lie point symmetry of the prolonged system (1), (11), and (16), saying

$$\begin{pmatrix}\sigma^u\\ \sigma^\phi\\ \sigma^f\\ \sigma^g\end{pmatrix}=\begin{pmatrix}2g\\ -\phi^2\\ -2\phi f\\ -2f^2-2\phi g\end{pmatrix}.\quad(17)$$

The result (17) indicates that the nonlocal symmetries (13) are localized in the properly prolonged system (1), (11), and (16) with the Lie point symmetry vector

$$V=2g\partial_u-\phi^2\partial_\phi-2\phi f\partial_f-2(f^2+\phi g)\partial_g\,.\quad(18)$$

In other words, the symmetries related to the truncated Painlevé expansion are just a special Lie point symmetry of the prolonged system.

Now we have obtained the localized nonlocal symmetries, an interesting question is what kind of finite transformation would correspond to the Lie point symmetry (18). We have the following theorem.

Theorem 3 If $\{u,\phi,f,g\}$ is a solution of the prolonged system (1), (11), and (16), then $\{\bar u,\bar\phi,\bar f,\bar g\}$ is given by

$$\bar\phi=\frac{\phi}{\epsilon\phi+1}\,,\quad \bar f=\frac{f}{(\epsilon\phi+1)^2}\,,$$
$$\bar g=\frac{g}{(\epsilon\phi+1)^2}-\frac{2\epsilon f^2}{(\epsilon\phi+1)^3}\,,$$
$$\bar u=u+\frac{2\epsilon g}{\epsilon\phi+1}-\frac{2\epsilon^2 f^2}{(\epsilon\phi+1)^2}\,,$$

with arbitrary group parameter ϵ.

Proof Using Lie's first theorem on vector (18) with the corresponding initial condition

$$\frac{\mathrm{d}\bar u(\epsilon)}{\mathrm{d}\epsilon}=2\bar g(\epsilon)\,,\quad \bar u(0)=u\,,$$
$$\frac{\mathrm{d}\bar\phi(\epsilon)}{\mathrm{d}\epsilon}=-\bar\phi^2(\epsilon)\,,\quad \bar\phi(0)=\phi\,,$$
$$\frac{\mathrm{d}\bar f(\epsilon)}{\mathrm{d}\epsilon}=-2\bar\phi(\epsilon)\bar f(\epsilon)\,,\quad \bar f(0)=f\,,$$
$$\frac{\mathrm{d}\bar g(\epsilon)}{\mathrm{d}\epsilon}=-2(\bar f^2(\epsilon)+\bar\phi(\epsilon)\bar g(\epsilon))\,,\quad \bar g(0)=g\,.$$

One can easily obtain the solutions of the above equations given in Theorem 3, thus the theorem is proved.

Actually, the above group transformation is equivalent to the truncated Painlevé expansion (3) since the singularity manifold equations (1), (11), and (16) are form invariant under the transformation $1+\epsilon\phi\to\phi$ (with $\epsilon f\to\phi_x, \epsilon g\to\phi_{xx}$).

3 Similarity Reductions with the Nonlocal Symmetries

In this section, we will discuss the symmetry reductions related to the nonlocal symmetries. In order to search for more similarity reductions of Eq. (1), we study Lie point symmetries of the prolonged systems instead of the single Eq. (1). According to the classical Lie point symmetry method, the Lie point symmetries for the whole prolonged systems possess the form

$$\sigma^u=Xu_x+Yu_y+Tu_t-U\,,$$
$$\sigma^\phi=X\phi_x+Y\phi_y+T\phi_t-\Phi\,,$$
$$\sigma^f=Xf_x+Yf_y+Tf_t-F\,,$$
$$\sigma^g=Xg_x+Yg_y+Tg_t-G\,,\quad(19)$$

where X, Y, T, U, Φ, F, G are function of x, y, t, u, ϕ, f, g, which means that the prolonged system (1), (11), and (16) are invariant under the transformations

$$u\to u+\epsilon\sigma^u\,,\quad \phi\to\phi+\epsilon\sigma^\phi\,,$$
$$f\to f+\epsilon\sigma^f\,,\quad g\to g+\epsilon\sigma^g\,,\quad(20)$$

with the infinitesimal parameter ϵ.

The symmetries σ^k $(k = u, \phi, f, g)$ are defined as the solution of the linearized equations of the prolonged systems (1), (11), and (16)

$$\sigma^u_{xt} + \sigma^u_{4x} + 6\sigma^u u_{xx} + 6u\sigma^u_{xx} + 12\sigma^u_x u_x + 3\sigma^u_{yy} + 3\sigma^u_{xy} = 0\,,$$
$$(\sigma^u_{xt} + \sigma^\phi_{xxxx} + 3\sigma^\phi_{yy} + 3\sigma^\phi_{xy})\phi_x^3$$
$$- (3\sigma^\phi_{xx}\phi_y + 4\sigma^\phi_{xxx}\phi_{xx} + \sigma^\phi_t\phi_{xx} + 3\sigma^\phi_y\phi_{xx}$$
$$+ \sigma^\phi_{xx}\phi_t + 4\sigma^\phi_{xx}\phi_{xxx})\phi_x^2 - (3\sigma^\phi_{xx}\phi_y^2$$
$$- 3\sigma^\phi_x\phi_{xx}\phi_y + 6\sigma^\phi_y\phi_{xx}\phi_y - 9\sigma^\phi_{xx}\phi_{xx}^2 - \sigma^\phi_x\phi_{xx}\phi_t$$
$$- 4\sigma^\phi_x\phi_{xx}\phi_{xxx})\phi_x + 6\sigma^\phi_x(\phi_y^2 - \phi_{xx}^2)\phi_{xx} = 0\,,$$
$$\sigma^f - \sigma^\phi_x = 0\,, \quad \sigma^g - \sigma^f_x = 0\,. \tag{21}$$

Substituting the expressions (19) into the symmetry equations (21) and collecting the coefficients of the independent partial derivatives of dependent variables u, ϕ, f, g. Then we obtain a system of overdetermined linear equations for the infinitesimals X, Y, T, U, Φ, F, G, which can be easily given by solving the determining equations

$$X = -\frac{1}{18}f_{1ttt}y^2 + \frac{1}{18}(6x + 3y)f_{1t} - \frac{1}{6}f_{2t}y + f_3\,,$$
$$Y = \frac{2}{3}f_{1t}y + f2\,, \quad T = f_1\,, \quad F = c_1\phi f + c_2 f\,,$$
$$\Phi = \frac{1}{2}c_1\phi^2 + c_2\phi + c_3,\ G = c_1(f^2 + g\phi) + c_2 g\,,$$
$$U = -\frac{1}{108}f_{1ttt}y^2 + \frac{1}{108}(6x - 3y)f_{1tt} - \frac{1}{36}f_{2tt}y$$
$$+ \frac{1}{108}(-72u + 9)f_{1t} + \frac{1}{6}f_{3t} - \frac{1}{12}f_{2t} - c_1 g\,, \tag{22}$$

where $f_1 \equiv f_1(t)$, $f_2 \equiv f_2(t)$, $f_3 \equiv f_3(t)$ are arbitrary functions of t, c_1, c_2, and c_3 are arbitrary constants. When $c_2 = c_3 = f_1 = f_2 = f_3 = 0$ and $c_1 = -2$, the obtained symmetry is just Eq. (17), and when $c_1 = 0$, the related symmetry is only the general Lie point symmetry of the original equation (1). To obtain more group invariant solutions, we would like to solve the symmetry constraint condition $\sigma^k = 0$ defined by Eq. (19) with Eq. (22), which is equivalent to solving the following characteristic equations

$$\frac{\mathrm{d}x}{X} = \frac{\mathrm{d}y}{Y} = \frac{\mathrm{d}t}{T} = \frac{\mathrm{d}u}{U} = \frac{\mathrm{d}\phi}{\Phi} = \frac{\mathrm{d}f}{F} = \frac{\mathrm{d}g}{G}\,. \tag{23}$$

To solve the characteristic equations, one special case is listed in the following.

Without loss of generality, we assume $f_1 = f_2 = 0$, and $f_3 = 1$. For simplicity, we introduce $\Delta^2 = c_2^2 - 2c_1c_3$. We find the similarity solutions after solving out the characteristic equations (23)

$$\phi = -\frac{c_2}{c_1} - \frac{\Delta}{c_1}\tanh\Big[\frac{1}{2}\Delta(F_1 + x)\Big]\,,$$
$$f = -F_2\,\mathrm{sech}^2\Big[\frac{1}{2}\Delta(F_1 + x)\Big]\,,$$
$$g = -F_3\,\mathrm{sech}^2\Big[\frac{1}{2}\Delta(F_1 + x)\Big]$$
$$- \frac{4c_1}{\Delta}F_4\frac{\mathrm{sech}^2[(1/2)\Delta(F_1 + x)]}{\mathrm{e}^{\Delta(F_1+x)} + 1}\,,$$
$$u = F_4 + \frac{2c_1}{\Delta}F_3\tanh\Big[\frac{1}{2}\Delta(F_1 + x)\Big]$$
$$- \frac{c_1^2}{\Delta^2}F_2^2\tanh\Big[\frac{1}{2}\Delta(F_1 + x)\Big]$$
$$\times\Big[\tanh\Big[\frac{1}{2}\Delta(F_1 + x)\Big] - 2\Big]\,, \tag{24}$$

where $F_1 = F_1(y, t)$, $F_2 = F_2(y, t)$, $F_3 = F_3(y, t)$, and $F_4 = F_4(y, t)$ are the group invariant functions while y and t are the similarity variables. Substituting Eq. (24) into the prolonged system (1), (11), and (16), the invariant functions F_1, F_2, F_3 and F_4 satisfy the reduction systems

$$F_2 = \frac{\Delta^2}{2c_1}\,,$$
$$F_3 = -\frac{\Delta^3}{2c_1} - \frac{\Delta^3}{c_1(\,\mathrm{e}^{\Delta(F_1+x)} + 1)}\,,$$
$$F_4 = \frac{1}{2}f_{4t}y + \frac{1}{4}f_4^2 - \frac{2}{3}f_{5t} + \frac{\Delta^2}{6}\,, \tag{25}$$

where F_1 satisfies the following reduction equation

$$F_1 = -f_4 y + f_5\,, \tag{26}$$

where $f_4 = f_4(t)$, $f_5 = f_5(t)$ are arbitrary functions with t. It is obvious that once the solutions F_1 are solved out with Eq. (26), the solutions F_2, F_3, and F_4 can be solved out directly from Eq. (25). So the explicit solutions for the generalized Kadomtsev–Petviashvili equation (1) are immediately obtained by substituting F_1, F_2, F_3, and F_4 into Eq. (24).

4 CRE Solvable and Soliton-Cnoidal Waves Solution

4.1 *CRE Solvable*

For the generalized Kadomtsev–Petviashvili equation (1), we aim to look for its truncated Painlevé expansion solution in the following possible form

$$u = u_0 + u_1R(w) + u_2R(w)^2\,,\ (w = w(x, y, t))\,, \tag{27}$$

where $R(w)$ is a solution of the Riccati equation

$$R_w = b_0 + b_1R + b_2R^2\,, \tag{28}$$

with b_0, b_1, b_2 being arbitrary constants. By vanishing all the coefficients of the power of $R(w)$ after substituting Eq. (27) with Eq. (28) into Eq. (1), we have seven overdetermined equations for only four undetermined functions u_0, u_1, u_2 and w, It is fortunate that the overdetermined system may be consistent, thus we obtain

$$u_1 = -2b_1b_2w_x^2 - 2b_2w_{xx}\,, \quad u_2 = -2b_2^2w_x^2\,,$$
$$u_0 = \frac{1}{6}(\delta - 12b_0b_2)w_x^2 - b_1w_{xx}$$

$$-\frac{1}{2}\Big(\Big(\frac{w_{xx}}{w_x}\Big)^2+\Big(\frac{w_y}{w_x}\Big)^2\Big)-\frac{2}{3}S_1-\frac{1}{6}P_1-\frac{1}{2}C_1\,, \quad (29)$$

and the function w must satisfy

$$\delta w_x w_{xx}+3(C_1C_{1x}+C_{1x}+C_{1y})+S_{1x}+P_{1x}=0\,,\quad (\delta=4b_0b_2-b_1^2)\,, \quad (30)$$

where

$$P_1=\frac{w_t}{w_x}\,,\ C_1=\frac{w_y}{w_x}\,,\ S_1=\frac{w_{xxx}}{w_x}-\frac{3}{2}\Big(\frac{w_{xx}}{w_x}\Big)^2. \quad (31)$$

From above, it shows that the Eq. (1) really has the truncated Painlevé expansion solution related to the Riccati equation (28). At this point, we call the expansion (27) is a consistent Riccati expansion (CRE) and the generalized Kadomtsev–Petviashvili equation (1) is CRE solvable.[25]

In summary, we have the following theorem:

Theorem 4 If w is a solution of

$$\delta w_x w_{xx}+3(C_1C_{1x}+C_{1x}+C_{1y})+S_{1x}+P_{1x}=0\,, \quad (32)$$

then:

$$u=u_0-(2b_1b_2w_x^2+2b_2w_{xx})R(w)-2b_2^2w_x^2R(w)^2 \quad (33)$$

is a solution of Eq. (1), with $R\equiv R(w)$ being a solution of the Riccati equation (28).

4.2 *Soliton-Cnoidal Wave Interaction Solutions*

Obviously, the Riccati equation (28) has a special solution $R(w)=\tanh(w)$, while the truncated Painlevé expansion solution (27) becomes

$$u=u_0+u_1\tanh(w)+u_2\tanh^2(w)\,, \quad (34)$$

where u_0, u_1, u_2, and w are determined by Eqs. (28), (29), and (30).

We know the solution (34) is just consistent with Theorem 4. As consistent tanh-function expansion (CTE) (34) is a special case of CRE, it is quite clear that a CRE solvable system must be CTE solvable, and vice verse. If a system is CTE solvable, some important interaction solitary wave solutions can be constructed directly. In order to say the relation clearly, we give out the following Bäcklund transformation.

Theorem 5(BT) If w is a solution of Eq. (30) with $\delta=4$, then

$$u=u_0-(2b_1b_2w_x^2+2b_2w_{xx})\tanh(w)-2b_2^2w_x^2\tanh^2(w), \quad (35)$$

is a solution of Eq. (1), where u_0 is determined by Eq. (29) with $b_0=1, b_1=0, b_2=-1$.

In order to obtain the solution of Eq. (1), we consider w in the form

$$w=k_1x+l_1y+d_1t+g\,, \quad (36)$$

where g is a function of x, y and t. It will lead to the interaction solutions between a soliton and other waves. By means of Theorem 5, some nontrivial solutions of the generalized Kadomtsev–Petviashvili equation (1) can be obtained from some quite trivial solutions of Eq. (30), which are listed as follows.

Case 1 In Eq. (30), we take a trivial solution for w, saying

$$w=kx+ly+dt+c\,, \quad (37)$$

with k, l, d, c being arbitrary constants. Then substituting Eq. (37) into Theorem 5 yields the following kink soliton and ring soliton solution of the generalized Kadomtsev–Petviashvili equation (1):

$$u=-(d+3l)\frac{1}{6k}-\frac{l^2}{2k^2}-\frac{2}{3}k^2+2k^2\,\mathrm{sech}^2(kx+ly+dt+c)\,. \quad (38)$$

Case 2 To find out the interaction solutions between soliton and cnoidal periodic wave, let

$$w=k_1x+l_1y+d_1t+W(X)\,,\quad (X\equiv k_2x+l_2y+d_2t)\,, \quad (39)$$

where $W_1\equiv W_1(X)=W_X$ satisfies

$$W_{1X}^2=a_0+a_1W_1+a_2W_1^2+a_3W_1^3+a_4W_1^4\,, \quad (40)$$

with a_0, a_1, a_2, a_3, a_4 being constants. Substituting Eq. (39) with Eq. (40) into Theorem 5, we have the relations

$$a_0=\frac{k_1^2a_2}{k_2^2}-\frac{2k_1^3a_3}{k_2^3}+(12k_1^4+k_1d_1+3k_1l_1+l_1^2)\frac{1}{k_2^4}-(k_1d_2+3k_1l_2+4l_1l_2)\frac{k_1}{k_2^5}-\frac{5k_1^2l_2^2}{k_2^6}\,,$$

$$a_1=\frac{2a_2k_1}{k_2}-\frac{3k_1^2a_3}{k_2^2}+(16k_1^3+d_1+3l_1)\frac{1}{k_2^3}+(6l_1l_2-k_1d_2-3k_1l_2)\frac{1}{k_2^4}-\frac{6k_1l_2^2}{k_2^5}\,,\ a_4=4\,, \quad (41)$$

which lead to the following explicit solutions of Eq. (1) in the form of

$$u=\frac{4}{3}(k_1+k_2W_1)^2-\frac{d_1+3l_1+d_2W_1+3l_2W_1+4k_2^3W_{1XX}}{6(k_1+k_2W_1)}+\frac{k_2^4W_{1X}^2-(l_1+l_2W_1)^2}{2(k_1+k_2W_1)^2}+2W_{1X}\tanh(k_1x+l_1y+d_1t+W)-2(k_1+k_2W_1)^2\tanh^2(k_1x+l_1y+d_1t+W)\,. \quad (42)$$

It is known that an equation by the definition of the elliptic functions can be written out in terms of Jacobi elliptic functions. The formula (42) exhibits the interactions between soliton and abundant cnoidal periodic waves. To show these soliton-cnoidal waves more intuitively, we just take a simple solution of Eq. (40) as

$$W_1=\mu_0+\mu_1\,\mathrm{sn}(mX,n)\,, \quad (43)$$

where $\mathrm{sn}(mX,n)$ is the usual Jacobi elliptic sine function. The modulus n of the Jacobi elliptic function satisfies:

$0 \le n \le 1$. When $n \to 1$, $\mathrm{sn}(\xi)$ degenerates as hyperbolic function $\tanh(\xi)$, when $n \to 0$, $\mathrm{sn}(\xi)$ degenerates as trigonometric function $\sin(\xi)$. Substituting Eq. (43) with Eq. (41) into Eq. (40) and setting the coefficients of $\mathrm{cn}(\xi)$, $\mathrm{dn}(\xi)$, $\mathrm{sn}(\xi)$ equal to zero, without loss of generality, takes $k_1 = k_2 = 1$, $l_1 = l_2$, yields

$$\begin{aligned} &a_2 = (5 - n^2)m^2 + 24(m+1)\,, \\ &a_3 = 8(m+2)\,, \quad \mu_0 = -1 + \frac{1}{2}m\,, \\ &\mu_1 = -\frac{1}{2}mn,\ d_2 = (n^2-1)m^3 + d_1\,. \end{aligned} \tag{44}$$

Hence, one kind of soliton-cnoidal wave solutions is obtained by taking Eq. (43) and

$$W = \mu_0 X + \mu_1 \int_{X_0}^{X} \mathrm{sn}(mY, n)\,\mathrm{d}Y\,, \tag{45}$$

with the parameter requirement (44) into the general solution (42).

The solution given in Eq. (42) with Eq. (41) denotes the analytic interaction solution between the soliton and the cnoidal periodic wave. In Fig. 1, we plot the interaction solution of the potential u when the value of the Jacobi elliptic function modulus $n \ne 1$. This kind of solution can be easily applicable to the analysis of interesting physical phenomenon. In fact, there are full of the solitary waves and the cnoidal periodic waves in the real physics world.

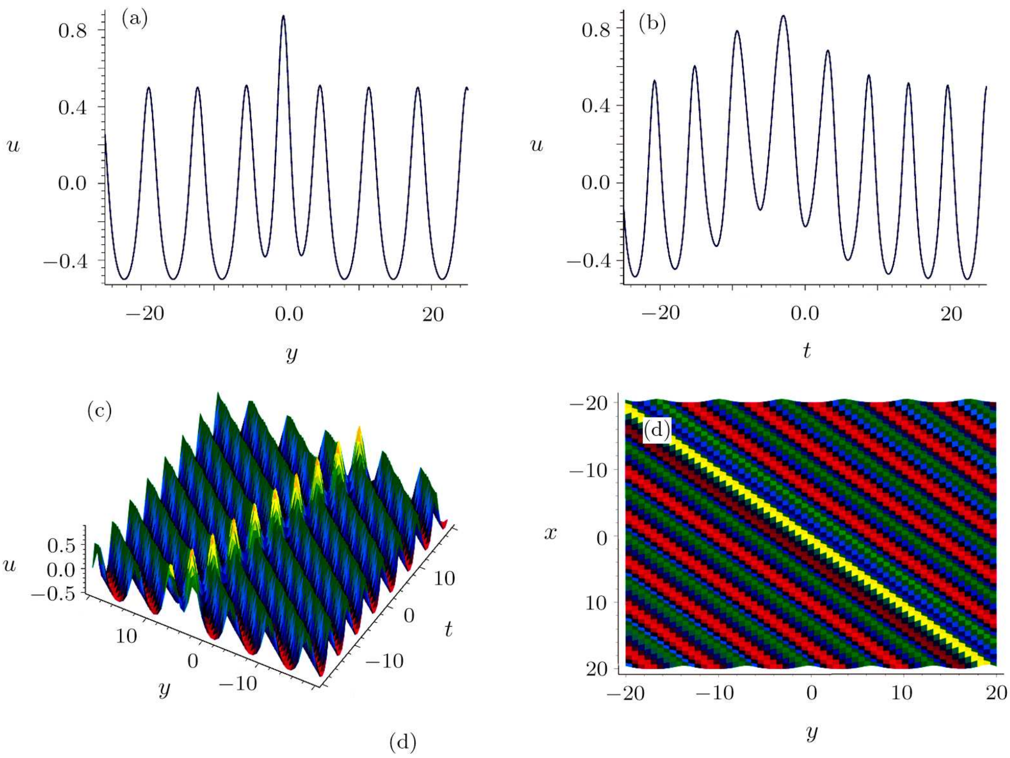

Fig. 1 (Color online) The type of soliton-cnoidal wave interaction solution for u with the parameters $m = 1$, $n = 1/2$, $k_2 = 1$, $l_1 = -1$, $\mu_0 = -3/2$, and $\mu_1 = 1/4$: (a) One-dimensional image at $x = 0$, $t = 1$; (b) One-dimensional image at $x = 0$, $y = 1$; (c) The three-dimensional plot; (d) Overhead view for u at $t = 0$.

5 Summary and Discussions

In summary, the generalized Kadomtsev–Petviashvili equation (1) is investigated by nonlocal symmetry method and consistent Riccati expansion (CRE) method.

On the one hand, applying the Painlevé expansion to the generalized Kadomtsev–Petviashvili equation, two BTs including auto-BT and non-auto-BT are obtained. By developing the truncated Painlevé expansion, the Schwartzian form of the generalized Kadomtsev–Petviashvili equation is found and the residual is demonstrated to be just the nonlocal symmetry. Meanwhile, the nonlocal symmetry is just related to the Möbious transformation symmetry by the linearized equation of non-auto-BT. Then the nonlocal symmetry is readily localized to Lie point symmetry by prolonging the original equation to a large system, the corresponding finite symmetry transformation and similarity reductions are found.

On the other hand, by means of the CRE method, the soliton-cnoidal wave solutions of the generalized Kadomtsev–Petviashvili equation are obtained. By a special form of CRE, i.e. the consistent tanh-function expansion (CTE), kink soliton+cnoidal periodic wave solution and ring soliton+cnoidal periodic wave solution are explicitly expressed by the Jacobi elliptic and the corresponding elliptic integral. The interactions between solitons and cnoidal periodic waves display some interesting and physical phenomena. The CRE method used here can be developed to find other kinds of solutions and integrable models. It can also be used to find interaction solutions among different kinds of nonlinear waves. The CRE method did provide us with the result which is quite nontrivial and difficult to be obtained by other traditional approaches.

In addition, the generalized Kadomtsev–Petviashvili equation (1) have been reported little in the current articles. So uncovering more integrable properties of the equation, such as the Darboux transformation, Hamiltonian structure and the conservation, are interesting and meaningful work. The details on the CRE method and other methods to solve interaction solutions among different kinds of nonlinear waves and the investigation of other integrability properties such as Hamiltonian structure of the generalized Kadomtsev–Petviashvili equation (1) deserves further study.

Acknowledgments

We would like to express our sincere thanks to S.Y. Lou, Y.Q. Li, and other members of our discussion group for their valuable comments.

References

[1] M.J. Ablowitz and P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, New York (1991).

[2] C. Rogers and W.K. Schief, *Bäcklund and Darboux Transformations: Geometry and Modern Applications in Soliton Theory*, Cambridge University Press, New York (2002).

[3] V.B. Matveev and M.A. Salle, *Darboux Transformations and Solitons*, Springer, Berlin (1991).

[4] C.H. Gu, H.S. Hu, and Z.X. Zhou, *Darboux Transformation in Soliton Theory and Its Geometric Applications*, Shanghai Scientific and Technical Publishers, Shanghai (1999).

[5] R. Hirota, *The Direct Method in Soliton Theory*, Cambridge University Press, New York (2004).

[6] J. Weiss, M. Tabor, and G. Carnevale, J. Math. Phys. **24** (1983) 522.

[7] J. Weiss, J. Math. Phys. **24** (1983) 1405.

[8] P.J. Olver and P. Rosenau, Phys. Lett. A **114** (1986) 107.

[9] Y. Chen and Z.Z. Dong, Nonlinear Anal. **71** (2009) e810.

[10] G.W. Bluman, A.F. Cheviakov, and S.C. Anco, *Applications of Symmetry Methods to Partial Differential Equations*, Springer, New York (2010).

[11] G.W. Bluman and J.D. Cole, J. Math. Mech. **18** (1969) 1025.

[12] I.S. Akhatov, R.K. Gazizov, and N.K. Ibragimov, J. Math. Sci. **55** (1991) 1401.

[13] F. Galas, J. Phys. A: Math. Gen. **25** (1992) L981.

[14] G.A. Guthrie, J. Phys. A: Math. Gen. **26** (1993) L905.

[15] S.Y. Lou, Phys. Lett. B **302** (1993) 261.

[16] S.Y. Lou, J. Phys. A: Math. Phys. **30** (1997) 4803.

[17] S.Y. Lou and X.B. Hu, J. Phys. A: Math. Gen. **30** (1997) L95.

[18] X.R. Hu, S.Y. Lou, and Y. Chen, Phys. Rev. E **85** (2012) 056607.

[19] S.Y. Lou, X.R. Hu, and Y. Chen, J. Phys. A: Math. Theor. **45** (2012) 155209.

[20] X.R. Hu and Y. Chen, Chin. Phys. B **24** (2015) 030201.

[21] X.P. Xin and Y. Chen, Chin. Phys. Lett. **30** (2013) 100202.

[22] X.P. Cheng, S.Y. Lou, C.L. Chen, and X.Y. Tang, Phys. Rev. E **89** (2014) 043202.

[23] S.Y. Lou, arXiv:1308.1140, (2013).

[24] X.N. Gao, S.Y. Lou, and X.Y. Tang, J. High Energy Phys. **05** (2013) 029.

[25] S.Y. Lou, Stud. Appl. Math. **134** (2015) 372.

[26] D. Yang, S.Y. Lou, and W.F. Yu, Commun. Theor. Phys. **60** (2013) 125.

[27] X.P. Cheng, Y.Q. Yang, and J.Y. Li, Commun. Theor. Phys. **61** (2014) 1.

[28] C.L. Chen and S.Y. Lou, Commun. Theor. Phys. **61** (2014) 545.

[29] Z.Y. Ma, J.X. Fei, and X.Y. Du, Commun. Theor. Phys. **64** (2015) 127.

[30] X.R. Hu and Y. Chen, Z. Naturforsch. **70** (2015) 729.

[31] B. Ren, X.Z. Liu, and P. Liu, Commun. Theor. Phys. **63** (2015) 125.

[32] B. Ren, Phys Scr. **90** (2015) 065206.

[33] X.R. Hu and Y.Q. Li, Appl. Math. Lett. **51** (2016) 20.

[34] X.R. Hu and Y. Chen, Chin. Phys. B **24** (2015) 090203.

[35] Q. Wang, Y. Chen, and H.Q. Zhang, Chaos Solitons, and Fractals. **25** (2005) 1019.

[36] Y.H. Wang and H. Wang, Phys. Scr. **89** (2014) 125203.

[37] Y.H. Wang and Y. Chen, Commun. Theor. Phys. **57** (2012) 217.

[38] Y.C. Hon and E.G. Fan, IMA. J. Appl. Math. **77** (2012) 236.

[39] Y.T. Gao and B. Tian, Int. J. Theor. Phys. **37** (1998) 2299.

[40] W.X. Ma and A. Pekcan, Z. Naturforsch. **66** (2011) 377.