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Nonlocal Symmetries and Exact Solutions for PIB Equation*

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Abstract *In this paper, the symmetry group of the (2+1)-dimensional Painlevé integrable Burgers (PIB) equations is studied by means of the classical symmetry method. Ignoring the discussion of the infinite-dimensional subalgebra, we construct an optimal system of one-dimensional group invariant solutions. Furthermore, by using the conservation laws of the reduced equations, we obtain nonlocal symmetries and exact solutions of the PIB equations.*

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1 Introduction

Symmetry is one of the most important concepts in the area of differential equations (DEs), especially in partial differential equations (PDEs). In the 19th century, Sophus Lie initiated his studies on continuous groups (Lie group). He showed that a point symmetry of a DE leads, in the case of an ODE, to reducing the order of the DE (irrespective of any imposed initial conditions) and, in the case of a PDE, to finding special solutions called invariant solutions of the DE. With the development of integrable systems and solion theory, the extensions of Lie's work to PDEs have focused on finding further applications of point symmetries to include linearization mappings and solutions of boundary value problems, extending the spaces of symmetries of given PDE system to include local symmetries[1−6] as well as nonlocal symmetries.[7−11]

Local symmetries admitted by a PDE are useful for finding invariant solutions. These solutions are obtained by using group invariants to reduce the number of independent variables. Local symmetries admitted by a nonlinear PDE are also useful to discover whether or not the equation can be linearized by an invertible mapping and construct an explicit linearization when one exists.

An obvious limitation of group-theoretic methods is that some PDEs of physical interest possess few symmetries. But they can admit nonlocal symmetries whose infinitesimal generators depend on integrals of the dependent variables in some specific manner.

There are several methods to obtain nonlocal symmetries of given PDE systems. In a number of cases the nonlocal symmetries may be easily obtained with the help of a recurrence operator.[12] But sometimes the recurrence operators of given system are difficult to obtain. Even if one got the recurrence operators of the system, one also can not obtain the nonlocal symmetries. In Ref. [13], Akhatov and Gazizov provided a method for constructing nonlocal symmetries of DEs based on the Lie-Bäcklund theory. Moreover, the finite symmetry transformation and similar reduction cannot be directly applied to nonlocal symmetries. Naturally, it is necessary to inquire as to whether one can transform nonlocal symmetries into Lie point symmetries by extending original system. In general, neither for the local nor for the nonlocal variables, the prolongation does not close. Bluman introduced the concept of potential symmetry (or nonlocal symmetry) for a PDE system. We introduce the concept in the 4th section. The potential symmetries of nonlinear PDEs have been studied in the literatures from several different points of view.

Symmetry group techniques based on local symmetries provide one method for constructing group invariant solutions of PDEs and linearizing nonlinear PDEs by invertible mappings. Local symmetries of potential system may yield nonlocal symmetries of the given system, and the existence of nonlocal symmetries leads to the construction of corresponding invariant solutions as well as to the linearization of nonlinear PDEs by non-invertible mappings.[14−16]

In this paper, taking the well known (2+1)-dimensional Painlevé integrable Burgers (PIB) equations for a special example, the PIB equation was derived from the generalized Painlevé integrability classification by Hong *et al.*,[17]

$$u_t = uu_y + avu_x + bu_{yy} + abu_{xx}\,, \quad u_x = v_y\,, \qquad (1)$$

some explicitly exact solutions of the PIB equation have been obtained via variable separation approach[17−18] and multiple Riccati equations rational expansion method.[19] We restudy this equation by using the nonlocal symmetry defined by Bluman's theory. First, we use Olver's

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method which only depends on fragments of the theory of Lie algebras to construct the optimal system[20−23] of the (2+1)-dimensional PIB equations, then use the conservation laws of the reduced equations to construct nonlocal symmetries.

This paper is arranged as follows: In Sec. 2, by using the classical Lie symmetry method, we get the vector fields of the PIB equations (1). Then the transformations leaving the solutions invariant, i.e., its symmetry groups are obtained. In Sec. 3, with the associated vector fields obtained in Sec. 2, we construct the one-parameter optimal system of group-invariant solutions. Based on the optimal system, some reductions of Eq. (1) are derived. In Sec. 4, using the conservation laws of the reduced equations, we obtain nonlocal symmetries and exact solutions. Finally, some conclusions and discussions are given in Sec. 5.

2 Symmetry Group of the PIB Equation

By applying the Lie symmetry method,[24−26] we consider the one-parameter group of infinitesimal transformations in(x,y,t,u,v) of Eq. (1) given by

$$\begin{aligned}
x^* &= x+\varepsilon X(x,y,t,u,v)+o(\varepsilon^2)\,,\\
y^* &= y+\varepsilon Y(x,y,t,u,v)+o(\varepsilon^2)\,,\\
t^* &= t+\varepsilon T(x,y,t,u,v)+o(\varepsilon^2)\,,\\
u^* &= u+\varepsilon \Psi(x,y,t,u,v)+o(\varepsilon^2)\,,\\
v^* &= v+\varepsilon \Phi(x,y,t,u,v)+o(\varepsilon^2)\,,
\end{aligned} \tag{2}$$

where ε is the group parameter. It is required that Eqs. (1) be invariant under the transformations (2), and this yields a system of overdetermined linear equations for the infinitesimals X, Y, T, Ψ, and Φ, which can be solved by virture of *Maple* to give,

$$\begin{aligned}
&X(x,y,t,u,v) = f(t)+\frac{(c_1t+c_2)x}{2}\,,\\
&Y(x,y,t,u,v) = \frac{(c_1y-2c_4)t}{2}+\frac{c_2y}{2}+c_5\,,\\
&T(x,y,t,u,v) = \frac{c_1t^2}{2}+c_2t+c_3\,,\\
&\Psi(x,y,t,u,v) = \frac{(-y-ut)c_1}{2}-\frac{c_2u}{2}+c_4\,,\\
&\Phi(x,y,t,u,v) = \frac{1}{2}\,\frac{-2((\mathrm{d}/\mathrm{d}t)f(t))-c_1x-(c_1t+c_2)av}{a}\,,
\end{aligned} \tag{3}$$

where c_1, c_2, c_3, c_4, c_5 are arbitrary constants, and $f(t)$ is an arbitrary function of t. Here we take the $f(t)$ as 1 and t for simplicity. The associated vector fields for the one-parameter Lie group of infinitesimal transformations v_1, v_2, v_3, v_4, v_5, v_6, v_7 are given as,

$$\begin{aligned}
&v_1 = \frac{tx}{2}\partial x+\frac{ty}{2}\partial y+\frac{t^2}{2}\partial t-\frac{y+ut}{2}\partial u-\frac{x+atv}{2a}\partial v\,,\\
&v_2 = \frac{x}{2}\partial x+\frac{y}{2}\partial y+t\partial t-\frac{u}{2}\partial u-\frac{v}{2}\partial v\,,\\
&v_3 = \partial t\,,\quad v_4=-t\partial y+\partial u\,,\quad v_5=\partial y\,,\\
&v_6 = \partial x\,,\quad v_7 = t\partial x-\frac{1}{a}\partial v\,.
\end{aligned} \tag{4}$$

Equations (4) show that the following transformations (defined by $\exp(\varepsilon v_i)$ $(i=1,2,\ldots,7)$) of variables x, y, t, u, v leave the solutions of Eqs. (1) invariant:

$$\begin{aligned}
&\exp(\varepsilon v_1):(x,y,t,u,v)\\
&\qquad\mapsto\Bigl(\frac{-2x}{t\varepsilon-2},\frac{-2y}{t\varepsilon-2},\frac{-2t}{t\varepsilon-2},-\frac{y\varepsilon+2u}{t\varepsilon-2},-\frac{x\varepsilon+2av}{a(t\varepsilon-2)}\Bigr)\,,\\
&\exp(\varepsilon v_2):(x,y,t,u,v)\\
&\qquad\mapsto(\,\mathrm{e}^{(1/2)\varepsilon}x,\,\mathrm{e}^{(1/2)\varepsilon}y,\,\mathrm{e}^{\varepsilon}t,\,\mathrm{e}^{(1/2)\varepsilon}u,\,\mathrm{e}^{(1/2)\varepsilon}v)\,,\\
&\exp(\varepsilon v_3):(x,y,t,u,v)\mapsto(x,y,t+\varepsilon,u,v)\,,\\
&\exp(\varepsilon v_4):(x,y,t,u,v)\mapsto(x,y-t\varepsilon,t,u+\varepsilon,v)\,,\\
&\exp(\varepsilon v_5):(x,y,t,u,v)\mapsto(x,y+\varepsilon,t,u,v)\,,\\
&\exp(\varepsilon v_6):(x,y,t,u,v)\mapsto(x+\varepsilon,y,t,u,v)\,,\\
&\exp(\varepsilon v_7):(x,y,t,u,v)\mapsto\Bigl(x+t\varepsilon,y,t,u,v+\frac{1}{a}\varepsilon\Bigr)\,.
\end{aligned}$$

Then the following theorem holds:

Theorem 1 If

$$\begin{cases} u=p(x,y,t)\\ v=q(x,y,t)\end{cases}$$

is a solution of Eqs. (1), then so are

$$\begin{aligned}
&u^{(1)}=\frac{2}{t\varepsilon+2}p\Bigl(\frac{2x}{t\varepsilon+2},\frac{2y}{t\varepsilon+2},\frac{2t}{t\varepsilon+2}\Bigr)-\frac{y\varepsilon}{t\varepsilon+2}\,,\\
&v^{(1)}=\frac{2}{t\varepsilon+2}q\Bigl(\frac{2x}{t\varepsilon+2},\frac{2y}{t\varepsilon+2},\frac{2t}{t\varepsilon+2}\Bigr)-\frac{x\varepsilon}{a(t\varepsilon+2)}\,,\\
&u^{(2)}=\,\mathrm{e}^{-(1/2)\varepsilon}p(\,\mathrm{e}^{-(1/2)\varepsilon}x,\,\mathrm{e}^{-(1/2)\varepsilon}y,\,\mathrm{e}^{-\varepsilon}t)\,,\\
&v^{(2)}=\,\mathrm{e}^{-(1/2)\varepsilon}q(\,\mathrm{e}^{-(1/2)\varepsilon}x,\,\mathrm{e}^{-(1/2)\varepsilon}y,\,\mathrm{e}^{-\varepsilon}t)\,,\\
&u^{(3)}=p(x,y,t-\varepsilon)\,,\quad v^{(3)}=q(x,y,t-\varepsilon)\,,\\
&u^{(4)}=p(x,y-t\varepsilon,t)-\varepsilon\,,\quad v^{(4)}=q(x,y-t\varepsilon,t)\,,\\
&u^{(5)}=p(x,y-\varepsilon,t)\,,\quad v^{(5)}=q(x,y-\varepsilon,t)\,,\\
&u^{(6)}=p(x-\varepsilon,y,t)\,,\quad v^{(6)}=q(x-\varepsilon,y,t)\,,\\
&u^{(7)}=p(x-t\varepsilon,y,t)\,,\quad v^{(7)}=q(x-t\varepsilon,y,t)-\frac{1}{a}\varepsilon\,.
\end{aligned}$$

3 Optimal System and Reductions of PIB Equations

By using the generators v_i of the Lie-point transformations in Eqs. (4), one can build up exact solutions of Eqs. (1) via the symmetry reduction approach. This allows one to lower the number of independent variables of the system of differential equations under consideration using the invariants associated with a given subgroup of the symmetry group. In the following we present some reductions leading to exact solutions of the equations of possible physical interest.

Firstly, we construct an optimal system to classify the group-invariant solutions of Eqs. (1). As it is said in the Ref. [20], the problem of classifying group-invariant solutions reduces to the problem of classifying subgroups of the full symmetry group under conjugation. And the problem of finding an optimal of subgroups is equivalent to that of finding an optimal system of subalgebras. Here, by using the method presented in Refs. [20–21], we construct the optimal system of one-dimensional subalgebras of Eqs. (1).

Applying the commutator operators $[v_m, v_n] = v_m v_n - v_n v_m$, we get the commutator table listed in Table 1 with the (i,j)-th entry indicating $[v_i, v_j]$. And it follows,

Table 1 Lie Bracket.

Lie	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	0	$-v_1$	$-v_2$	0	$1/2v_4$	$-1/2v_7$	0
v_2	v_1	0	$-v_3$	$1/2v_4$	$-1/2v_5$	$-1/2v_6$	$1/2v_7$
v_3	v_2	v_3	0	$-v_5$	0	0	v_6
v_4	0	$-1/2v_4$	v_5	0	0	0	0
v_5	$-1/2v_4$	$1/2v_5$	0	0	0	0	0
v_6	$-1/2v_7$	$1/2v_6$	0	0	0	0	0
v_7	0	$-1/2v_7$	$-v_6$	0	0	0	0

Proposition The operators v_i $(i = 1, 2, \ldots, 7)$ form a Lie algebra, which is a seven dimensional symmetry algebra.

To compute the adjoint representation, we use the Lie series in conjunction with the above commutator table. Applying the formula $\mathrm{Ad}(\exp(\varepsilon v_i))v_j = v_j - \varepsilon[v_i, v_j] + (1/2)\varepsilon^2[v_i,[v_i,v_j]] - \cdots$ and Table 1, one can have the adjoint representation listed in Table 2 with the (i,j)-th entry indicating $\mathrm{Ad}(\exp(\varepsilon v_i))v_j$.

Following Ovsiannikov,[20] one calls two subalgebras v_2 and v_1 of a given Lie algebra equivalent if one can find an element g in the Lie group so that $\mathrm{Ad}g(v_1) = v_2$; where $\mathrm{Ad}g$ is the adjoint representation of g on $\bar{v}$. Given a nonzero vector, for example,

$$\bar{v} = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 + a_5 v_5 + a_6 v_6 + a_7 v_7 \,.$$

Our task is to simplify as many of the coefficients a_i as possible though judicious applications of adjoint maps to $\bar{v}$. In this way, omitting the detailed computation, one can get the following theorem by the complicated computation:

Table 2 Adjoint Representation.

Lie	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	v_1	$v_2 + \varepsilon v_1$	$v_3 + \varepsilon v_2 + (\varepsilon^2)2/v_1$	v_4	$v_5 - (1/2)\varepsilon v_4$	$v_6 + (1/2)\varepsilon v_7$	v_7
v_2	$\mathrm{e}^{-\varepsilon} v_1$	v_2	$\cos(\varepsilon)v_3 + \sin(\varepsilon)v_3$	$\mathrm{e}^{-(1/2)\varepsilon} v_4$	$\mathrm{e}^{(1/2)\varepsilon} v_5$	$\mathrm{e}^{(1/2)\varepsilon} v_6$	$\mathrm{e}^{-(1/2)\varepsilon} v_7$
v_3		$v_2 - \varepsilon v_3$	v_3	$v_4 + \varepsilon v_5$	v_5	v_6	$v_7 - \varepsilon v_6$
v_4	v_1	$v_2 + (1/2)\varepsilon v_4$	$v_3 - \varepsilon v_5$	v_4	v_5	v_6	v_7
v_5	$v_1 + (1/2)\varepsilon v_4$	$v_2 - (1/2)\varepsilon v_5$	v_3	v_4	v_5	v_6	v_7
v_6	$v_1 + (1/2)\varepsilon v_7$	$v_2 - (1/2)\varepsilon v_6$	v_3	v_4	v_5	v_6	v_7
v_7	v_1	$v_2 + (1/2)\varepsilon v_7$	$v_2 + \varepsilon v_6$	v_4	v_5	v_6	v_7

Theorem 2 The operators generate an optimal system S_1 given by

(a1) $v_2, a_1 \neq 0\,;$

(a2) $v_3 + a_4 v_4 + a_7 v_7, \quad a_1 = a_2 = 0, a_3 \neq 0\,;$

(b1) $v_1 + a_3 v_3 + a_6 v_6, \quad a_1 \neq 0\,;$

(b2) $\alpha v_5 + a_7 v_7, a_1 = a_2 = a_3 = 0, \quad a_4 \neq 0\,;$

(c1) $v_1 + a_5 v_5 + a_6 v_6, \quad a_1 \neq 0\,;$

(c2) $v_6, a_1 = a_2 = a_3 = a_4 = a_5 = 0, \quad a_6 \neq 0\,;$

(d1) $v_1 + a_5 v_5, \quad a_1 \neq 0\,;$

(d2) $v_7, a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0, \quad a_7 \neq 0\,;$

(e1) $v_1, a_1 \neq 0\,.$

We can set the coefficients of v_i as 1, -1, 0. Note that in the group of equivalence transformations, there also include discrete transformations. Making use of S_1, we discuss the reductions and solutions of Eqs. (1) with the coefficients as 1. Omitting the computation, one can get the reductions in Table 3 using the general method.

Table 3 Invariants and Reduction.

	Invariants parameters	Reduction
a1	$u = \frac{1}{\sqrt{t}}U(\xi,\eta), \quad v = \frac{1}{\sqrt{t}}V(\xi,\eta),$ $\xi = \frac{x}{\sqrt{t}}, \quad \eta = \frac{y}{\sqrt{t}}.$	$\frac{U}{2} + \frac{1}{2}\xi U_\xi + \frac{1}{2}\eta U_\eta + UU_\eta + aVU_\xi + bU_{\eta\eta} + abU_{\xi\xi} = 0,$ $U_\xi - V_\eta = 0.$
a2	$u = U(\xi,\eta)\,, \quad v = -\frac{1}{a}t + V(\xi,\eta),$ $\xi = x - \frac{1}{2}t^2, \quad \eta = y.$	$UU_\eta + aU_\xi V + bU_{\eta\eta} + abU_{\xi\xi} = 0,$ $U_\xi - V_\eta = 0.$
b1	$u = \frac{-yt}{t^2+2} + \frac{U(\xi,\eta)}{\sqrt{t^2+2}}, \quad v = -\frac{1}{a} - \frac{(x-t)t}{a(t^2+2)} + \frac{1}{\sqrt{t^2+2}}V(\xi,\eta),$ $\xi = \frac{x-t}{\sqrt{t^2+2}}, \quad \eta = \frac{y}{\sqrt{t^2+2}}.$	$bU_{\eta\eta} + 2\eta + abU_{\xi\xi} + aU_\xi V + UV_\eta = 0,$ $U_\xi - V_\eta = 0.$
b2	$u = U(\xi,\eta), \quad v = V(\xi,\eta) - \frac{x}{at},$ $\xi = \frac{x}{t} - y, \quad \eta = t.$	$\eta U_\eta + \eta UU_\xi - aVU_\xi - b\eta U_{\xi\xi} - \frac{abU_{\xi\xi}}{\eta} - \xi U_\xi = 0,$ $U_\xi + \eta V_\xi + \frac{\eta}{a} = 0.$

Table 3 (Continued)

	Invariants parameters	Reduction
c1	$u=-\frac{1+ty}{t^2}-\frac{1}{t^2}+\frac{1}{t}U(\xi,\eta),\quad v=-\frac{1+tx}{at^2}-\frac{1}{at^2}+\frac{1}{t}V(\xi,\eta),$ $\xi=\frac{1+tx}{t^2},\quad \eta=\frac{1+ty}{t^2}.$	$2-UU_\eta-aU_\xi V-bU_{\eta\eta}-abU_{\xi\xi}=0,$ $U_\xi-V_\eta=0.$
c2	$u=U(\xi,\eta),\quad v=V(\xi,\eta),$ $\xi=y,\quad \eta=t.$	$U_\eta-UU_\xi-bU_{\xi\xi}=0,$ $V_\xi=0.$
d1	$u=-\frac{2+ty}{t^2}+\frac{1}{t}U(\xi,\eta),\quad v=-\frac{x}{ta}+\frac{1}{t}V(\xi,\eta),$ $\xi=\frac{x}{t},\quad \eta=\frac{1+ty}{t^2}.$	$2-UU_\eta-aU_\xi V-bU_{\eta\eta}-abU_{\xi\xi}=0,$ $U_\xi-V_\eta=0.$
d2	$u=U(\xi,\eta)\,,\quad V=-\frac{x}{a\eta}+V(\xi,\eta),$ $\xi=y,\quad \eta=t.$	$U_\eta-UU_\xi-bU_{\xi\xi}=0,$ $V_\xi=0.$
e1	$u=-\frac{y}{t}+\frac{1}{t}U(\xi,\eta),\quad v=-\frac{x}{at}+\frac{1}{t}V(\xi,\eta),$ $\xi=\frac{x}{t},\quad \eta=\frac{y}{t}.$	$UU_\eta+aU_\xi V+bU\eta\eta+abU\xi\xi=0,$ $U_\xi-V_\eta=0.$

4 Nonlocal Symmetries of the Reduced Equations

The potential symmetry approach is an algorithmic procedure for seeking nonlocal symmetries of PDE systems. This approach requires the existence of a conservation law of a given system. Each conservation law allows the introduction of one or more auxiliary potential variables which are nonlocally defined with respect to the original dependent variables. The resulting potential system yields nonlocal symmetries of the given system if it admits local symmetry generators that do not project onto local symmetry generators of the given system.

In this section, we seek the nonlocal symmetries of the reduced equation instead of seeking the nonlocal symmetries of the PIB equation directly. Through the complex calculation, we can know that there are no nonlocal symmetries except the case c2 and the case d2. As for the other cases, one can obtain a lot of exactly solutions using some effective method. Here we omit the solving process.

For the case c2:

$$U_\eta-UU_\xi-bU_{\xi\xi}=0\,,\quad V_\xi=0\,,\tag{5}$$

it is easy to get the solution of the second equation of Eq. (5): $V=V(x,t)+\tilde{C}$.

We seek the symmetries of the first equation of the Eq. (5). In general, one should obtain conservation laws of this equation firstly. For a PDE system, nontrivial conservation laws arise from linear combinations of the equations of the PDE system with multipliers (factors, characteristics)[7] that yield nontrivial divergence expression.

Theorem 3 A set of non-singular multipliers

$$\{\Lambda_\sigma(x,U,\partial U,\ldots,\partial^l U)\}_{\sigma=1}^N$$

yields a local conservation laws for a PDE system $R\{x;u\}$ if and only if the set of identities

$$E_U(\Lambda_\sigma(x,U,\partial U,\ldots,\partial^l U)R(x,U,\partial U,\ldots,\partial^k U))\equiv 0\,,$$

holds for arbitrary function $U(x)$, where E_U is Euler operator with respect to U defined as

$$E_U=\frac{\partial}{\partial U}-D_i\frac{\partial}{\partial U_i}+\cdots+(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial U_{i_1\cdots i_s}}+\cdots,$$

and D is total derivative.

Using this theorem, we seek multipliers of the first equation of Eq. (5), consider zeroth-order multipliers $\Lambda(\xi,\eta,U)$, first order multipliers $\Lambda(\xi,\eta,U,U_\xi)$ and second order multipliers $\Lambda(\xi,\eta,U,U_\xi,U_{\xi\xi})$ respectively. But one only obtains $\Lambda=c$, where c is an arbitrary constant. So the conservation laws have the form: $D_\eta(U)=D_\xi((1/2)U^2+bU_\xi)$.

The conservation laws yields a pair of potential equation $UW\{\xi,\eta;U,W\}$ given by

$$W_\xi=U\,,\quad W_\eta=\Bigl(\frac{1}{2}U^2+bU_\xi\Bigr)\,,\tag{6}$$

for some auxiliary potential variable $W=W(\xi,\eta)$. In Eqs. (6), potential variable W is a nonlocal variable, i.e., it cannot be expressed as a local function of give variable (ξ,η,U) and partial derivatives of U.

The point symmetry maps any solution of $UW\{\xi,\eta;U,W\}$ to a solution of $UW\{\xi,\eta;U,W\}$. Suppose $UW\{\xi,\eta;U,W\}$ is invariant under one-parameter Lie group of point transformations with corresponding infinitesimal generator

$$\tilde{v}=\alpha(\xi,\eta,U,W)\frac{\delta}{\delta\xi}+\beta(\xi,\eta,U,W)\frac{\delta}{\delta\eta}+A(\xi,\eta,U,W)\frac{\delta}{\delta U}+B(\xi,\eta,U,W)\frac{\delta}{\delta W}\,.$$

Definition The point symmetry of the potential system $UW\{\xi,\eta;U,W\}$ defines a potential symmetry of Eq. (6) if and only if the infinitesimals $(\alpha(\xi,\eta,U,W),\beta(\xi,\eta,U,W),A(\xi,\eta,U,W))$ depend explicitly on one or more components of W.

Using Lie group method one can obtain the corresponding infinitesimal generator. Omitting the detailed computation, one can get the following results by the complicated computation:

$$A(\xi,\eta,U,W)=\frac{1}{2b^{3/2}\,\mathrm{e}^{\sqrt{C_{10}}\xi/\sqrt{b}}}[(2b\sqrt{C_{10}}-\sqrt{b}U)\,\mathrm{e}^{C_{10}\eta}\,\mathrm{e}^{-W/2b}C_7C_8\,\mathrm{e}^{2\sqrt{C_{10}}\xi/\sqrt{b}}$$
$$-\,b^{3/2}(C_1U\eta+C_2U+C_1\xi+2C_4)\,\mathrm{e}^{\sqrt{C_{10}}\xi/\sqrt{b}}-C_9\,\mathrm{e}^{C_{10}\eta}(2b\sqrt{C_{10}}+\sqrt{b}U)\,\mathrm{e}^{-W/2b}C_7]\,,$$
$$B(\xi,\eta,U,W)=-\frac{1}{2}\Big(\eta b+\frac{1}{2}\xi^2\Big)C_1-C_4\xi+C_6+\frac{C_7\,\mathrm{e}^{C_{10}\eta}(C_8(\,\mathrm{e}^{\sqrt{C_{10}}\xi/\sqrt{b}})^2+C_9)\,\mathrm{e}^{-W/2b}}{\mathrm{e}^{\sqrt{C_{10}}\xi/\sqrt{b}}}\,,$$
$$\alpha(\xi,\eta,U,W)=\frac{1}{2}(C_1\eta+C_2)\xi+C_4\eta+C_5\,,\quad \beta(\xi,\eta,U,W)=\frac{1}{2}C_1\eta^2+C_2\eta+C_3\,.$$

Base on the definition, the Eqs. (5) exist potential symmetries. Suppose $C_1=0$, $C_2=1$, $C_3=0$, $C_4=0$, $C_5=0$, $C_6=0$, $C_7=1$, $C_8=1$, $C_9=1$, $C_{10}=0$, one can get

$$A(\xi,\eta,U,W)=-\frac{1}{2}\frac{U(2\,\mathrm{e}^{-W/2b}+b)}{b}\,,$$
$$B(\xi,\eta,U,W)=2\,\mathrm{e}^{-(1/2)(W/b)}\,,$$
$$\alpha(\xi,\eta,U,W)=\frac{1}{2}\xi\,,\quad \beta(\xi,\eta,U,W)=\eta\,,$$

then the vector field expression is

$$\tilde{v}'=\frac{1}{2}\xi\frac{\delta}{\delta\xi}+\eta\frac{\delta}{\delta\eta}-\frac{1}{2}\frac{U(2\,\mathrm{e}^{-(1/2)(W/b)}+b)}{b}\frac{\delta}{\delta U}+2\,\mathrm{e}^{-(1/2)(W/b)}\frac{\delta}{\delta W}\,.\tag{7}$$

We use the standard algorithm to seek exactly solutions of Eqs. (6) and the algorithm is omitted here. In order to obtain invariant variables, we should solve the characteristic equations, we write the corresponding characteristic equations in the form

$$\frac{2\,\mathrm{d}\xi}{\xi}=\frac{\mathrm{d}\eta}{\eta}=\frac{-2b\,\mathrm{d}U}{U(2\,\mathrm{e}^{-(1/2)(W/b)}+b)}=\frac{\mathrm{d}W}{2\,\mathrm{e}^{-(1/2)(W/b)}}\,.\tag{8}$$

Solving Eqs. (8), one can obtain

$$U=\frac{f(X)}{\sqrt{\eta}(\ln(\eta)+g(X))}\,,$$
$$W=-2b\ln\Big(\frac{b}{\ln(\eta)+g(X)}\Big)\,,\quad X=\frac{\xi}{\sqrt{\eta}}\,,\tag{9}$$

where f, g are arbitrary functions of the corresponding variables. As a translated canonical coordinate $\hat{X}=2\ln\xi$, which is a solution of $\tilde{v}'(\hat{X})=1$.

Variables $\{X,\hat{X},f(X,\hat{X}),g(X,\hat{X})\}$ are canonical coordinates. In the potential system $UW\{\xi,\eta;U,W\}$ perform a local change of variables $(\xi,\eta;U,W)\rightarrow(X,\hat{X},f(X,\hat{X}),g(X,\hat{X}))$ to obtain a locally equivalent system $\tilde{U}\tilde{W}\{X,\hat{X},f,g\}$.

By substituting Eq. (9) into Eqs. (6), we obtain the reduction of Eqs. (6),

$$2-Xg'-f'=0\,,\quad f=2bg'\,.\tag{10}$$

In order to solve Eq. (10), one should substitute the second equation into the first equation of Eqs. (10) and obtain the following result,

$$2-Xg'-2bg''=0\,,\tag{11}$$

setting $g'=h$ and Eq. (11) transforma into a first order ordinary differential equation as $1-Xh-2bh'=0$. Solving this equation one can get

$$h=\,\mathrm{e}^{-X^2/4b}\Big(\frac{1}{b}\int\,\mathrm{e}^{X^2/4b}\,\mathrm{d}X+\tilde{C}_1\Big)\,,\tag{12}$$

where $\tilde{C}_1$ is integral constant. In the Eq. (12), the integration $\int\,\mathrm{e}^{X^2/4b}\,\mathrm{d}X$ can not be expressed as linear combination of some elementary function. But it can be expressed as Error function, i.e.

$$\int\,\mathrm{e}^{X^2/4b}\,\mathrm{d}X=\frac{\sqrt{\pi}\,\mathrm{erf}\,((1/2)\sqrt{-(1/b)}X)}{\sqrt{-1/b}}\,.$$

Error function is expressed as

$$\mathrm{erf}\,(\theta)=\frac{2}{\sqrt{\pi}}\int_0^\theta\,\mathrm{e}^{-\delta^2}\,\mathrm{d}\delta\,.$$

So the solutions of Eqs. (10) are

$$f=\Big(\frac{2\sqrt{\pi}\,\mathrm{erf}\,((1/2)\sqrt{-1/b}X)}{\sqrt{-1/b}}+\tilde{C}_1\Big)\,\mathrm{e}^{-X^2/4b}\,,$$
$$g=\int\frac{(2\sqrt{\pi}\,\mathrm{erf}\,((1/2)\sqrt{-1/b}X)+\tilde{C}_1\sqrt{-1/b})\,\mathrm{e}^{-X^2/4b}}{2b\sqrt{-1/b}}\,\mathrm{d}X+\tilde{C}_2.\tag{13}$$

Thus, we can obtain a new solution of Eq. (6) by substituting Eqs. (13) into Eq. (9),

$$U=\frac{(2\sqrt{\pi}\,\mathrm{erf}\,((1/2)/\sqrt{-(1/b)}X)/\sqrt{-1/b}+\tilde{C}_1)\,\mathrm{e}^{-X^2/4b}}{\sqrt{\eta}(\ln(\eta)+\int[(2\sqrt{\pi}\,\mathrm{erf}\,((1/2)\sqrt{-(1/b)}X)+\tilde{C}_1\sqrt{-1/b})\,\mathrm{e}^{-X^2/4b}/2b\sqrt{-1/b}]\,\mathrm{d}X+\tilde{C}_2)}\,,$$
$$W=-2b\ln\Big(\frac{b}{\ln(\eta)+\int[(2\sqrt{\pi}\,\mathrm{erf}\,((1/2)\sqrt{-(1/b)}X)+\tilde{C}_1\sqrt{-1/b})\,\mathrm{e}^{-X^2/4b}/2b\sqrt{-1/b}]\,\mathrm{d}X+\tilde{C}_2}\Big),$$

where $X=\xi/\sqrt{\eta}$.

And the new solution of the PIB equation will be the following form:

$$u(x,y,t)=\frac{(2\sqrt{\pi}\,\mathrm{erf}\,((1/2)\sqrt{-(1/b)}X)/\sqrt{-1/b}+\tilde{C}_1)\,\mathrm{e}^{-X^2/4b}}{\sqrt{\eta}(\ln(\eta)+\int[(2\sqrt{\pi}\,\mathrm{erf}\,((1/2)\sqrt{-(1/b)}X)+\tilde{C}_1\sqrt{-1/b})\,\mathrm{e}^{-X^2/4b}/2b\sqrt{-1/b}]\,\mathrm{d}X+\tilde{C}_2)}\,,$$
$$v(x,y,t)=V(x,t)+\tilde{C}_3\,, \tag{14}$$

where $X=y/\sqrt{t}$, $\eta=t$.

On the other hand, we can discuss the nonlocally related subsystem of potential system $UW\{\xi,\eta;U,W\}$. The dependent variable U can be eliminated from the potential system $UW\{\xi,\eta;U,W\}$ to yield the subsystem $W\{\xi,\eta;W\}$ given by

$$\frac{\partial}{\partial\eta}W-\frac{1}{2}\Big(\frac{\partial}{\partial\xi}W\Big)^2-b\Big(\frac{\partial^2}{\partial\xi^2}W\Big)=0\,, \tag{15}$$

which is nonlocally related to the given PDE but locally related to the potential system $UW\{\xi,\eta;U,W\}$.

We solve the subsystem $W\{\xi,\eta;W\}$ and obtain the solutions of PIB equations. One can obtain the corresponding infinitesimal generator by the complicated computation using the Lie group method:

$$\tilde{v}=\Big(\frac{\xi C'_1\eta-2C'_4\eta+C'_2\xi}{2}+C'_9\Big)\frac{\delta}{\delta\xi}+\Big(\frac{C'_1\eta^2}{2}+C'_2\eta+C'_3\Big)\frac{\delta}{\delta\eta}$$
$$+\Big(\frac{1}{4}\frac{4C'_8\,\mathrm{e}^{C'_{10}b\eta}((\,\mathrm{e}^{\sqrt{C'_{10}}\xi})^2C'_6+C'_7)\,\mathrm{e}^{-m/2b}-2\,\mathrm{e}^{\sqrt{C'_{10}}\xi}((\eta b+\xi^2/2)C'_1-2C'_4\xi-2C'_5)}{\mathrm{e}^{\sqrt{C'_{10}}\xi}}\Big)\frac{\delta}{\delta W}\,.$$

Suppose $C'_1=0$, $C'_2=1$, $C'_3=0$, $C'_4=1$, $C'_5=0$, $C'_6=0$, $C'_7=0$, $C'_9=0$, $C'_{10}=0$, one can obtain:

$$\tilde{v}''=\Big(-\eta+\frac{1}{2}\xi\Big)\frac{\delta}{\delta\xi}+\eta\frac{\delta}{\delta\eta}+\xi\frac{\delta}{\delta W}\,. \tag{16}$$

Solving the corresponding characteristic equations, one can obtain:

$$W=2\xi+2\eta+F(K)\,, \tag{17}$$

where $K=(\xi+2\eta)/\sqrt{\eta}$.

By substituting Eq. (17) into Eq. (15), we obtain the reduction of Eq. (15),

$$\frac{1}{2}KF'+\frac{1}{2}F'^2+bF''=0\,. \tag{18}$$

If we set $F'=G$, then Eq. (18) transforms into the following form:

$$G'=-\frac{K}{2b}G-\frac{1}{2b}G^2\,, \tag{19}$$

one can see that the Eq. (19) is Riccati type equation.

By setting $G(K)=-2bH(K)$, Eq. (19) becomes the following form:

$$H'=H^2-\frac{1}{2b}KH\,. \tag{20}$$

Theorem 4 If $\tilde{H}$ is the solution of the second order ODE $\tilde{H}''+(K/2b)\tilde{H}'=0$, then $-\tilde{H}'/\tilde{H}$ is the solution of the Riccati equation (20).

It is easy to solve the equation $\tilde{H}''+(K/2b)/\tilde{H}'=0$. The solution of the Eq. (20) has the form:

$$H=\frac{-\,\mathrm{e}^{-K^2/4b}}{\int\,\mathrm{e}^{-K^2/4b}\mathrm{d}K}\,.$$

By using the formulas $G(K)=-2bH(K)=F'$, one can get :

$$F'=\frac{2b\,\mathrm{e}^{-K^2/4b}}{\int\,\mathrm{e}^{-K^2/4b}\mathrm{d}K}\,,$$

and with the help of the first equation of Eqs. (6) and the Eq. (17), one can obtain $U=2+F'K_\xi$, i.e.

$$U=2+\frac{2b\,\mathrm{e}^{-K^2/4b}}{\sqrt{\eta}\int\,\mathrm{e}^{-K^2/4b}\mathrm{d}K}\,.$$

So we can get the new solution of PIB equation:

$$u(x,y,t)=2+\frac{2b\,\mathrm{e}^{-K^2/2b}}{\sqrt{\eta}\int\,\mathrm{e}^{-K^2/4b}\mathrm{d}K}\,,$$
$$v(x,y,t)=V(x,t)+\tilde{C}_4\,, \tag{21}$$

where $K=(y+2t)/\sqrt{t}$, $\eta=t$.

5 Conclusion and Discussion

In this paper, the symmetries of the PIB equations are investigated by means of the classical Lie symmetry method. The symmetry algebra and group of Eqs. (1) are obtained. Specially, the most general one-parameter groups of symmetry are given out as the composition of transforms in the seven various one-subgroups $\exp(\epsilon v_1),\ldots,\exp(\epsilon v_7)$. Next, the one-dimensional subalgebras of the Lie algebra of Eqs. (1) are classified. Finally, the nonlocal symmetries and nonlocal solutions of PIB equations are obtained.

Using nonlocal symmetries to construct explicit solutions of PDEs is of considerable interest and value. It would be possible to extend this approach to many other PDEs. However, there is not a universal way to obtain the nonlocal symmetries. One can use various methods to obtain valuable results which is worthy of our further study.

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