Darboux Transformation and N-soliton Solution for Extended Form of Modified Kadomtsev–Petviashvili Equation with Variable-Coefficient*

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Abstract *The extended form of modified Kadomtsev–Petviashvili equation with variable-coefficient is investigated in the framework of Painlevé analysis. The Lax pairs are obtained by analysing two Painlevé branches of this equation. Starting with the Lax pair, the N-times Darboux transformation is constructed and the N-soliton solution formula is given, which contains 2n free parameters and two arbitrary functions. Furthermore, with different combinations of the parameters, several types of soliton solutions are calculated from the first order to the third order. The regularity conditions are discussed in order to avoid the singularity of the solutions. Moreover, we construct the generalized Darboux transformation matrix by considering a special limiting process and find a rational-type solution for this equation.*

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1 Introduction

Integrable systems are differential and difference equations with various mathematica structures and wide applications in physics and engineering. Most integral equations have multi-soliton solutions. Among these equations, the Kadomtsev–Petviashvili (KP) equation has been considered as a ubiquitous and important physical model. It is a (2+1)-dimensional partial differential equation which describes nonlinear wave motion:[1]

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0\,. \tag{1}$$

This equation can be applied to model long wavelengths water waves with weakly non-linear restoring forces and frequency dispersion. It is a two-dimensional generalization of one-dimensional Korteweg-de Vries (KdV) equation.

The modified Kadomtsev–Petviashvili (mKP) equation:

$$u_t + u_{xxx} + 3\partial_x^{-1}u_{yy} - 6u^2u_x - 6u_x\partial_x^{-1}u_y = 0 \tag{2}$$

describes water waves in (x, y) plane when the nonlinearity is higher than the KP equation. Equation (2) can be considered as a model for the propagation of the water waves in fluid dynamics,[2–3] ionacoustic waves in a plasma with nonisothermal electrons,[4–6] nonlinear Alfvén waves in a cold collision-free plasma,[7] and electromagnetic wave in an isotropic charge-free infinite ferromagnetic thin film.[8–9] The Darboux transformation for Eq. (2) was obtained in Ref. [10] via Painlevé analysis, and the line soliton solutions for this equation were constructed by means of the Hirota bilinear method.[11] The Miura transformations between mKP and KP equation were also studied in Ref. [12].

Generally, there is a considerable amount of work in the extended forms of nonlinear evolution equations. These works focus on the effect of the additional terms on the integrability, the dispersion relations of the equation, the amplitudes, the velocity and other physical phenomena of the solutions.

An extended form of the mKP equation (2)

$$u_t + u_{xxx} + 3\partial_x^{-1}u_{yy} - 6u^2u_x - 6u_x\partial_x^{-1}u_y + \alpha u_y + \beta u_x = 0 \tag{3}$$

is investigated in Ref. [3], where α and β are arbitrary constants, and u_x, u_y are the potentials in the x- and the y-directions, respectively. This extended form of the mKP equation is obtained by adding the terms αu_y and βu_x to Eq. (2).

The celebrated KP-like equations possess certain important properties and physical applications from water waves to plasma physics as well as field theories. Because of the assumptions of the constant coefficients, the physical models in which they arise tend to be highly idealized, for example, in the propagation of small-amplitude surface waves in a fluid with constant depth. However, the

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variable-coefficient generalizations of the KP-like equation are better able to provide us with more certain realistic physical satiation in these models, such as the dynamics of nonlinear waves in a real ocean,[13−15] which is inhomogeneous, and the waves are comparatively influenced by the refraction and geometric divergence. In fact, the KP-like models with variable-coefficient are among the most interesting variable-coefficient nonlinear evolution equations.

Based on the facts we have clarified above, and in order to possess the soliton solutions with more parametric freedom, we consider the following variable-coefficient version for Eq. (3):

$$u_t + u_{xxx} + 3\partial_x^{-1}u_{yy} - 6u^2u_x - 6u_x\partial_x^{-1}u_y + \alpha(t)u_y + \beta(t)u_x = 0\,, \tag{4}$$

where $\alpha(t)$ and $\beta(t)$ are two arbitrary functions.

The main proposal in our paper is to construct the Lax-pair and N-times Darboux transformation for the extended form of the mKP equation with variable-coefficient. Singularity analysis and the Painlevé property are considered to be the important methods for determining the integrability of nonlinear partial differential equations (PDEs). Based on the Painlevé property, we can use the singular manifold method to obtain the Bäcklund transformations and Lax pairs of a nonlinear PDE.[16−18] However, this method has a disadvantage when it is applied to the PDEs with Painlevé expansion branches more than one. Afterwards, the modification version for this method, which is called the generalized singular manifold method proposed in Ref. [19] by Estevez and Gordoa. It solves PDEs that include as many singular manifolds as Painlevé branches. This modification procedure was applied to Boussinesq equation, the modified Korteweg de Vries equation[20] and the Mikhailov–Shabat systems,[19] and the relation between singular manifolds and Hirota's τ functions[21] was clearly established in these papers.

Following the method proposed in Ref. [10], we would like to find the Lax-pairs for Eq. (4). First of all, we implement the Painlevè analysis to Eq. (4), so we can write its solution as the form of Painlevè expansion. Then, by comparing the coefficients in the leading term of the expansion, we derive two Painlevè branches with the corresponding singular manifold being introduced, which constitutes an auto-Bäcklund transformation between two pairs of solutions. After a series of complicated calculation and procedures of linearization, we derive the Lax-pair for Eq. (4). Secondly, with this Lax-pair, we construct the Darboux transformation for this equation.

Darboux transformation is a powerful tool in the construction of solutions for integral partial differential equations, including multi-solitons, breather solutions and other interesting solutions. It is essentially a kind of gauge transformation which can keep the form of the Lax-pair invariant. Especially, we investigate the N-times Darboux transformation for Eq. (4). Moreover, the N-soliton solution formula are also derived in the form of Wronskian determinant. Finally, using the idea inherited from Ref. [23], we construct a special generalized Darboux transformation for this equation where the Lax-pair does not contain a spectral parameter. As a result, the N-times formula for the rational form solutions are calculated.

The paper is organized as follows. In Sec. 2, the Lax pairs for the extended mKP equation with variable coefficients are calculated by analysing its two Painlevé branches. In Sec. 3, we construct the N-times Darboux transformation for this equation. The N-times generalized Darboux transformation are constructed in Sec. 4. The final section contains some conclusions discussions.

2 Painlevé Analysis and the Construction of Lax Pair

2.1 *Painlevé Analysis and Auto-Bäcklund Transformations*

We can consider Eq. (4) as a two component system with the following form:

$$\begin{aligned}&u_t + u_{xxx} + 3\omega_y - 6u^2u_x - 6\omega u_x + \alpha(t)u_y + \beta(t)u_x = 0\,,\\ &\omega_x - u_y = 0\,.\end{aligned} \tag{5}$$

In terms of the singularity analysis, Eq. (5) enjoys the Painlevé property, which means that its solution can be written as a Painlevé expansion such as

$$u = \sum_{i=0}^{\infty} u_j(x,y,t)[\chi(x,y,t)]^{j-\alpha}, \tag{6a}$$

$$\omega = \sum_{i=0}^{\infty} \omega_j(x,y,t)[\chi(x,y,t)]^{j-\beta}. \tag{6b}$$

The analysis of the leading term provides:

$$\alpha = 1, \quad \beta = 1, \quad u_0 = a\chi_x, \quad \omega_0 = a\chi_y\,, \tag{7}$$

where $a = \pm 1$ and $\chi(x,y,z)$ is an arbitrary function. Therefore, there exist two Painlevé branches which can appear simultaneously in the truncated expansion for Eq. (5). For each branch, the auto-Bäcklund transformation can be introduced as

$$u' = u + \frac{\phi_x}{\phi} - \frac{\sigma_x}{\sigma}, \quad \omega' = \omega + \frac{\phi_y}{\phi} - \frac{\sigma_y}{\sigma}\,, \tag{8}$$

where, ϕ is the singular manifold for $a = 1$ and σ for $a = -1$. Obviously, there exists two pairs of solutions (u,ω) and (u',ω') of Eq. (5),

$$\begin{aligned}&u_t + u_{xxx} + 3\omega_y - 6u^2u_x - 6\omega u_x + \alpha(t)u_y + \beta(t)u_x = 0\,,\\ &\omega_x - u_y = 0\,,\end{aligned} \tag{9}$$

and

$$\begin{aligned}&u'_t + u'_{xxx} + 3\omega'_y - 6u'^2u'_x - 6\omega' u'_x + \alpha(t)u'_y + \beta(t)u'_x = 0\,,\\ &\omega'_x - u'_y = 0\,.\end{aligned} \tag{10}$$

Assume that we can decouple the crossed terms $(\phi_x/\phi)(\sigma_x/\sigma)$ through the following expression

$$\frac{\phi_x}{\phi}\frac{\sigma_x}{\sigma} = A\frac{\phi_x}{\phi} + B\frac{\sigma_x}{\sigma}\,, \tag{11}$$

and define

$$v_1 = \frac{\phi_{xx}}{\phi_x},\quad w_1 = \frac{\phi_t}{\phi_x},\quad \tau_1 = \frac{\phi_y}{\phi_x}\,,$$
$$v_2 = \frac{\sigma_{xx}}{\sigma_x},\quad w_2 = \frac{\sigma_t}{\sigma_x},\quad \tau_2 = \frac{\sigma_y}{\sigma_x}\,. \tag{12}$$

Substitute (8) into (10), and use (11) to decouple the crossed product, then the set of equations can be provided,

$$A = \frac{1}{2}(2u + v_1 + \tau_1),\quad B = \frac{1}{2}(-2u + v_2 - \tau_2)\,,$$
$$6(\omega - u_x + u^2) = w_1 + 4v_{1x} + v_1^2 + 3\tau_1^2 + \alpha(t)\tau_1 + \beta(t)\,,$$
$$6(\omega + u_x + u^2) = w_2 + 4v_{2x} + v_2^2 + 3\tau_2^2 + \alpha(t)\tau_2 + \beta(t)\,,$$
$$\Big(w_1 + v_{1x} - \frac{v_1^2}{2} + \frac{3}{2}\tau_1^2 + \alpha(t)\tau_1\Big)_x + 3\tau_{1y} = 0\,,$$
$$\Big(w_2 + v_{2x} - \frac{v_2^2}{2} + \frac{3}{2}\tau_2^2 - \alpha(t)\tau_2\Big)_x + 3\tau_{2y} = 0\,. \tag{13}$$

From (13), we obtain the following relations:

$$(AB)_x = AB(\tau_2 - \tau_1)\,,$$
$$[AB(2A - 2B + v_1 - v_2 - 3\tau_1 - 3\tau_2) - \alpha(t)]_x$$
$$= AB(w_2 - w_1)\,. \tag{14}$$

In this case, taking the derivative of (11) with respect to x, y and t, and using (11) again, we have:

$$A_x = A(v_2 - A - B),\quad B_x = B(v_1 - A - B)\,,$$
$$A_y = [A(\tau_2 + B)]_x,\quad B_y = [B(\tau_1 - A)]_x\,,$$
$$A_t = [Aw_2 + AB(2A - 2B + v_1 - v_2 - 3\tau_1 - 3\tau_2 - \alpha(t))]_x\,,$$
$$B_t = [-Bw_1 + AB(2A - 2B + v_1 - v_2 - 3\tau_1 - 3\tau_2 - \alpha(t))]_x\,. \tag{15}$$

2.2 *Construction of Lax Pairs*

In order to linearize Eq. (15), the following changes should be introduced,

$$A = \frac{\psi_{2x}}{\psi_2},\quad B = \frac{\psi_{1x}}{\psi_1}\,, \tag{16}$$

in which case Eq. (15) can be written as the following two Lax pairs:

$$\psi_{1xx} + 2u\psi_{1x} + \psi_{1y} = 0\,,$$
$$\psi_{1t} + 4\psi_{1xxx} + 6\Big[u_x + u^2 - \omega - \frac{1}{3}\alpha(t)u + \frac{1}{6}\beta(t)\Big]\psi_{1x} + (12u - \alpha(t))\psi_{1xx}\,, \tag{17}$$
$$\psi_{2xx} - 2u\psi_{2x} - \psi_{2y} = 0\,,$$
$$\psi_{2t} + 4\psi_{2xxx} + 6\Big[u_x + u^2 - \omega - \frac{1}{3}\alpha(t)u + \frac{1}{6}\beta(t)\Big]\psi_{2x} - (12u - \alpha(t))\psi_{2xx}\,. \tag{18}$$

3 Darboux Transformation and N-soliton Solutions

3.1 *N-times Darboux Transformation*

From Eq. (17), we can obtain the Lax pair for Eq. (5)

$$\phi_{1y} = -\phi_{1xx} - 2u\phi_{1x},\quad \phi_{1t} = A(u)\phi\,, \tag{19}$$
$$A(u) = -4\partial^3 - (12u - \alpha(t))\partial^2 - 6\Big(u_x + u^2 - (\partial^{-1}u_y) - \frac{1}{3}\alpha(t)u + \frac{1}{6}\beta(t)\Big)\partial\,,$$

where $\partial = \partial_x$ and $\partial\partial^{-1} = \partial^{-1}\partial = 1$.

The elementary Darboux transformation of the spectral problem (19) is:

$$T[1]\phi \triangleq \phi[1] = \phi - \frac{\phi_1}{\phi_{1x}}\phi_x\,, \tag{20}$$

and the potential satisfies the relation:

$$u[1] = u + \partial\ln\Big(\frac{\phi_{1x}}{\phi_1}\Big)\,, \tag{21}$$

where $u[1]$ satisfies Eq. (4), and $(u[1], \phi[1])$ also satisfies (19):

$$\phi_y[1] = -\phi_{xx}[1] - 2u[1]\phi_x[1]\,,$$
$$\phi_t[1] = A(u[1])\phi[1]\,. \tag{22}$$

Remark 1 Denoting $u[i], \phi[i]$ and $\phi_j[i]$ as the action of i-times repeated Darboux transformation (20), (21) on the seed solutions: u, ϕ, ϕ_j, then:

$$\phi_{j,y}[i] = -\phi_{j,xx}[i] - 2u[i]\phi_{j,x}[i]\,,$$
$$\phi_{j,t}[i] = A(u[i])\phi_j[i]\,. \tag{23}$$

Before we construct the N-times Darboux transformation, the following lemma is needed:

Lemma 1 For arbitrary integral l, k $(1 \leqslant l \leqslant n-1, 1 \leqslant k \leqslant l-1)$, we have

$$W_2(\phi_{l+1}[l], \phi_{l+2}[l], \ldots, \phi_{l+k}[l]) = (-1)^k\frac{\phi_l^k[l-1]}{\phi_{l,x}^{k+1}[l-1]}W_2(\phi_l[l-1], \phi_{l+1}[l-1], \ldots, \phi_{l+k}[l-1])\,,$$
$$W_1(\phi_{l+1}[l], \phi_{l+2}[l], \ldots, \phi_{l+k}[l]) = (-1)^k\frac{\phi_l^{k-1}[l-1]}{\phi_{l,x}^{k}[l-1]}W_1(\phi_l[l-1], \phi_{l+1}[l-1], \ldots, \phi_{l+k}[l-1])\,,$$
$$W_1(\phi[l], \phi_{l+1}[l], \ldots, \phi_{l+k}[l]) = (-1)^k\frac{\phi_l^k[l-1]}{\phi_{l,x}^{k+1}[l-1]}W_1(\phi[l-1], \phi_l[l-1], \ldots, \phi_{l+k}[l-1])\,.$$

The proof of this Lemma was given in Ref. [22]. And the following theorem gives the N-times Darboux transformation for Eq. (5):

Theorem 1 $\phi_1,\phi_2,\ldots,\phi_N$ is the solution of (19), and u is the solution of extended form of mKP equation with variable coefficients, then the N-times Darboux transformation is:

$$\phi[N]=\frac{W_1(\phi,\phi_1,\phi_2,\ldots,\phi_N)}{W_2(\phi_1,\phi_2,\ldots,\phi_N)}\,, \tag{24}$$

$$u[N]=u+\partial\ln\frac{W_2(\phi_1,\phi_2,\ldots,\phi_N)}{W_1(\phi_1,\phi_2,\ldots,\phi_N)}\,, \tag{25}$$

and $(\phi[N],u[N])$ satisfies:

$$\begin{aligned}&\phi_y[N]=-\phi_{xx}[N]-2u[N]\phi_x[N]\,,\\&\phi_t[N]=A(u[N])\phi[N]\,,\end{aligned} \tag{26}$$

where, w_1,w_2 are two types of Wronskian determinant defined by:

$$W_1(\phi_1,\phi_2,\ldots,\phi_k)=\det(A),\quad A_{i,j}=\frac{\mathrm{d}^{j-1}}{\mathrm{d}x^{j-1}}\phi_i\,, \tag{27}$$

$$W_2(\phi_1,\phi_2,\ldots,\phi_k)=\det(B),\quad B_{i,j}=\frac{\mathrm{d}^{j}}{\mathrm{d}x^{j}}\phi_i\,. \tag{28}$$

Hence, with the conclusion of Lemma 1, we have:

$$\begin{aligned}\phi[N]&=\phi[N-1]-\frac{\phi_N[N-1]}{\phi_{N,x}[N-1]}\phi_x[N-1]\\&=\frac{W_1[\phi[N-1],\phi_N[N-1]]}{W_2(\phi_N[N-1])}=\cdots\\&=\frac{W_1(\phi_1,\phi_2,\ldots,\phi_N)}{W_2(\phi_1,\phi_2,\ldots,\phi_N)}\,,\end{aligned} \tag{29}$$

with,

$$\begin{aligned}u[N]&=u[N-1]+\partial\ln\frac{\phi_{N,x}[N-1]}{\phi_N[N-1]}\\&=u[N-2]+\partial\ln\frac{\phi_{N-1,x}[N-2]}{\phi_{N-1}[N-2]}\\&\quad+\partial\ln\frac{W_2(\phi_N[N-1])}{W_1(\phi_N[N-1])}\\&=u[N-2]+\partial\ln\frac{W_2[\phi_{N-1}[N-2],\phi_N[N-2]]}{W_1[\phi_{N-1}[N-2],\phi_N[N-2]]}\\&=\cdots=u+\partial\ln\frac{W_2(\phi_1,\phi_2,\ldots,\phi_N)}{W_1(\phi_1,\phi_2,\ldots,\phi_N)}\,.\end{aligned} \tag{30}$$

This completes the proof of Theorem 1. □

3.2 *N-soliton Solutions Formula*

(i) One-soliton solution

Next, using formulation (30) we intend to find the soliton solution for Eq. (4). Starting from the seed solution $u=c$, where c is a constant, then the solution for (19) is:

$$\begin{aligned}\phi_1&=\phi_{k_1}+\phi_{p_1}\,,\\\phi_{k_1}&=\mathrm{e}^{k_1x+(-2ck_1-k_1^2)y-\int(6c^2k_1-2c\alpha(t)k_1+12ck_1^2-\alpha(t)k_1^2+\beta(t)k_1+4k_1^3)\,\mathrm{d}t+\mathcal{P}_1(k_1)}\,,\\\phi_{p_1}&=\mathrm{e}^{-p_1x+(2cp_1-p_1^2)y-\int(-6c^2p_1+2c\alpha(t)p_1+12cp_1^2-\alpha(t)p_1^2-\beta(t)p_1-4p_1^3)\,\mathrm{d}t+\mathcal{F}_1(p_1)}\,,\end{aligned} \tag{31}$$

where, k_1 and p_1 are arbitrary constants. $\mathcal{P}_1(k_1)$ and $\mathcal{F}_1(p_1)$ are polynomials of k_1 and p_1, respectively.

In the case of vacuum seed where $c=0$, we take $\mathcal{P}_1(k_1)=\mathcal{F}_1(p_1)=0$ for convenience, then solution (31) is reduced to

$$\begin{aligned}\phi_{k_1}&=\mathrm{e}^{k_1x-k_1^2y+\int(-4k_1^3+\alpha(t)k_1^2-k_1\beta(t))\,\mathrm{d}t}\,,\\\phi_{p_1}&=\mathrm{e}^{-p_1x-p_1^2y+\int(4p_1^3+\alpha(t)p_1^2+p_1\beta(t))\,\mathrm{d}t}\,.\end{aligned}$$

Therefore, relation (21) gives one-soliton solution with arbitrary function $\alpha(t),\beta(t)$:

$$u[1]=\frac{k_1^2\phi_{k_1}+p_1^2\phi_{p_1}}{k_1\phi_{k_1}-p_1\phi_{p_1}}-\frac{k_1\phi_{k_1}-p_1\phi_{p_1}}{\phi_{k_1}+\phi_{p_1}}\,. \tag{32}$$

It is noted that the condition $k_1\neq p_1$ and $k_1p_1<0$ must be satisfied. Moreover, to make sure the amplitude is positive, one needs $k_1>0$, $p_1<0$. Solution (32) is equivalent to the following form:

$$u[1]=\gamma_1(2\cosh[\theta_1(x,y,t)+\varphi_1]+\nu_1)^{-1}, \tag{33}$$

where, $\theta_1(x,y,t)=(k_1+p_1)x+(p_1^2-k_1^2)y+\int -4(k_1^3+p_1^3)+\alpha(t)(k_1^2-p_1^2)-\beta(t)(k_1+p_1)\,\mathrm{d}t$, $\gamma_1=(k_1+p_1)^2/\sqrt{-k_1p_1}$, $\nu_1=(k_1-p_1)/\sqrt{-k_1p_1}$, and $\varphi_1=(1/2)\ln(-k_1/p_1)$ is the phase position.

The amplitude of soliton solution is $\gamma_1/(2+\nu_1)$. The center trajectory is

$$\theta_1(x,y,t)+\varphi_1=0\,.$$

If k_1 and p_1 satisfy the relation $(k_1-p_1)^2+4k_1p_1=0$, then $u[1]$ becomes

$$u[1]=\frac{\gamma_1}{4}\mathrm{sech}^2\Big[\frac{\theta_1(x,y,t)}{2}+\frac{\varphi_1}{2}\Big]\,,$$

when $y=0$, $\alpha(t)$ and $\beta(t)$ are taken as constants, so the wave velocity relative to x-axis is $v_u=-4(k_1^2-k_1p_1+p_1^2)+\alpha(t)(k_1-p_1)-\beta(t)$.

Solving equations

$$\begin{aligned}&\int(-4k_1^3+\alpha(t)k_1^2-k_1\beta(t))\,\mathrm{d}t=c_1\omega_1(t)\,,\\&\int(4p_1^3+\alpha(t)p_1^2+p_1\beta(t))\,\mathrm{d}t=c_2\omega_2(t)\,,\end{aligned}$$

it yields

$$\alpha(t)=-\frac{-4k_1^3p_1+4k_1p_1^3-c_1p_1\omega_1'(t)-c_2k_1\omega_2'(t)}{k_1p_1(k_1+p_1)}\,,$$

$$\beta(t) = -\frac{4k_1^3p_1^2 + 4k_1^2p_1^3 + c_1p_1^2\omega_1'(t) - c_2k_1^2\omega_2'(t)}{k_1p_1(k_1+p_1)}.$$

In addition, choosing different functions $\alpha(t)$, $\beta(t)$, we can obtain several types of soliton solutions, which are listed in the following.

a) Taking $k_1 = 1$, $p_1 = -1/2$, $c_1 = 1$, $\omega_1'(t) = \cos(\mu_1 t)$, $c_2 = 1$, $\omega_2'(t) = \cos(\mu_2 t)$, we obtain the "periodic-type" soliton $u[1]$:

$$u[1](x,y,t) = \frac{\exp[\frac{1}{2}x + \frac{3}{4}y + \frac{\sin(\mu_1 t)}{\mu_1} + \frac{\sin(\mu_2 t)}{\mu_2}]}{4\exp[x + \frac{2\sin(\mu_1 t)}{\mu_1}] + 6\exp[\frac{1}{2}x + \frac{3}{4}y + \frac{\sin(\mu_1 t)}{\mu_1} + \frac{\sin(\mu_2 t)}{\mu_2}] + 2\exp[\frac{3}{2}y + \frac{2\sin(\mu_2 t)}{\mu_2}]}. \tag{34}$$

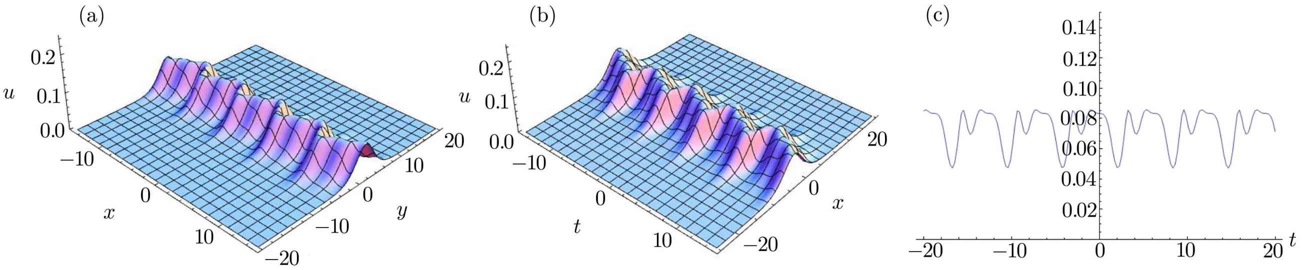

Fig. 1 (a)–(b) 3-D plot for the one "periodic-type" soliton when $x = 0$, $y = 0$, respectively; (c) Transverse plot of solution (34) when $x = y = 0$.

Let $\mu_1 = 1$, $\mu_2 = 2$, Figs. 1(a)–1(c) show the image of the solution as $x = 0$, $y = 0$, and $x = y = 0$. Moreover, this "periodic-type" soliton solution is actually a kind of periodic solution, which has the period 2π.

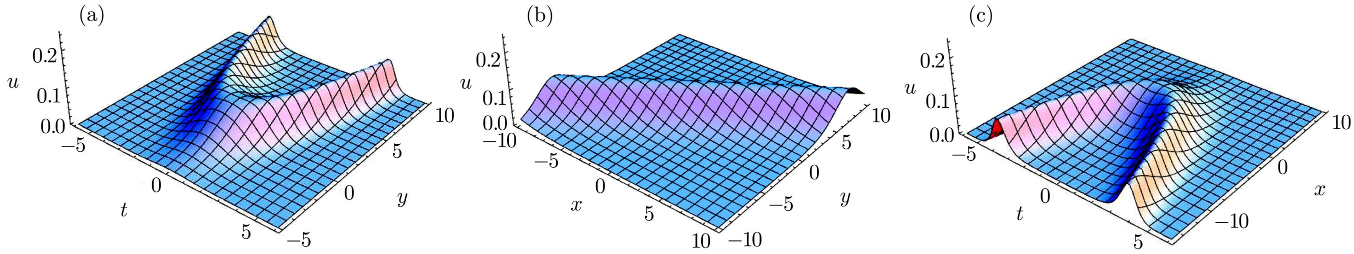

Fig. 2 (a)–(c) 3-D plot for the one "parabolic-type" soliton (35) when $x = 0$, $t = 0$ and $y = 0$, respectively.

b) Taking $k_1 = 1$, $p_1 = -1/2$, $c_1 = 1$, $\omega_1'(t) = a_1 t$, $c_2 = 1$, $\omega_2'(t) = a_2 t$, then we obtain a "parabolic-type" soliton:

$$u[1](x,y,t) = \frac{\exp[\frac{1}{2}x + \frac{3}{4}y + \frac{1}{2}t^2a_1 + \frac{1}{2}t^2a_2]}{4\exp[x + t^2a_1] + 2\exp[\frac{3}{2}y + t^2a_2] + 6\exp[\frac{1}{2}x + \frac{3}{4}y + \frac{1}{2}t^2a_1 + \frac{1}{2}t^2a_2]}. \tag{35}$$

When $a_1 = 2$, $a_2 = 1$, Fig. 2(a)–Fig. 2(c) show the images of solution (35) as $x = 0$, $t = 0$, and $y = 0$.

c) Taking $k_1 = 14$, $p_1 = -1/2$, $c_1 = 1$, $\omega_1'(t) = \operatorname{sech}^2(\eta_1 t)$, $c_2 = 1$, $\omega_2'(t) = \operatorname{sech}^2(\eta_2 t)$, then it yields the "tanh-type" soliton solution:

$$u[1](x,y,t) = \frac{\exp[\frac{1}{2}x + \frac{3}{4}y + \frac{\tanh(t\eta_1)}{\eta_1} + \frac{\tanh(t\eta_2)}{\eta_2}]}{4\exp[x + \frac{2\tanh(t\eta_1)}{\eta_1}] + 6\exp[\frac{1}{2}x + \frac{3}{4}y + \frac{\tanh(t\eta_1)}{\eta_1} + \frac{\tanh(t\eta_2)}{\eta_2}] + 2\exp[\frac{3}{2}y + \frac{2\tanh(t\eta_2)}{\eta_2}]}. \tag{36}$$

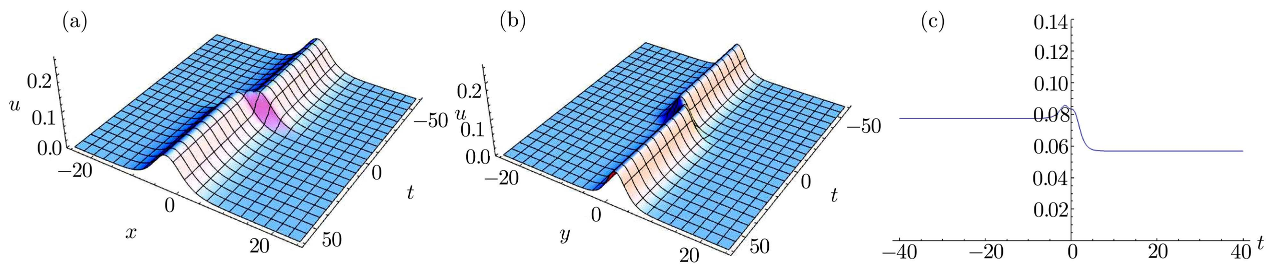

Fig. 3 (a)–(b) 3-D plot for the one "tanh-type" soliton when $y = 0$, $x = 0$, respectively; (c) Transverse plot of solution (36) when $x = y = 0$.

If we choose $\eta_1 = 1/2$, $\eta_2 = 1$, there exist Figs. 3(a)– 3(c) to show the image of this solution as $y = 0$, $x = 0$, and $x = y = 0$.

(ii) Two-soliton solution

General expressions for two-soliton solutions can be easily obtained by taking $N = 2$ in (30). However, in order to guarantee for the regularity of the two-soliton solution, we need to do some regularity analysis. In terms of the potential relation (30), it must be ensued that functions $W_1(\phi_1, \ldots, \phi_N)$ and $W_2(\phi_1, \ldots, \phi_N)$ are certainly not zero. Hence, the choice of the parameters k_1, k_2, p_1, p_2 must satisfy the following regularity condition:

$$k_2 - k_1 \gtreqless 0, \quad k_1 - p_2 \gtreqless 0, \quad k_2 + p_1 \gtreqless 0, \quad p_1 - p_2 \gtreqless 0\,,$$

with,

$$p_1 p_2 (p_1 - p_2) \gtreqless 0, \quad k_1 k_2 (k_2 - k_1) \gtreqless 0, \quad k_1 p_2 (k_1 + p_2) \gtreqless 0, \quad -p_1 k_2 (p_1 + k_2) \gtreqless 0\,.$$

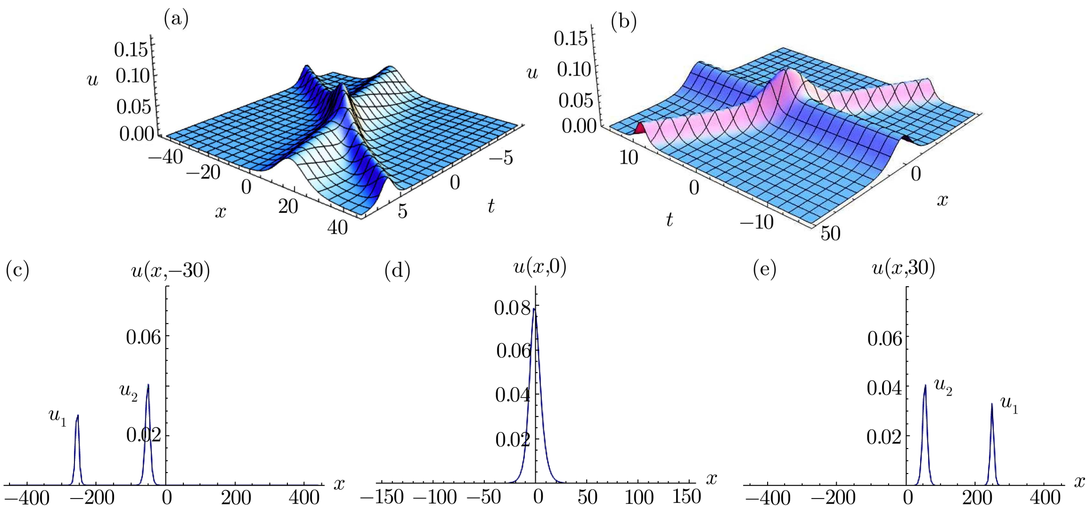

Fig. 4 (a) 3D-plot of two-soliton when $y = 0$ and $\alpha(t) = \beta(t) = 0$; (b) 3D-plot of two-soliton when $y = 0$ and $\alpha(t) = 1$, $\beta(t) = -1$; (c)–(e) The interaction progress of two solitons when t is orderly taken as -30, 0, 30.

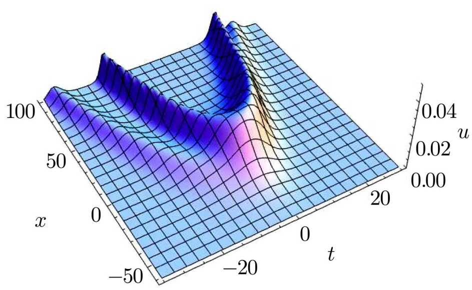

Fig. 5 3D-plot of two "parabolic-type" solitons interactions.

With the above condition, taking $k_1 = 1/2$, $k_2 = 2/3$, $p_1 = -1/4$, $p_2 = -1$, $\alpha(t) = 0$, $\beta(t) = 0$, one can derive the elementary 2-soliton solutions interactions depicted in Fig. 4(a). While Fig. 4(b) corresponds to the case when $\alpha(t) = 1$, $\beta(t) = -1$. Figures 4(c)–4(e) describe the interaction progress of two solitons when t is orderly taken as $-30, 0, 30$. It is easy to verify that the wave velocity admits $v_{u_1} > v_{u_2}$, hence soliton u_1 is chasing after soliton u_2, which the amplitude is higher than u_1. Then they have a collision near the moment $t = 0$, and break into two original single soliton soon after the interaction.

When the parameters are taken as: $k_1 = 0.2$, $p_1 = -0.1$, $k_2 = 0.4$, $p_2 = -0.3$, $c_1 = 1$, $\omega'(t) = t$, $\beta(t) = t$, $\alpha(t) = (c_1\omega'(t) + k_1\beta(t) + 4k_1^3)/k_1^2$, one obtains the "Parabolic-Type" soliton interactions in Fig. 5.

If the parameters are taken as: $k_1 = 0.2$, $p_1 = -0.1$, $k_2 = 0.4$, $p_2 = -0.3$, $c_1 = 1$, $\omega'(t) = \sin(t/6)$, $\beta(t) = \sin(t)$, $\alpha(t) = (c_1\omega'(t) + k_1\beta(t) + 4k_1^3)/k_1^2$, it will appear two-periodic soliton interactions shown in Fig. 6(a). Figures 6(b) and 6(c) are the density plots for another two soliton solutions where the variable coefficients are taken as $\omega_1(t) = \cos(t/6)$, $\omega_2(t) = \cos(t/6)$ and $\omega_1(t) = \cos(t/6)$, $\omega_2(t) = \cos(t/8)$, separately.

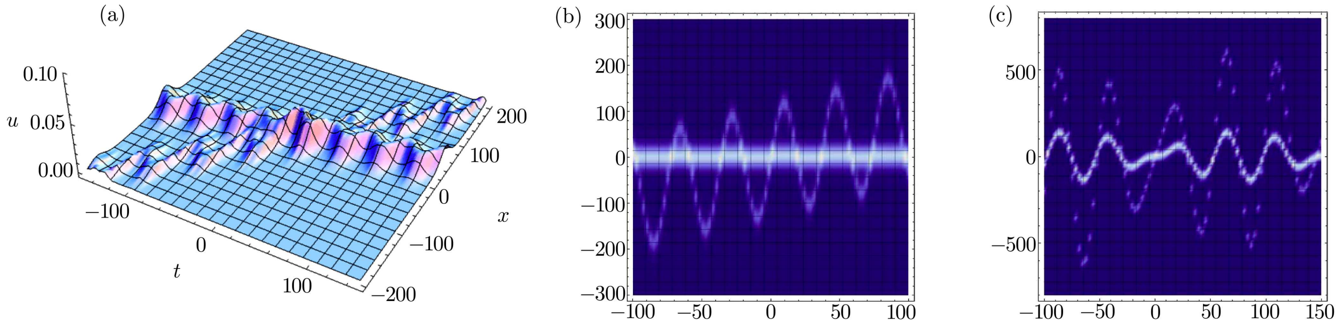

Fig. 6 (a) 3D-plot of two periodic soliton interctions; (b)–(c) Density plot for the two periodic solitons.

(iii) Three-soliton solution

In order to analysis the singularities for the 3-soliton solution, we especially define the 3×3 Vandermonde determinate $\mathrm{Vd}[f_1, f_2, f_3]$ as:

$$\mathrm{Vd}[f_1, f_2, f_3] := \begin{vmatrix} 1 & 1 & 1 \\ f_1 & f_2 & f_3 \\ f_1^2 & f_2^2 & f_3^2 \end{vmatrix} . \tag{37}$$

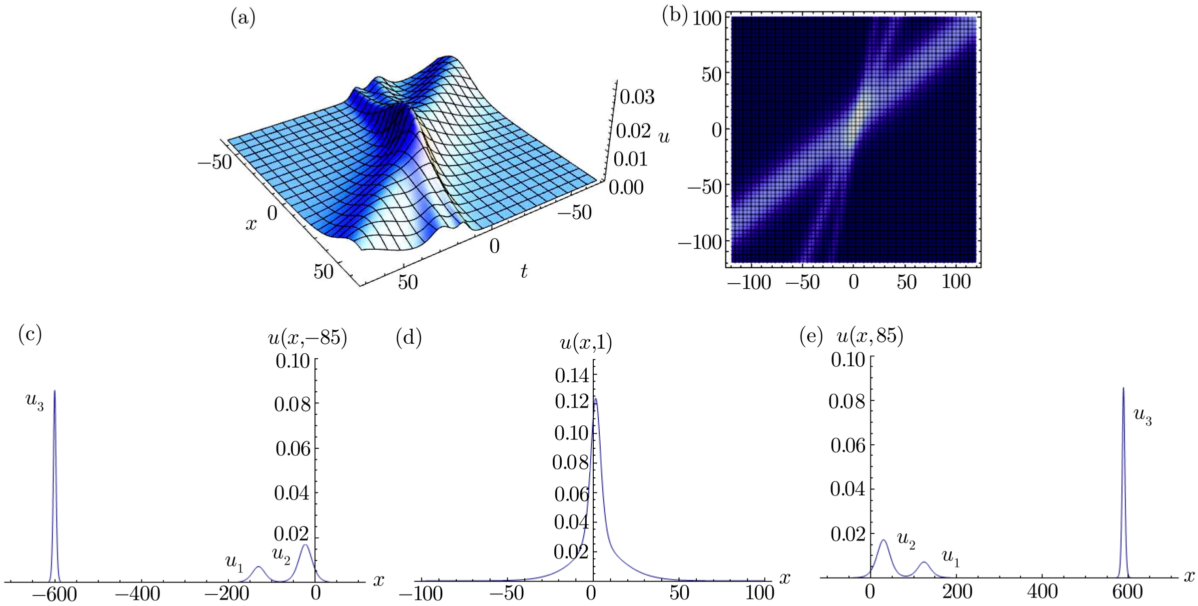

Fig. 7 (a) 3-D plot for elementary 3-soliton solutions; (b) Density plot of (a); (c)–(e) Transverse plot of (a) when $x = y = 0$, and the interaction progress of 3-solitons when t is orderly taken as -85, 1, 85.

Then the parameters $k_1, k_2, k_3, p_1, p_2, p_3$ must satisfy the following regularity condition:

$$\begin{aligned} &(k_1k_2k_3)\mathrm{Vd}[k_1, k_2, k_3] \gtreqless 0, \quad (k_1k_2p_3)\mathrm{Vd}[k_1, -p_3, k_2] \gtreqless 0\,, \\ &(k_1p_2k_3)\mathrm{Vd}[k_1, k_3, -p_2] \gtreqless 0, \quad (p_1k_2k_3)\mathrm{Vd}[-p_1, k_2, k_3] \gtreqless 0\,, \\ &(p_1p_2k_3)\mathrm{Vd}[p_1, -k_3, p_2] \gtreqless 0, \quad (p_1k_2p_3)\mathrm{Vd}[-k_2, p_1, p_3] \gtreqless 0\,, \\ &(k_1p_2p_3)\mathrm{Vd}[p_2, -k_1, p_3] \gtreqless 0, \quad (p_1p_2p_3)\mathrm{Vd}[p_1, p_2, p_3] \gtreqless 0\,, \end{aligned}$$

simultaneously with,

$$\begin{aligned} &\mathrm{Vd}[k_1, k_2, k_3] \gtreqless 0, \quad \mathrm{Vd}[k_1, -p_3, k_2] \gtreqless 0, \quad \mathrm{Vd}[k_1, k_3, -p_2] \gtreqless 0, \quad \mathrm{Vd}[-p_1, k_2, k_3] \gtreqless 0\,, \\ &\mathrm{Vd}[p_1, -k_3, p_2] \gtreqless 0, \quad \mathrm{Vd}[-k_2, p_1, p_3] \gtreqless 0, \quad \mathrm{Vd}[p_2, -k_1, p_3] \gtreqless 0, \quad \mathrm{Vd}[p_2, p_1, p_3] \gtreqless 0\,. \end{aligned}$$

Hence we choose $k_1 = 0.2$, $k_2 = 0.4$, $k_3 = 0.6$, $p_1 = -0.3$, $p_2 = -0.5$, $p_3 = -0.7$, $\alpha(t) = 0$, $\beta(t) = 0$, which satisfied the above regularity condition. The elementary 3-soliton solutions interactions are dipicted in Fig. 7(a). Moreover, one can see in Figs. 7(c)–7(e) that soliton u_3 is running after u_1 and u_2 while u_1 is going after u_2, simultaneously, which is rooted in the relation of the wave velocity: $v_{u_3} > v_{u_1} > v_{u_2}$.

Next, taking variable coefficients as $\alpha(t) = t$, $\beta(t) = t$, and $y = 0$, then Fig. 8 shows the interactions of three "parabolic-type" soliton solutions.

Moreover, in the case of non-vacuum seed where $c \neq 0$, we can derive the soliton solution on the nonzero plane. Specifically, Figs. 9(a) and 9(b) separately show the one-soliton solution and the two-soliton solution with the constant background $c = 2$ and $c = 1$ where the parameters are taken as $k_1 = 1/2$, $p_1 = -1/4$, $k_2 = 2/3$, $p_2 = -1$, $\alpha(t) = 1$, $\beta(t) = 2$.

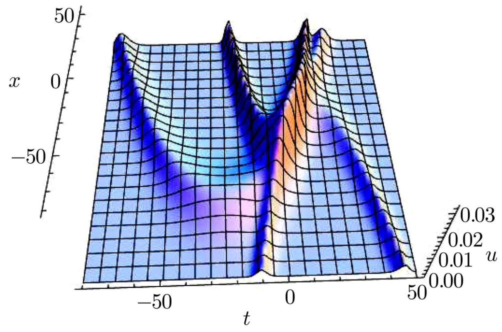

Fig. 8 3-D plot for the interactions of three "parabolic-type" soliton solutions.

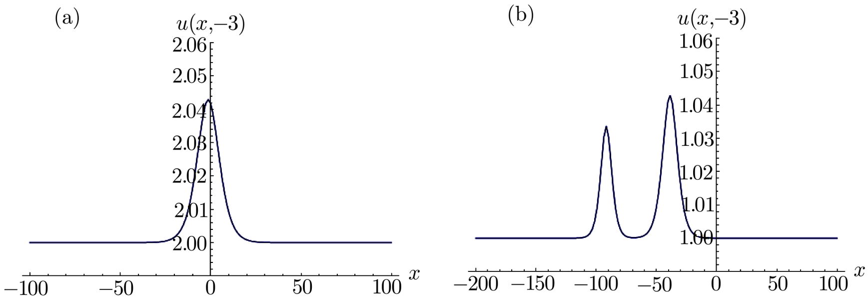

Fig. 9 (a)–(b) One-soliton solution and two-soliton solutions with different constant backgrounds $c = 2$ and $c = 1$, seperately.

4 Generalized Darboux Transformation and Rational Type Solutions

From (31) we can regard $\phi_1 = \phi_1(k_1, p_1)$ as a function on variable k_1, p_1, which plays the same role as spectral parameters, then $\phi_1(k_1 + \epsilon, p_1 + \epsilon)$ also admits (19) when $u = 0$, where ϵ is a small perturbation of the parameters. Because of the property exits for the DT (20):

$$T[1]\phi_1 = \phi_1 - \frac{\phi_1}{\phi_{1x}}\phi_{1x} = 0\,, \tag{38}$$

it is obvious to see that:

$$T[1]\phi_1(k_1 + \epsilon, p_1 + \epsilon) \to 0, \ \text{as } \epsilon \to 0\,.$$

Let $\phi_2 = \phi_1(k_1 + \epsilon, p_1 + \epsilon)$, we may consider the limitation:

$$\lim_{\epsilon\to 0} \frac{T[1]\phi_2}{\epsilon} = \lim_{\epsilon\to 0} \frac{T[1]\phi_1(k_1, p_1) + ((\mathrm{d}/\mathrm{d}\epsilon)T[1]\phi_2|_{\epsilon\to 0})\epsilon + o(\epsilon)}{\epsilon} = \lim_{\epsilon\to 0} \frac{\mathrm{d}}{\mathrm{d}\epsilon}\phi_2[1]\,. \tag{39}$$

Denoting $\phi_1[1] = (\mathrm{d}/\mathrm{d}\epsilon)\phi_2[1]|_{\epsilon\to 0}$, so $\phi_1[1]$ can be used to the next step Darboux transformation:

$$\phi[2] = \phi[1] - \frac{\phi_1[1]}{\phi_{1,x}[1]}\phi_x[1]\,. \tag{40}$$

It should be noted that one can only consider the small perturbation on single parameter:

$$\phi_1(k_1 + \epsilon, p_1), \ \text{or } \phi_1(k_1, p_1 + \epsilon)\,,$$

and then the above limiting process can be still implemented.

Following this idea, we look back to the Wronskian solution expression:

Taking $N = 2$ as a concrete example:

$$u[2] = u + \partial \ln \frac{W_2(\phi_1, \phi_2)}{W_1(\phi_1, \phi_2)} = u + \partial \ln \frac{\det\begin{pmatrix} \phi_1' & \phi_1'' \\ \phi_2' & \phi_2'' \end{pmatrix}}{\det\begin{pmatrix} \phi_1 & \phi_1' \\ \phi_2 & \phi_2' \end{pmatrix}}\,. \tag{41}$$

Taylor expansion at $\epsilon \to 0$:

$$\begin{vmatrix} \phi_1 & \phi_1' \\ \phi_2 & \phi_2' \end{vmatrix} = \begin{vmatrix} \phi_1 & \phi_1' \\ \phi_1 + \phi_1^{[1]}\epsilon + o(\epsilon) & (\phi_1 + \phi_1^{[1]}\epsilon + o(\epsilon))' \end{vmatrix} = \begin{vmatrix} \phi_1 & \phi_1' \\ \phi_1^{[1]} & (\phi_1^{[1]})' \end{vmatrix} \epsilon + o(\epsilon)\,,$$

$$\begin{vmatrix} \phi_1' & \phi_1'' \\ \phi_2' & \phi_2'' \end{vmatrix} = \begin{vmatrix} \phi_1' & \phi_1'' \\ \phi_1' + (\phi_1^{[1]})'\epsilon + o(\epsilon) & \phi_1'' + (\phi_1^{[1]})''\epsilon + o(\epsilon) \end{vmatrix} = \begin{vmatrix} \phi_1' & \phi_1'' \\ (\phi_1^{[1]})' & (\phi_1^{[1]})'' \end{vmatrix} \epsilon + o(\epsilon)\,,$$

where, $\phi_1^{[1]} = (\mathrm{d}/\mathrm{d}\epsilon)(\phi_1(k_1+\epsilon, p_1+\epsilon))|_{\epsilon\to 0}$.

Hence,

$$u[2] = u + \partial \ln \frac{\begin{vmatrix} \phi_1' & \phi_1'' \\ (\phi_1^{[1]})' & (\phi_1^{[1]})'' \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_1' \\ \phi_1^{[1]} & (\phi_1^{[1]})' \end{vmatrix}}\,, \tag{42}$$

which is a new solution to Eq. (4).

In general, we consider N distinct functions:

$$\phi_j = \phi_1(k_j, p_j)\,, \quad k_j = k_1 + \epsilon_j\,, \quad p_j = p_1 + \epsilon_j\,, \quad j = 2, 3, \ldots, N\,.$$

Taking the limit $k_j \to k_1$, $p_j \to p_1$ in $u[N]$, then the N-times iterated DT of the potential can be reduced to:

$$\begin{aligned} u[N] &= u + \lim_{\epsilon_2,\ldots,\epsilon_N \to 0} \partial \ln \frac{W_2(\phi_1, \phi_2, \ldots, \phi_N)}{W_1(\phi_1, \phi_2, \ldots, \phi_N)} \\ &= u + \partial \ln \frac{W_2(\phi_1, \phi_1^{[1]}, \ldots, \phi_1^{[N-1]})}{W_1(\phi_1, \phi_1^{[1]}, \ldots, \phi_1^{[N-1]})}\,, \end{aligned} \tag{43}$$

where,

$$\phi_1^{[j-1]} = \lim_{\epsilon_j \to 0} \frac{\mathrm{d}^{j-1}}{\mathrm{d}\epsilon_j^{j-1}} \phi_1(k_1+\epsilon_j, p_1+\epsilon_j)\,, \quad j = 1, \ldots, N\,. \tag{44}$$

Taking $N = 2$ in above formula, we find a rational type solution:

$$u[2] = u + \partial \ln \frac{W_2(\phi_1, \phi_1^{[1]})}{W_1(\phi_1, \phi_1^{[1]})}\,, \tag{45}$$

with W_1 and W_2 having the following forms:

$$\begin{aligned} W_1 &= \mathrm{e}^{\varphi_k} - \mathrm{e}^{\varphi_p} + \mathcal{F}_1(x,y,t)\,\mathrm{e}^{(\varphi_k+\varphi_p)/2}\,, \\ W_2 &= k_1^2\,\mathrm{e}^{\varphi_k} - p_1^2\,\mathrm{e}^{\varphi_p} + \mathcal{F}_2(x,y,t)\,\mathrm{e}^{(\varphi_k+\varphi_p)/2}, \\ \varphi_k &= 2xk_1 - 2yk_1^2 + 2\int(-4k_1^3 + k_1^2\alpha[t] - k_1\beta[t])\,\mathrm{d}t\,, \\ \varphi_p &= -2xp_1 - 2yp_1^2 + 2\int(4p_1^3 + p_1^2\alpha[t] + p_1\beta[t])\,\mathrm{d}t\,, \end{aligned}$$

with,

$$\mathcal{C}_1 = \int[(k_1 - p_1)\alpha(t) - \beta(t)]\,\mathrm{d}t\,,$$

$$\begin{aligned} \mathcal{F}_1(x,y,t) &= 2(k_1+p_1)[x + (p_1 - k_1)y \\ &\quad - 6(k_1^2 + p_1^2)t + \mathcal{C}_1]\,, \\ \mathcal{F}_2(x,y,t) &= k_1^2 - p_1^2 + 2k_1p_1(k_1+p_1)[-x + (k_1 - p_1)y \\ &\quad + 6(k_1^2 + p_1^2)t - \mathcal{C}_1]\,. \end{aligned}$$

Different with the soliton solution we have discussed in the previous section, the rational type solution (45) which derived via the above generalized DT procedure is actually the combination of exponential functions and rational polynomials with integral of arbitrary functions. It is obvious to notice from W_1 to W_2 that solution (45) has some singularity, therefore, the regularity condition is difficult and cumbersome to discuss.

5 Conclusion

In conclusion, we investigate the extended form of the mKP equation with variable coefficients in the framework of Painlevé analysis. The Lax-pairs for the extended form of the mKP equation are obtained via analysing its two Painlevé branches. Meanwhile, the N-times Darboux transformation is constructed with the N-soliton solution formula given, which has $2n$ free parameters and two arbitrary functions. In particular, according to different choices of parameters, several types of soliton solutions are calculated from the first to the third order, while the regularity conditions are also discussed to avoid the singularity of the soliton solutions. In addition, using a special limiting process, we construct the generalized Darboux transformation matrix for the extended form of mKP equation with variable coefficients, and the rational-type solution is also calculated.

It should be noted that the extended form of mKP equation is real equation, so we believe that the binary Darboux transformation can be implemented into this equation to derive other physical solutions, including the lump solution or the rogue wave solution, etc. This work is in progress and will be reported in our further paper.

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