Commun. Theor. Phys. (Beijing, China) **51** (2009) pp. 470–474

Vol. 51, No. 3, March 15, 2009

Function Projective Synchronization in Discrete-Time Chaotic System with Uncertain Parameters*

CHEN Yong† and LI Xin

Shanghai Key Laboratory of Trustworthy Computing, East China Normal University, Shanghai 200062, China

Nonlinear Science Center and Department of Mathematics, Ningbo University, Ningbo 315211, China

(Received February 15, 2008; Revised December 30, 2008)

Abstract *The function projective synchronization of discrete-time chaotic systems is presented. Based on backstepping design with three controllers, a systematic, concrete and automatic scheme is developed to investigate function projective synchronization (FPS) of discrete-time chaotic systems with uncertain parameters. With the aid of symbolic-numeric computation, we use the proposed scheme to illustrate FPS between two identical 3D Hénon-like maps with uncertain parameters. Numeric simulations are used to verify the effectiveness of our scheme.*

PACS numbers: 05.45.Xt
Key words: function projective synchronization, backstepping design, discrete-time chaotic system, Hénon-like map

1 Introduction

Chaos synchronization has received considerable attention since the pioneering works of Fujisaka and Yamada,[1] Pecora, and Carroll,[2] Pyragas,[3] Ott, Grebogi, and Yorke.[4] Up to now, there exist many types of chaos synchronization in dynamical systems such as complete synchronization, partial synchronization, phase synchronization, lag synchronization, anticipated synchronization, generalized slag, anticipated, and completed synchronization, synchronization, antiphase synchronization, etc.[5−8] In particular, amongst all kinds of chaos synchronization, projective synchronization in partially linear systems reported by Mainieri and Rehacek[9] is one of the most noticeable ones where the drive and response vectors evolve in a proportional scale — the vectors become proportional. Recently, some researchers[10−13] extended the projective synchronization to non-partially-linear systems. Many powerful methods have been reported to investigate some types of chaos synchronization in continuous-time systems. In fact, many mathematical models of neural networks, biological process, physical process and chemical process, etc., were defined using discrete-time dynamical systems.[14−18] Recently, more and more attentions were paid to the chaos control and synchronization in discrete-time dynamical systems.[18−22]

Backstepping design[18,23−25] has become a systematic and powerful method for the construction of both feedback controllers and associated Lyapunov functions. The design method has been applied to investigate control and synchronization of many continuous-time dynamical systems.[25−28] Up until now, some articles have been reported to extend the backstepping design to deduce some proper controllers to investigate chaos control and synchronization in some discrete-time dynamical systems.[18−22] The synchronization of chaotic systems with uncertain parameters was investigated in Refs. [29] ∼ [34].

More recently, in Refs. [13] and [35] we have the proposed function projective synchronization (FPS) in the continuous-time systems where the drive and response vectors evolve in a proportional scale function matrix. Based on the FPS method and symbolic computation *Maple*, in Ref. [13] the function projective synchronization of two identical chaotic systems (two identical classic Lorenz systems) is achieved up to a scaling function matrix with different initial values. In Ref. [35], the function projective synchronization of two different systems (the unified chaotic system and the Rössler system) is achieved up to a scaling function matrix f with different initial values.

In this paper, on the lines of the function synchronization thought,[13,35] we would like to define a type of function projective synchronization in discrete-time dynamical systems with uncertain parameters. Here based on the backstepping design method, we present a systematic and automatic algorithm to investigate simultaneously FPS, via controllers between discrete-time drive system and response system. With the aid of symbolic-numeric computation, the proposed scheme is used to illustrate FPS between two identical Hénon-like maps with uncertain parameters. Moreover numerical simulations are used to verify the effectiveness of the proposed scheme.

*The project supported by the National Natural Science Foundation of China under Grant Nos. 10735030 and 90718041, Shanghai Leading Academic Discipline Project under Grant No. B412, Zhejiang Provincial Natural Science Foundations of China under Grant No. Y604056, Doctoral Foundation of Ningbo City under Grant No. 2005A61030 and Program for Changjiang Scholars and Innovative Research Team in University under Grant No. IRT0734

†E-mail: ychen@sei.ecnu.edu.cn

This paper is arranged as follows. In Sec. 2, We introduce FPS in discrete-time systems to investigate function projective synchronization in two identical Hénon-like maps. Finally, some conclusions and discussions are given in Sec. 3.

2 FPS of Two Identical Hénon-like Maps with Two Uncertain Parameters

In the following, the definitions of function projective synchronization in discrete-time dynamical systems is introduced, a Lyapunov stability theory for discrete-time dynamical systems is given.

Definition For two discrete-time (chaotic or hyperchaotic) dynamical systems (i) $x(k+1) = F(x(k))$ and (ii) $y(k+1) = G(y(k)) + u(x(k), y(k))$, where $(x(k), y(k)) \in R^{m+m}, k \in Z/Z^{-}$, and $u(x(k), y(k)) \in R^m$, let (iii) $e(k) = (e_1(k), e_2(k), \ldots, e_m(k)) = (y_1(k) - f_1(x(k))x_1(k),\, y_2(k) - f_2(x(k))x_2(k), \ldots, y_m(k) - f_m(x(k))x_m(k))$ be boundary vector functions, if there exists proper controllers $u(x(k), y(k)) = (u_1(x(k), y(k)), u_2(x(k), y(k)), \ldots, u_m(x(k), y(k)))^{\mathrm{T}}$ such that $\lim_{k\to\infty}(e(k)) = 0$, we say that there exists **function projective synchronizaton (FPS)** between the systems (i) and (ii).

Based on the Lyapunov stability theory, for the error discrete-time (iii) generated by drive system (i) and response system (ii), let $L(e_1(k), e_2(k), \ldots, e_m(k))|_{e_i(k)\equiv 0 (i=1,2,\ldots,m)=0}$, if $\triangle L(k) = L(k+1) - L(k) \leq 0$, with the equality holding if and only if $e_i(k) \equiv 0$ $(i = 1, 2, \ldots, m)$, it is said that systems (i) and (ii) are function projective synchronized.

In this letter based on the backstepping design method, we would like to present a systematic, generalized and constructive scheme to seek the controllers such that two identical 3D Hénon-like maps with strict-feed form are function projective synchronized.

Consider the 3D discrete-time Henon-like map

$$
\begin{aligned}
x_1(k+1) &= 1 + x_3(k) - \alpha x_2^2(k)\,,\\
x_2(k+1) &= 1 + \beta x_2(k) - \alpha x_1^2(k)\,,\\
x_3(k+1) &= \beta x_1(k)
\end{aligned} \tag{1}
$$

as the drive system, and the response system is followed as:

$$
\begin{aligned}
y_1(k+1) &= 1 + y_3(k) - \alpha_1(k) y_2^2(k) + u_1(x, y)\,,\\
y_2(k+1) &= 1 + \beta_1(k) y_2(k) - \alpha_1(k) y_1^2(k) + u_2(x, y)\,,\\
y_3(k+1) &= \beta_1(k) y_1(k) + u_3(x, y)\,,
\end{aligned} \tag{2}
$$

where $\alpha_1(k)$ and $\beta_1(k)$ are uncertain parameters, which estimate the parameters of α and β. And u_1, u_2, and u_3 are the controllers such that two chaotic systems can be synchronized in the sense of FPS. In the following, we would like to realize the FPS of two identical Hénon-like maps with two uncertain parameters by backstepping design method.

Let the error states be $e_1 = x_1 - 2y_1$, $e_2 = x_2 + y_2$, $e_3 = x_3 - (1 + \tanh^2 x_3) y_3$, $e_4(k) = \alpha_1(k) - \alpha$, $e_5(k) = \beta_1(k) - \beta$. Then from (3.1) and (3.2), we have the discrete-time error dynamical system

$$
\begin{aligned}
e_1(k+1) &= -1 + x_3(k) - \alpha x_2^2(k) - 2y_3(k) + 2\alpha_1(k) y_2^2(k) - 2u_1(x, y)\,,\\
e_2(k+1) &= 2 + \beta x_2(k) - \alpha x_1^2(k) + \beta_1(k) y_2(k) - \alpha_1(k) y_1^2(k) + u_2(x, y)\,,\\
e_3(k+1) &= \beta x_1(k) - (1 + \tanh(\beta x_1(k))^2)(\beta_1(k) y_1(k) + u_3(x, y))\,.
\end{aligned} \tag{3}
$$

In the following based on the backstepping design and the improved ideas of Refs. [25–27], we give a systematic and constructive algorithm to derive the controllers $u(x, y)$ step by step such that systems (1) and (2) are synchronized together.

Step 1 Let the first partial Lyapunov function be $L_1(k) = |e_1(k)|$ and the second error variable be

$$
e_2(k) = e_1(k+1) - c_{11} e_1(k) \tag{4}
$$

where $c_{11} \in R$. Then we have the derivative of $L_1(k)$

$$
\Delta L_1(k) = |e_1(k+1)| - |e_1(k)| \leq (|c_{11}| - 1)|e_1(k)| + |e_2(k)|\,. \tag{5}
$$

Step 2 Let the second partial Lyapunov function candidate be $L_2(k) = L_1(k) + d_1|e_2(k)|$ and the third error variable be

$$
e_3(k) = e_2(k+1) - c_{21} e_1(k) - c_{22} e_2(k)\,, \tag{6}
$$

where $d_1 > 1, c_{21}, c_{22} \in R$. Therefore, from (4) and (6) we have the derivative $L_2(k)$,

$$
\Delta L_2(k) = L_2(k+1) - L_2(k) \leq (d_1|c_{21}| + |c_{11}| - 1)|e_1(k)| + (d_1|c_{22}| + 1 - d_1)|e_2(k)| + d_1|e_3(k)|\,. \tag{7}
$$

Step 3 Let the third partial Lyapunov function candidate be $L_3(k) = L_2(k) + d_2|e_3(k)|$ and the forth error state be

$$
e_4(k) = e_3(k+1) - c_{31} e_1(k) - c_{32} e_2(k) - c_{33} e_3(k)\,, \tag{8}
$$

where $d_2 > d_1 > 1, c_{31}, c_{32}, c_{33} \in R$. Therefore, from (6) and (8) we have the derivative $L_3(k)$,

$$
\Delta L_3(k) = L_3(k+1) - L_3(k)
$$

$$\leq (d_2|c_{31}| + d_1|c_{21}| + |c_11| - 1)|e_1(k)| + (d_2|c_{32}| + d_1(|c_{22}| - 1) + 1)|e_2(k)|$$
$$+ (d_2|c_{33}| + d_1 - d_2)|e_3(k)| + d_2|e_4(k)|\,. \tag{9}$$

Step 4 Let the fourth partial Lyapunov function candidate be $L_4(k) = L_3(k) + d_3|e_4(k)|$ and the fourth error state be

$$e_5(k) = e_4(k+1) - c_{41}e_1(k) - c_{42}e_2(k) - c_{43}e_3(k) - c_{44}e_4(k)\,, \tag{10}$$

where $d_3 > d_2 > d_1 > 1, c_{41}, c_{42}, c_{43}, c_{44} \in R$. Therefore, from (8) and (10) we have the derivative $L_4(k)$,

$$\Delta L_4(k) = L_4(k+1) - L_4(k)$$
$$\leq (d_3|c_{41}| + d_2|c_{31}| + d_1|c_{21}| + |c_11| - 1)|e_1(k)| + (d_3|c_{42}| + d_2|c_{32}| + d_1(|c_{22}| - 1) + 1)|e_2(k)|$$
$$+ (d_3|c_{43}| + d_2|c_{33}| + d_1 - d_2)|e_3(k)| + (d_3|c_{44}| + d_2 - d_3)|e_4(k)| + d_3|e_5(k)|\,. \tag{11}$$

Step 5 Let the Lyapunov function be $L(k) = L_4(k) + d_4|e_5(k)|$. From the above steps we have

$$e_5(k+1) - c_{51}e_1(k) - c_{52}e_2(k) - c_{53}e_3(k) - c_{54}e_4(k) + c_{55}e_5(k) = 0\,, \tag{12}$$

where $d_4 > d_3 > d_2 > d_1 > 1, c_{51}, c_{52}, c_{53}, c_{54}, c_{55} \in R$. Then from (10), (11), and (12), we obtain the derivative of the Lyapunov function $L(k)$,

$$\Delta L(k) = L(k+1) - L(k)$$
$$\leq (d_4|c_{51}| + d_3|c_{41}| + d_2|c_{31}| + d_1|c_{21}| + |c_11| - 1)|e_1(k)| + (d_4|c_{52}| + d_3|c_{42}| + d_2|c_{32}|$$
$$+ d_1(|c_{22}| - 1) + 1)|e_2(k)| + (d_4|c_{53}| + d_3|c_{43}| + d_2|c_{33}| + d_1 - d_2)|e_3(k)| + (d_4|c_{54}|$$
$$+ d_3|c_{44}| + d_2 - d_3)|e_4(k)| + (d_3 - d_4 + d_4|c_{55}|)|e_5(k)|\,. \tag{13}$$

With the aid of symbolic computation, from (4), (6), (8), (10), and (12), we can determine the scalar controllers of $u(x, y)$ in the form

$$u_1(x,y) = -1/2 + 1/2\,x_3(k) - 1/2\,\alpha\,(x_2(k))^2 - y_3(k) + \alpha_1(k)(y_2(k))^2 - 1/2\,c_{11}x_1(k)$$
$$+ c_{11}y_1(k) - 1/2\,x_2(k) - 1/2\,y_2(k)\,,$$
$$u_2(x,y) = -2 - \beta\,x_2(k) + \alpha\,(x_1(k))^2 - \beta_1(k)y_2(k) + \alpha_1(k)(y_1(k))^2 + c_{21}x_1(k) - 2\,c_{21}y_1(k)$$
$$+ c_{22}x_2(k) + c_{22}y_2(k) + x_3(k) - y_3(k) - y_3(k)(\tanh(x_3(k)))^2\,,$$
$$u_3(x,y) = \frac{A}{B}\,,$$

where

$$A = \beta\,x_1(k) - \beta_1(k)y_1(k) - (\tanh(\beta\,x_1(k)))^2\beta_1(k)y_1(k) - c_{31}x_1(k) + 2c_{31}y_1(k) - c_{32}x_2(k) - c_{32}y_2(k)$$
$$- c_{33}x_3(k) + c_{33}y_3(k) + c_{33}y_3(k)(\tanh(x_3(k)))^2 - \alpha_1(k) + \alpha\,,$$
$$B = 1 + (\tanh(\beta\,x_1(k)))^2\,,$$

and

$$\alpha_1(k+1) = \alpha + c_{41}x_1(k) - 2\,c_{41}y_1(k) + c_{42}x_2(k) + c_{42}y_2(k) + c_{43}x_3(k)$$
$$- c_{43}y_3(k) - c_{43}y_3(k)(\tanh(x_3(k)))^2 + c_{44}\alpha_1(k) - c_{44}\alpha + \beta_1(k) - \beta\,,$$
$$\beta_1(k+1) = \beta + c_{51}x_1(k) - 2\,c_{51}y_1(k) + c_{52}x_2(k) + c_{52}y_2(k) + c_{53}x_3(k)$$
$$- c_{53}y_3(k) - c_{53}y_3(k)(\tanh(x_3(k)))^2 + c_{54}\alpha_1(k) - c_{54}\alpha + c_{55}\beta_1(k) - c_{55}\beta\,.$$

From (13), we know that the right-hand side of (13) is negative-definite, if the parameters d_i $(i = 1, 2, 3, 4)$ and c_{ij} $(1 \leq j \leq i \leq 4)$ satisfy

$$ld_1|c_{21}| + d_2|c_{31}| + d_3|c_{41}| + d_4|c_{51}| + |c_{11}| < 1\,,$$
$$d_1|c_{22}| + d_2|c_{32}| + d_3|c_{42}| + d_4|c_{52}| < d_1 - 1\,,$$
$$d_2|c_{33}| + d_3|c_{43}| + d_4|c_{53}| < d_2 - d_1\,,$$
$$d_3|c_{44}| + d_4|c_{54}| < d_3 - d_2\,,$$
$$|c_{55}| < \frac{d_4 - d_3}{d_4}\,,$$

then $\Delta L(k)$ is negative-definite, which denotes that the resulting close-loop discrete-time system

$$\begin{pmatrix} e_1(k+1) \\ e_2(k+1) \\ e_3(k+1) \\ e_4(k+1) \\ e_5(k+1) \end{pmatrix} = \begin{pmatrix} c_{11} & 1 & 0 & 0 & 0 \\ c_{21} & c_{22} & 1 & 0 & 0 \\ c_{31} & c_{32} & c_{33} & 0 & 0 \\ c_{41} & c_{42} & c_{43} & c_{44} & 0 \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} \end{pmatrix} \begin{pmatrix} e_1(k) \\ e_2(k) \\ e_3(k) \\ e_4(k) \\ e_5(k) \end{pmatrix}$$

is globally asynptotically stable and $\lim_{k\to+\infty} e_i(k) = 0$, that is to say, the Hénon map discrete-time systems (1) and (2) are function projective synchronized.

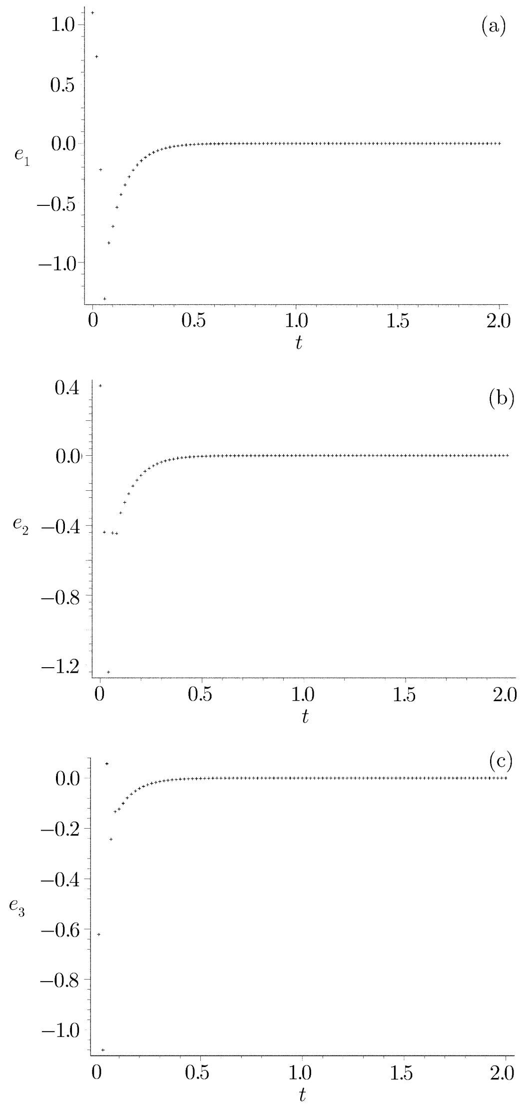

Fig. 1 The orbits of the error states.

In the following we use numerical simulations to verify the effectiveness of the obtained controllers $u(x, y)$. Here we take $\alpha = 1.4$, $\beta = 0.2$, $c_{11} = 0.3$, $c_{21} = 0.02$, $c_{22} = 0.4$, $c_{31} = 0.05$, $c_{32} = 0.1$, $c_{33} = -0.2$, $c_{41} = 0.01$, $c_{42}, c_{43} = 0.03$, $c_{44} = 0.04$, $c_{51} = 0.01$, $c_{52} = 0.02$, $c_{53} = 0.03$, $c_{54} = 0.04$, $c_{55} = 0.05$, $d_1 = 2$, $d_2 = 3$, $d_3 = 5$, $d_4 = 6$ and the initial values $[x_1(0) = 0.1, x_2(0) = 0.2, x_3(0) = -0.5]$, $[y_1(0) = -0.5, y_2(0) = 0.2, y_3(0) = 0.1]$, and $\alpha_1(0) = 0.1$, $\beta_1(0) = 0.1$ respectively. The graphs of the error states are shown in Figs. 1(a)-1(c), and simulations of the two parameters $\alpha_1(k), \beta_1(k)$ are displayed in Figs. 2(a) and 2(b). Finally we give the attractors after being synchronized with controllers are displayed in Fig. 3.

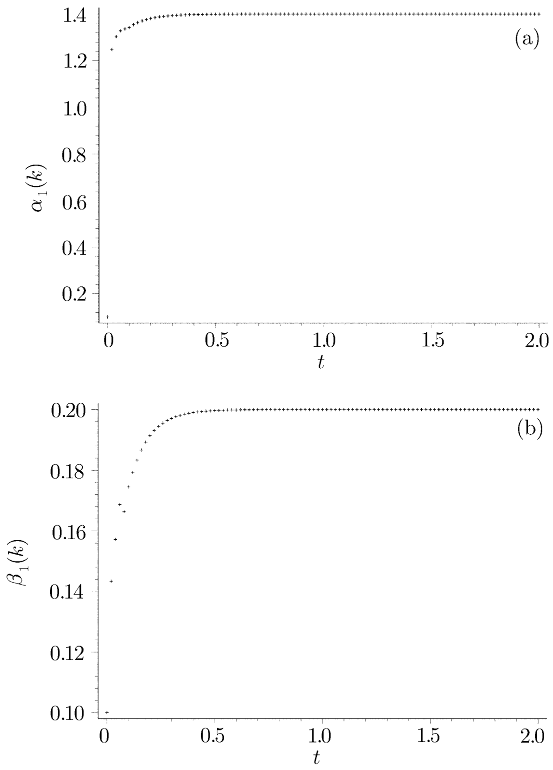

Fig. 2 The orbits of uncertain parametes.

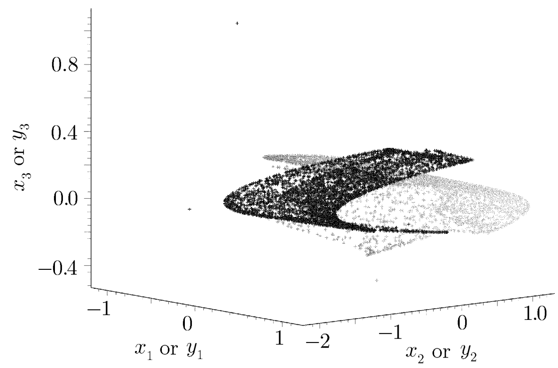

Fig. 3 The two attractors after being synchronized with $(f_1(x), f_2(x), f_3(x)) = (2, -1, 1+\tanh(x_3(k))^2)$: the dark one is the response system with the controllers, and the other is the drive system.

3 Summary and Conclusions

In summary, we have defined function projective synchronization in discrete-time dynamical systems. And then based on backstepping design with controllers, a systematic and automatic scheme is developed investigate FPS between the discrete-time drive systems and response systems with strict-feedback forms. With the aid of symbolic-numeric computation, we use the proposed scheme to illustrate FPS between two identical 3D Hénon-like maps with uncertain parameters. Numerical simulations are used to verify the effectiveness of the proposed scheme.

References

[1] H. Fujisaka and T. Yamada, Prog. Theor. Phys. **69** (1983) 32.

[2] L.M. Pecora and T.L. Carroll, Phys. Rev. Lett. **64** (1990) 821; T.L. Carroll and L.M. Pecora, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **38** (1991) 453

[3] K. Pyragas, Phys. Lett. A **170** (1992) 421; K. Pyragas, Phys. Lett. A **181** (1993) 203.

[4] E. Ott, C. Grebogi, and J.A. Yorke, Phys. Rev. Lett. **64** (1990) 1196.

[5] G. Chen and X. Dong, *From Chaos to Order*, World Scientific, Singapore (1998).

[6] X. Li and Y. Chen, Commun. Theor. Phys. **48** (2007) 132; X. Li and Y. Chen, Commun. Theor. Phys. **48** (2007) 864.

[7] X.Y. Wang and M.J. Wang, Int. J. Mod. Phys. C **18(3)** (2007) 399.

[8] Z.Y. Yan, Chaos **15** (2005) 023902; Phys. Lett. A **334** (2005) 406.

[9] R. Mainieri and J. Rehacek, Phys. Rev. Lett. **82** (1999) 3042.

[10] J. Yan and C. Li, Chaos, Solitons and Fractals **26** (2005) 1119.

[11] G. Wen and D. Xu, Chaos, Solitons and Fractals **26** (2005) 71.

[12] G.H. Li, Chaos, Solitons and Fractals **32** (2007) 1786.

[13] Y. Chen and X. Li, Int. J. Modern Phys. C **18** (2007) 883.

[14] T. Yamakawa, *et al.*, *Proc. of the 2nd International Conference on Fuzzy Logic and Neural Networks*, Iizuka, Japan, July **17-22** (1992) pp. 145–148.

[15] M. Henon, Commun. Math. Phys. **50** (1976) 69; N.F. Rulkov, Phys. Rev. Lett. **86** (2001) 183.

[16] M. Itoh, T. Yang, and L.O. Chua, Int. J. Bifurcation Chaos Appl. Sci. Eng. **11** (2001) 551; K. Konishi and H. Kokame, Phys. Lett. A **248** (1998) 359.

[17] G.M. Zaslavsky, M. Edelman, and B.A. Niyazov, Chaos **7** (1997) 159; M. Itoh, *et al.*, IEICE Trans. Fundamentals E **77** (1994) 2092.

[18] Z.Y. Yan, Chaos **16** (2006) 013119; Z.Y. Yan, Phys. Lett. A **342** (2005) 309.

[19] P.C. Yeh and P.V. Kokotovic, Int. J. Control **62** (1995) 303.

[20] J. Lu, *et al.*, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **48** (2001) 1359 .

[21] S.S. Ge, G.Y. Li, and T.H. Lee, Automatica **39** (2003) 807.

[22] L. Huang, M. Wang, and R. Feng, Chaos, Solitons and Fractals **23** (2005) 617.

[23] I. Kanellakopoulos, P.V. Kokotovic, and A.S. Morse, IEEE Trans. Autom. Control **36** (1991) 1241.

[24] M. Krstic, *et al.*, *Nonlinear Adaptive Control Design*, Wiley, New York (1995).

[25] C. Wang and S.S. Ge, Chaos, Solitons and Fractals **12** (2001) 1199.

[26] Q. Wang and Y. Chen, Appl. Math. Comp. **181** (2006) 48.

[27] C. Wang and S.S. Ge, Int. J. Bifurcation Chaos Appl. Sci. Eng. **11** (2001) 1115.

[28] S.H. Chen, D. Wang, L. Chen, Q. Zhang, and C. Wang, Chaos **14** (2004) 539.

[29] Y. Wang, Z.H. Guan, and H.O. Wang, Phys. Lett. A **312** (2003) 34.

[30] J. Lu, X. Wu, and X. Han, Phys. Lett. A **329** (2004) 327; X. Han, *et al.*, Chaos, Solitons and Fractals **22** (2004) 221.

[31] M.T. Yassen, Chaos, Solitons and Fractals **23** (2005) 1245; M.T. Yassen, Appl. Math. Comput. **135** (2001) 113.

[32] D.B. Huang and R.W. Guo, Chaos **14** (2004) 152.

[33] J.Q. Lu and J.D. Cao, Chaos **15** (2005) 043901.

[34] J.H. Park, Chaos, Solitons and Fractals **25** (2005) 333.

[35] Y. Chen and X. Li, Z. Naturforsch **62a** (2007) 29.