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ADM-Padé technique for the nonlinear lattice equations

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ABSTRACT

ADM-Padé technique is a combination of Adomian decomposition method (ADM) and Padé approximants. We solve two nonlinear lattice equations using the technique which gives the approximate solution with higher accuracy and faster convergence rate than using ADM alone. Bell-shaped solitary solution of Belov–Chaltikian (BC) lattice and kink-shaped solitary solution of the nonlinear self-dual network equations (SDNEs) are presented. Comparisons are made between approximate solutions and exact solutions to illustrate the validity and the great potential of the technique.

1. Introduction

The study of differential–difference equations (DDEs) has received considerable attention in recent years [1–15]. The DDEs play an important role in modelling complicated physical phenomena (particle vibrations in lattices, current flow in electrical networks, pulses in biological chains, etc.).

ADM-Padé technique, which is a combination of Adomian decomposition method (ADM) [16–18] and Padé approximants [19,20], has been used to solve DDEs and PDEs by various researchers. Abassy [21] solved Burgers and good Boussinesq equations. Basto [22] approximated the theoretical solution of the Burgers equation. Wazwaz solved the Thomas–Fermi equation [23] and approximated Volterra's population model [24]. Wang [14] derived the solitary solution of the discrete hybrid equation.

In this paper, we solve two nonlinear lattice equations using ADM-Padé technique. Firstly, we consider Belov–Chaltikian (BC) lattice defined by

$$\begin{aligned}\frac{\partial u_n}{\partial t} &= u_n(u_{n+1} - u_{n-1}) - v_n + v_{n-1},\\ \frac{\partial v_n}{\partial t} &= v_n(u_{n+2} - u_{n-1}),\end{aligned} \tag{1}$$

which were found in the study of lattice analogues of W-algebras by Belov and Chaltikian [5]. BC lattice equations have rich mathematical structures. Belov and Chaltikian established the bi-Hamiltonian structure of this system [5]. Sahadevan and his co-worker not only derived a sequence of conserved densities and generalized symmetries [6] but also obtained (2×2) matrix recursion operator [7]. In Ref. [8], Hu and Zhu derived a Bäcklund transformation, nonlinear superposition formula and obtained multi-soliton solutions of BC lattice. Secondly, we consider the nonlinear self-dual network which is one of the typical integrable nonlinear lattice system and describes the propagation of electrical signals in a cascade of four-terminal nonlinear LC self-dual circuits. The nonlinear self-dual network equations (SDNEs) read

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$$\begin{aligned}\frac{\partial u_n}{\partial t} &= (1+\gamma u_n^2)(v_n - v_{n+1}),\\ \frac{\partial v_n}{\partial t} &= (1+\gamma v_n^2)(u_{n-1} - u_n),\end{aligned} \tag{2}$$

where $\gamma = \pm 1,\ u_n$ and v_n are the voltage and current in the nth capacitance and inductance of the network, respectively. Zhang et al. [15] achieved a series of bright solitons, dark solitons, and kink solitons besides the known bright soliton and kink soliton by the real exponential approach. We solve the two nonlinear lattice equations using ADM-Padé technique converting truncated series solutions of ADM into diagonal Padé approximants. The solitary solutions are obtained with higher accuracy and faster convergence rate than using ADM alone. Comparisons are made between numerical solutions and exact solutions to illustrate the validity and the great potential of the technique.

The paper is organized as follows. In next section, ADM-Padé technique for the differential–difference equations is outlined. In Section 3, Belov–Chaltikian lattice is studied. In Section 4, the nonlinear self-dual network equations are studied. Finally, some discussions and conclusions are given.

2. The description of ADM-Padé technique

2.1. The description of ADM for solving the DDEs

For the purposes of the illustration of the decomposition method, we consider a system of nonlinear differential–difference equations as follows:

$$\begin{aligned}&L_i(u_i(n,t)) + R_i(u_1(n,t), u_1(n+1,t), u_1(n-1,t), \ldots, u_k(n,t), u_k(n+1,t), u_k(n-1,t), \ldots)\\ &\quad + N_i(u_1(n,t), u_1(n+1,t), u_1(n-1,t), \ldots, u_k(n,t), u_k(n+1,t), u_k(n-1,t), \ldots) = g_i,\\ &\quad i = 1,2,\ldots,k,\end{aligned} \tag{3}$$

where $u_i(n,t)$ is the unknown function with respect to the discrete spatial variable n and the temporal variable t, L_i is the highest-order derivative which is assumed to be invertible, R_i is the remaind linear operator, N_i is the nonlinear operator and g_i is the source term. So we have

$$\begin{aligned}L_i(u_i(n,t)) = g_i &- R_i(u_1(n,t), u_1(n+1,t), u_1(n-1,t), \ldots, u_k(n,t), u_k(n+1,t), u_k(n-1,t), \ldots)\\ &- N_i(u_1(n,t), u_1(n+1,t), u_1(n-1,t), \ldots, u_k(n,t), u_k(n+1,t), u_k(n-1,t), \ldots),\\ &i = 1,2,\ldots,k.\end{aligned} \tag{4}$$

Applying the inverse operator L_i^{-1} on both sides of (4) gives

$$\begin{aligned}L_i^{-1}L_i(u_i(n,t)) = L_i^{-1}g_i &- L_i^{-1}R_i(u_1(n,t), u_1(n+1,t), u_1(n-1,t), \ldots, u_k(n,t), u_k(n+1,t), u_k(n-1,t), \ldots)\\ &- L_i^{-1}N_i(u_1(n,t), u_1(n+1,t), u_1(n-1,t), \ldots, u_k(n,t), u_k(n+1,t), u_k(n-1,t), \ldots),\\ &i = 1,2,\ldots,k.\end{aligned} \tag{5}$$

Using the initial conditions, we get

$$\begin{aligned}u_i(n,t) = f_i &- L_i^{-1}R_i(u_1(n,t), u_1(n+1,t), u_1(n-1,t), \ldots, u_k(n,t), u_k(n+1,t), u_k(n-1,t), \ldots)\\ &- L_i^{-1}N_i(u_1(n,t), u_1(n+1,t), u_1(n-1,t), \ldots, u_k(n,t), u_k(n+1,t), u_k(n-1,t), \ldots),\\ &i = 1,2,\ldots,k,\end{aligned} \tag{6}$$

where the function f_i represents the term arising from integrating the source term g_i and from using the given initial conditions or boundary conditions. Then, $u_i(n,t)(1 \leqslant i \leqslant k)$ can be represented as a series

$$u_i(n,t) = \sum_{m=0}^{\infty} u_{i,m}(n,t). \tag{7}$$

The nonlinear term N_i $(1 \leqslant i \leqslant k)$ will be decomposed by the infinite series of the Adomian polynomials

$$N_i(u_1,\ldots,u_k) = \sum_{m=0}^{\infty} A_{i,m}, \tag{8}$$

where $A_{i,m}$'s are obtained by writing

$$v_i(\lambda) = \sum_{m=0}^{\infty} \lambda^m u_{i,m}(n,t), \tag{9}$$

$$N_i(v_1(\lambda), v_2(\lambda), \ldots, v_k(\lambda)) = \sum_{m=0}^{\infty} \lambda^m A_{i,m}, \tag{10}$$

where λ is a parameter for convenience. We deduce

$$A_{i,m} = \frac{1}{m!}\left[\frac{d^m}{d\lambda^m} N_i(v_1(\lambda), \ldots, v_k(\lambda))\right]_{\lambda=0}. \tag{11}$$

To determine the components $u_{i,m}(n,t)(1 \leqslant i \leqslant k)$, we employ the recurrence relation

$$\begin{aligned} &u_{i,0}(n,t) = f_i, \\ &u_{i,m+1}(n,t) = -L_i^{-1}R_i(u_{1,m}(n,t), u_{1,m}(n+1,t), u_{1,m}(n-1,t), \ldots, u_{k,m}(n,t), \\ &\qquad\qquad u_{k,m}(n+1,t), u_{k,m}(n-1,t), \ldots) - L_i^{-1}A_{i,m}, \quad (m \geqslant 0). \end{aligned} \tag{12}$$

So the r-term partial sum

$$\phi_{i,r} = \sum_{m=0}^{r-1} u_{i,m}(n,t), \quad 1 \leqslant i \leqslant k \tag{13}$$

can serve as a practical solution.

2.2. The Padé approximants on the series solution

When we obtain the truncated series solution of order at least $(L+M)$ in t by ADM, we will use it to obtain Padé$[L/M](x,t)$ approximate solution for $u(x,t)$. The Padé approximants [19,20] are a particular type of rational fraction approximation to the value of a function. The idea is to match the Taylor series expansion as far as possible.

We denote the L, M Padé approximant to $A(x) = \sum_{i=0}^{\infty} a_i x^i$ by

$$[L/M] = \frac{P_L(x)}{Q_M(x)}, \tag{14}$$

where $P_L(x)$ is a polynomial of degree at most L and $Q_M(x)$ is a polynomial of degree at most M. We impose the normalization condition

$$Q_M(0) = 1.0. \tag{15}$$

such that P_L and Q_M have no common factors. This means that the formal power series $A(x)$ equals the $[L/M]$ approximant through $L+M+1$ terms. By using the conclusion given in [19], we know that the $[L/M]$ approximant is uniquely determined.

Suppose $f(x)$ is the ratio of two polynomials

$$f(x) = \frac{p(x)}{q(x)}, \tag{16}$$

where

$$\begin{aligned} p(x) &= p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \cdots + p_L x^L, \\ q(x) &= 1 + q_1 x + q_2 x^2 + q_3 x^3 + \cdots + q_M x^M, \end{aligned} \tag{17}$$

the truncated sum $\sum_{i=0}^{K} a_i x^i$ is given. Let

$$A_K(x) = \sum_{i=0}^{K} a_i x^i, \tag{18}$$

then

$$\begin{aligned} f(x) &= \sum_{i=0}^{\infty} a_i x^i \\ &= \sum_{i=0}^{K} a_i x^i + \sum_{i=K+1}^{\infty} a_i x^i \\ &= A_K(x) + \sum_{i=K+1}^{\infty} a_i x^i \\ &= A_K(x) + O(x^{K+1}). \end{aligned} \tag{19}$$

From $f(x) = \frac{p(x)}{q(x)}$, we have

$$\frac{p(x)}{q(x)} = A_K(x) + O(x^{K+1}), \tag{20}$$

that is

$$A_K(x) - \frac{p(x)}{q(x)} = O(x^{K+1}), \tag{21}$$

If $K \geqslant L + M$, (21) is the definition that $f(x) = \frac{p(x)}{q(x)}$ is a Padé approximant of $A_K(x)$. Because $q(0) = 1 \neq 0$, the Padé approximant is unique for given L and M. So (21) means that for a function equal to the ratio of two polynomials such as (16), the Padé approximant of its truncated Taylor series $A_K(x)$, which is uniquely determined for given L and M, gives the original function $f(x) = \frac{p(x)}{q(x)}$ when $K \geqslant L + M$.

To determine the coefficients of $p(x)$ and $q(x)$, we may multiply (21) by $q(x)$, which linearizes the coefficient equations. We can write out (21) in more detail as

$$\begin{cases} a_{L+1} + a_L q_1 + \cdots + a_{L-M+1} q_M = 0, \\ a_{L+2} + a_{L+1} q_1 + \cdots + a_{L-M+2} q_M = 0, \\ \ldots \\ a_{L+M} + a_{L+M-1} q_1 + \cdots + a_L q_M = 0, \end{cases} \tag{22}$$

and

$$\begin{cases} a_0 = p_0, \\ a_1 + a_0 q_1 = p_1, \\ a_2 + a_1 q_1 + a_0 q_2 = p_2, \\ \ldots \\ a_L + a_{L-1} q_1 + \cdots + a_0 q_L = p_L. \end{cases} \tag{23}$$

From (22), we can solve all the q's. Once the q's are known, Eq. (23) gives an explicit formula for the unknown p's. The construction of $[L/M]$ approximants involves only algebraic operations [19,20]. Each choice of L, degree of the numerator and M, degree of the denominator, leads to an approximant. The major difficulty in applying the technique is how to direct the choice in order to obtain the best approximant. This needs the use of a criterion for the choice depending on the shape of the solution. A criterion which has worked well here is the choice of $[L/M]$ approximants such that $L = M$. We construct the approximants by built-in utilities of Maple in the following sections.

If ADM truncated Taylor series of the exact solution with enough terms and the solution can be written as the ratio of two polynomials with no common factors, then the Padé approximants for the truncated series provide the exact solution. Even when the exact solution cannot be expressed as the ratio of two polynomials, the Padé approximants for the ADM truncated series usually greatly improve the accuracy and enlarge the convergence domain of the solutions.

3. The soliton solution of the Belov–Chaltikian lattice

Consider Eq. (1) with the initial conditions:

$$\begin{aligned} u(n,0) &= f_1, \\ v(n,0) &= f_2. \end{aligned} \tag{24}$$

We rewrite Eq. (1) in operator form:

$$\begin{aligned} L_t(u_n) &= u_n u_{n+1} - u_n u_{n-1} - v_n + v_{n-1}, \\ L_t(v_n) &= v_n u_{n+2} - v_n u_{n-1}, \end{aligned} \tag{25}$$

where L_t is a first-order differential operator and L_t^{-1} is a integrate operator defined by

$$L_t^{-1} \equiv \int_0^t (\cdot) dt. \tag{26}$$

Operating L_t^{-1} on both sides of Eq. (25) and using the initial conditions, we obtain

$$\begin{aligned} u_n &= f_1 + L_t^{-1}(u_n u_{n+1} - u_n u_{n-1}) - L_t^{-1} v_n + L_t^{-1} v_{n-1}, \\ v_n &= f_2 + L_t^{-1}(v_n u_{n+2} - v_n u_{n-1}). \end{aligned} \tag{27}$$

Therefore,

$$\begin{aligned} u_n &= f_1 + L_t^{-1}(M(u_n, u_{n+1}) - N(u_n, u_{n-1})) - L_t^{-1} v_n + L_t^{-1} v_{n-1}, \\ v_n &= f_2 + L_t^{-1}(P(v_n, u_{n+2}) - Q(v_n, u_{n-1})), \end{aligned} \tag{28}$$

where

$$\begin{aligned} M(u_n, u_{n+1}) &= u_n u_{n+1}, \\ N(u_n, u_{n-1}) &= u_n u_{n-1}, \\ P(v_n, u_{n+2}) &= v_n u_{n+2}, \\ Q(v_n, u_{n-1}) &= v_n u_{n-1}, \end{aligned} \tag{29}$$

can be expressed in terms of Adomian polynomial as follows:

$$M(u_n, u_{n+1}) = \sum_{m=0}^{\infty} A_m, \tag{30}$$

$$N(u_n, u_{n-1}) = \sum_{m=0}^{\infty} B_m, \tag{31}$$

$$P(v_n, u_{n+2}) = \sum_{m=0}^{\infty} C_m, \tag{32}$$

$$Q(v_n, u_{n-1}) = \sum_{m=0}^{\infty} D_m. \tag{33}$$

Therefore, u_n and v_n can be written as

$$u_n = \sum_{m=0}^{\infty} u_{n,m}, \tag{34}$$

$$v_n = \sum_{m=0}^{\infty} v_{n,m}. \tag{35}$$

According to (11), we can get the first few components of Adomian polynomial as follows:

$$\begin{aligned} A_0 &= u_{n,0}u_{n+1,0}, \\ B_0 &= u_{n,0}u_{n-1,0}, \\ C_0 &= v_{n,0}u_{n+2,0}, \\ D_0 &= v_{n,0}u_{n-1,0}, \\ A_1 &= u_{n,0}u_{n+1,1} + u_{n,1}u_{n+1,0}, \\ B_1 &= u_{n,0}u_{n-1,1} + u_{n,1}u_{n-1,0}, \\ C_1 &= v_{n,0}u_{n+2,1} + v_{n,1}u_{n+2,0}, \\ D_1 &= v_{n,0}u_{n-1,1} + v_{n,1}u_{n-1,0}, \\ A_2 &= u_{n,0}u_{n+1,2} + u_{n,1}u_{n+1,1} + u_{n,2}u_{n+1,0}, \\ B_2 &= u_{n,0}u_{n-1,2} + u_{n,1}u_{n-1,1} + u_{n,2}u_{n-1,0}, \\ C_2 &= v_{n,0}u_{n+2,2} + v_{n,1}u_{n+2,1} + v_{n,2}u_{n+2,0}, \\ D_2 &= v_{n,0}u_{n-1,2} + v_{n,1}u_{n-1,1} + v_{n,2}u_{n-1,0}, \\ &\ldots \end{aligned}$$

Other polynomials can be determined in a similar manner. To determine the components $u_{n,m}$ and $v_{n,m},\ m \geqslant 0$, we employ the recurrence relation

$$\begin{aligned} u_{n,0} &= f_1, \\ u_{n,m} &= L^{-1}(A_{m-1} - B_{m-1}) - L^{-1} v_{n,m-1} + L^{-1} v_{n-1,m-1}, \\ v_{n,0} &= f_2, \\ v_{n,m} &= L^{-1}(C_{m-1} - D_{m-1}). \end{aligned} \tag{36}$$

So the r-term approximate solutions are evaluated as follows:

$$\phi_{n,r} = \sum_{m=0}^{r-1} u_{n,m}, \tag{37}$$

$$\psi_{n,r} = \sum_{m=0}^{r-1} v_{n,m}. \tag{38}$$

Now, we will give the numerical solution of Belov–Chaltikian lattice. The exact solutions of Eq. (1) is

$$\begin{aligned} u_n &= \left(\ln\frac{g(n+\frac{1}{2},t,z)}{g(n-\frac{1}{2},t,z)}\right)_t, \\ v_n &= \frac{g(n+\frac{5}{2},t,z)g(n-\frac{3}{2},t,z)}{g(n+\frac{3}{2},t,z)g(n-\frac{1}{2},t,z)}, \end{aligned} \tag{39}$$

where $g(n,t,z) = 1+\exp(\eta),\ \eta = pn + qz + rt + \eta^0$, $q = \lambda^{-2}(e^{2p}-1)$, $r = \lambda^{-1}(e^p-1)$, $\lambda = \sqrt{\frac{e^{\frac{1}{2}p}-e^{\frac{3}{2}p}}{e^{\frac{3}{2}p}-e^{-\frac{3}{2}p}}}$, p and η^0 are constant, z is an auxiliary variable and for a reason of simple calculus, we select $p = 0.3I,\ z = 1$ and $\eta^0 = 0$. We suppose the initial conditions of the problem is the above exact solution at $t = 0$. So we get

$$\begin{aligned} u_{n,0} &= u_n(0), \\ v_{n,0} &= v_n(0), \end{aligned} \tag{40}$$

the remaining components $u_{n,m}$ and $v_{n,m}, m \geqslant 1$ can be obtained using Maple. Therefore, we can obtain five-term approximate solutions $\phi_{n,5} = \sum_{m=0}^{4} u_{n,m}$ and $\psi_{n,5} = \sum_{m=0}^{4} v_{n,m}$.

Using ADM-Padé technique at $n = 1$, the rational approximations [2/2] are

$$\begin{aligned} u[2/2]_1 =& \frac{0.1783720957\times10^{-10} - 0.6691905964\times10^{-1}I - (0.1250265938\times10^{-2} + 0.1897390643\times10^{-8}I)t}{1 + (0.3127423789\times10^{-7} + 0.4240612604I)t - (0.1722521359\times10^{-1} - 0.1385328681\times10^{-7}I)t^2} \\ & - \frac{(0.9131421779\times10^{-10} - 0.2851516463\times10^{-2}I)t^2}{1 + (0.3127423789\times10^{-7} + 0.4240612604I)t - (0.1722521359\times10^{-1} - 0.1385328681\times10^{-7}I)t^2}, \\ v[2/2]_1 =& \frac{0.8961078955 - 0.2433538024\times10^{-9}I + (0.2085414300\times10^{-7} + 0.3583005444I)t}{1 + (0.2341608030\times10^{-7} + 0.3547135175I)t + (0.3581914902\times10^{-2} + 0.1048425152\times10^{-7}I)t^2} \\ & + \frac{(0.7800570275\times10^{-2} + 0.1027546800\times10^{-7}I)t^2}{1 + (0.2341608030\times10^{-7} + 0.3547135175I)t + (0.3581914902\times10^{-2} + 0.1048425152\times10^{-7}I)t^2}. \end{aligned} \tag{41}$$

Figs. 1 and 2 show the truncated series solutions $(|\phi_{n,5}|, |\psi_{n,5}|)$ of ADM give a good approximant in small interval of convergence $(-2 \leqslant t \leqslant 2)$ and outside it, high error is obtained, while the ADM-Padé solutions $(|u[2/2]_1|, |v[2/2]_1|)$ enlarge the convergence domain $(-4.8 \leqslant t \leqslant 4.8$ for $|u[2/2]_1|, -4.5 \leqslant t \leqslant 4.5$for $|v[2/2]_1|)$. For any fixed n, we find that the ADM-Padé [2/2] solutions greatly improve the accuracy of the truncated series solution of ADM. Figs. 3 and 4 show the ADM-Padé solutions $(|u[2/2]_n|, |v[2/2]_n|)$ and the ADM solutions $(|\phi_{n,5}|, |\psi_{n,5}|)$ in the interval of convergence at $n = 0, 1, \ldots, 10$, respectively. Fig. 5 shows the exact solutions $(|u_n|, |v_n|)$ in the corresponding interval at $n = 0, 1, \ldots, 10$. In Table 1, we compare the absolute error of the modulus of ADM-Padé[2/2] solutions

$$||u_n| - |u[2/2]_n||,\ ||v_n| - |v[2/2]_n||,$$

with the absolute error of the modulus of ADM solutions

$$||u_n| - |\phi_{n,5}||,\ ||v_n| - |\psi_{n,5}||,$$

at some points for $n = 1$. the graphic and numerical illustrations show the Padé technique enlarges the domain of convergence of the solutions.

4. The soliton solution of the nonlinear self-dual network equations

Consider Eq. (2) with the initial conditions

$$\begin{aligned} u(n,0) &= f_1, \\ v(n,0) &= f_2. \end{aligned} \tag{42}$$

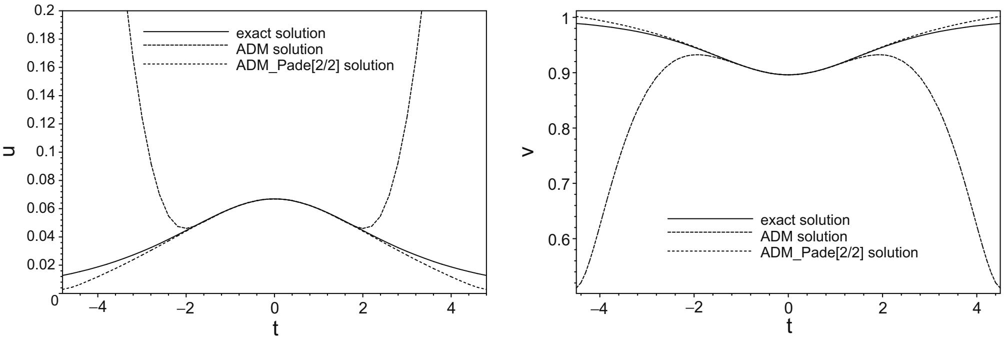

Fig. 1. The comparison among the ADM solutions $(|\phi_{n,5}|, |\psi_{n,5}|)$, the ADM-Padé solutions $(|u[2/2]_n|,\ |v[2/2]_n|)$ and the exact solutions $(|u_n|, |v_n|)$ of Belov–Chaltikian lattice at $n = 1$.

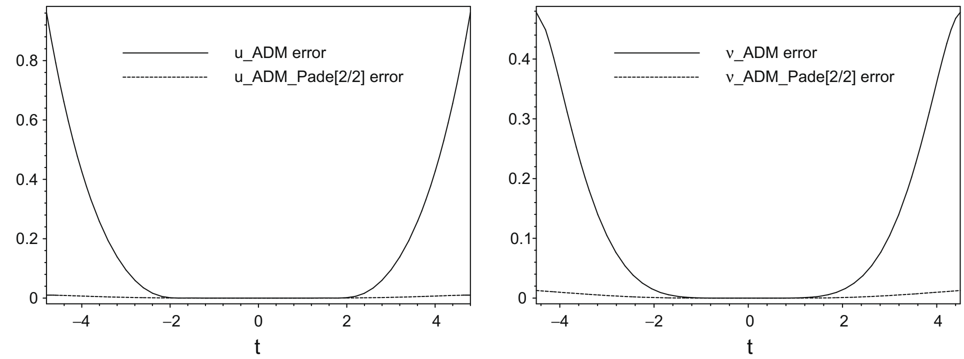

Fig. 2. The comparison of the absolute error between the ADM solutions $(|\phi_{n,5}|, |\psi_{n,5}|)$ and the ADM-Padé solutions $(|u[2/2]_n|, |v[2/2]_n|)$ of Belov–Chaltikian lattice at $n = 1$.

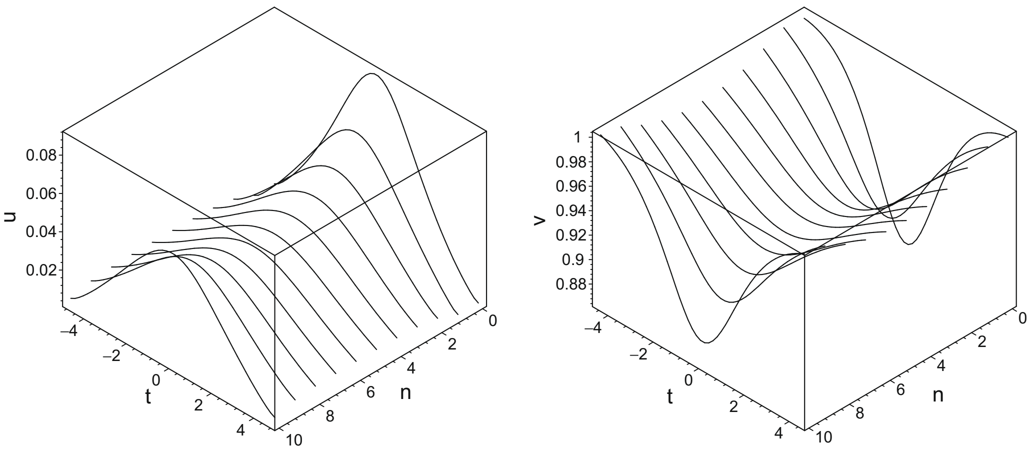

Fig. 3. The ADM-Padé solutions $(|u[2/2]_n|, |v[2/2]_n|)$ of Belov–Chaltikian lattice in the interval of convergence at $n = 0, 1, \ldots, 10$.

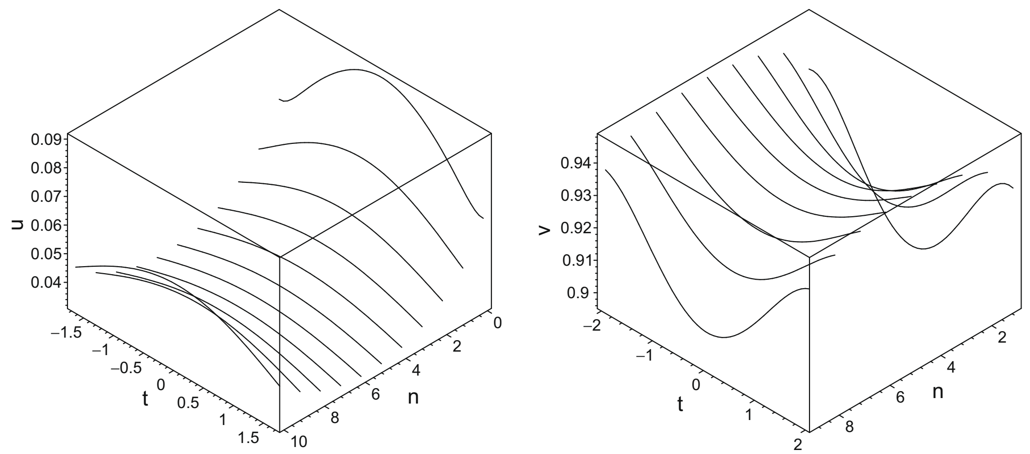

Fig. 4. The ADM solutions $(|\phi_{n,5}|, |\psi_{n,5}|)$ of Belov–Chaltikian lattice in the interval of convergence at $n = 0, 1, \ldots, 10$.

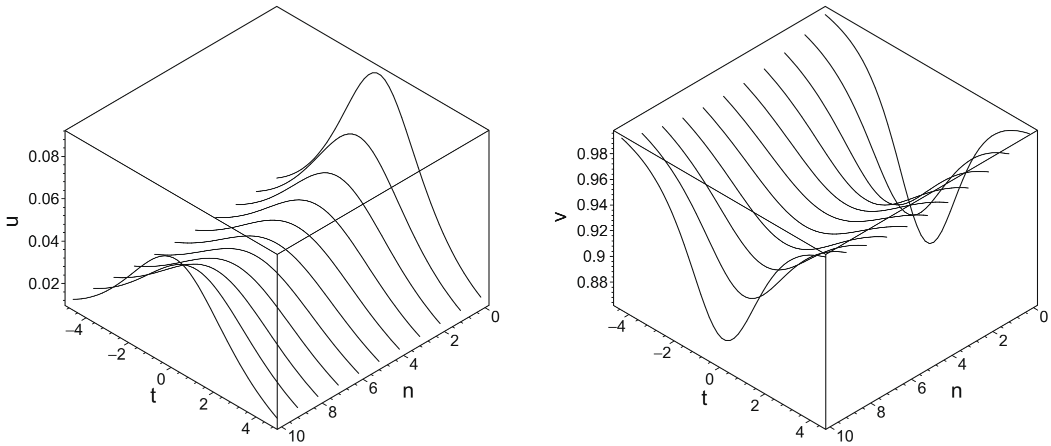

Fig. 5. The exact solutions $(|u_n|, |v_n|)$ of Belov–Chaltikian lattice in the corresponding interval at $n = 0, 1, \ldots, 10$.

Table 1
The comparison of the absolute error between the ADM solutions $(|\phi_{n,5}|, |\psi_{n,5}|)$ and the ADM-Padé solutions $(|u[2/2]_n|, |v[2/2]_n|)$ of Belov–Chaltikian lattice at $n = 1$.

t	$\|u\|$		$\|v\|$	
	ADM error	ADM-Padé[2/2] error	ADM error	ADM-Padé[2/2] error
−4	$4.481343251 \times 10^{-1}$	$7.708549336 \times 10^{-3}$	$5.231797874 \times 10^{-1}$	$1.133206457 \times 10^{-2}$
−2	$1.988626898 \times 10^{-2}$	$6.944387756 \times 10^{-4}$	$2.286686407 \times 10^{-2}$	$9.58456423 \times 10^{-4}$
0	0	3×10^{-11}	0	9×10^{-10}
2	$1.988626915 \times 10^{-2}$	$6.944387480 \times 10^{-4}$	$2.286686552 \times 10^{-2}$	$9.584549913 \times 10^{-4}$
4	$4.481343303 \times 10^{-1}$	$7.708544844 \times 10^{-3}$	$5.231797992 \times 10^{-1}$	$1.133205733 \times 10^{-2}$

We rewrite Eq. (2) in operator form:

$$\begin{aligned} L_t(u_n) &= \gamma u_n^2 v_n - \gamma u_n^2 v_{n+1} + v_n - v_{n+1}, \\ L_t(v_n) &= \gamma v_n^2 u_{n-1} - \gamma v_n^2 u_n + u_{n-1} - u_n, \end{aligned} \tag{43}$$

where L_t is a first-order differential operator and L_t^{-1} is a integrate operator defined by

$$L_t^{-1} \equiv \int_0^t (\cdot) dt. \tag{44}$$

Operating L_t^{-1} on both sides of Eq. (43) and using the initial conditions, we obtain

$$\begin{aligned} u_n &= f_1 + \gamma L_t^{-1}(u_n^2 v_n - u_n^2 v_{n+1}) + L_t^{-1} v_n - L_t^{-1} v_{n+1}, \\ v_n &= f_2 + \gamma L_t^{-1}(v_n^2 u_{n-1} - v_n^2 u_n) + L_t^{-1} u_{n-1} - L_t^{-1} u_n. \end{aligned} \tag{45}$$

Therefore,

$$\begin{aligned} u_n &= f_1 + \gamma L_t^{-1}(M(u_n, v_n) - N(u_n, v_{n+1})) + L_t^{-1} v_n - L_t^{-1} v_{n+1}, \\ v_n &= f_2 + \gamma L_t^{-1}(P(u_{n-1}, v_n) - Q(u_n, v_n)) + L_t^{-1} u_{n-1} - L_t^{-1} u_n, \end{aligned} \tag{46}$$

where

$$\begin{aligned} M(u_n, v_n) &= u_n^2 v_n, \\ N(u_n, v_{n+1}) &= u_n^2 v_{n+1}, \\ P(u_{n-1}, v_n) &= v_n^2 u_{n-1}, \\ Q(u_n, v_n) &= v_n^2 u_n, \end{aligned} \tag{47}$$

can be expressed in terms of Adomian polynomial as follows:

$$M(u_n, v_n) = \sum_{m=0}^{\infty} A_m, \tag{48}$$

$$N(u_n, v_{n+1}) = \sum_{m=0}^{\infty} B_m, \tag{49}$$

$$P(u_{n-1}, v_n) = \sum_{m=0}^{\infty} C_m, \tag{50}$$

$$Q(u_n, v_n) = \sum_{m=0}^{\infty} D_m. \tag{51}$$

Therefore, u_n and v_n can be written as

$$u_n = \sum_{m=0}^{\infty} u_{n,m}, \tag{52}$$

$$v_n = \sum_{m=0}^{\infty} v_{n,m}. \tag{53}$$

According to (11), we can get the first few components of Adomian polynomial as follows:

$$\begin{aligned}
A_0 &= u_{n,0}^2 v_{n,0},\\
B_0 &= u_{n,0}^2 v_{n+1,0},\\
C_0 &= v_{n,0}^2 u_{n-1,0},\\
D_0 &= v_{n,0}^2 u_{n,0},\\
A_1 &= 2u_{n,0}u_{n,1} v_{n,0} + u_{n,0}^2 v_{n,1},\\
B_1 &= 2u_{n,0}u_{n,1} v_{n+1,0} + u_{n,0}^2 v_{n+1,1},\\
C_1 &= 2 v_{n,0} v_{n,1} u_{n-1,0} + v_{n,0}^2 u_{n-1,1},\\
D_1 &= 2 v_{n,0} v_{n,1} u_{n,0} + v_{n,0}^2 u_{n,1},\\
A_2 &= 2u_{n,0}u_{n,2} v_{n,0} + u_{n,1}^2 v_{n,0} + 2u_{n,0}u_{n,1} v_{n,1} + u_{n,0}^2 v_{n,2},\\
B_2 &= 2u_{n,0}u_{n,2} v_{n+1,0} + u_{n,1}^2 v_{n+1,0} + 2u_{n,0}u_{n,1} v_{n+1,1} + u_{n,0}^2 v_{n+1,2},\\
C_2 &= 2 v_{n,0} v_{n,2} u_{n-1,0} + v_{n,1}^2 u_{n-1,0} + 2 v_{n,0} v_{n,1} u_{n-1,1} + v_{n,0}^2 u_{n-1,2},\\
D_2 &= 2 v_{n,0} v_{n,2} u_{n,0} + v_{n,1}^2 u_{n,0} + 2 v_{n,0} v_{n,1} u_{n,1} + v_{n,0}^2 u_{n,2},\\
&\cdots
\end{aligned}$$

Other polynomials are determined in a similar manner. To determine the components $u_{n,m}$ and $v_{n,m},\ m \geqslant 0$, we employ the recurrence relations

$$\begin{aligned}
u_{n,0} &= f_1,\\
u_{n,m} &= \gamma L_t^{-1}(A_{m-1} - B_{m-1}) + L_t^{-1} v_{n,m-1} - L_t^{-1} v_{n+1,m-1},\\
v_{n,0} &= f_2,\\
v_{n,m} &= \gamma L_t^{-1}(C_{m-1} - D_{m-1}) + L_t^{-1} u_{n-1,m-1} - L_t^{-1} u_{n,m-1}.
\end{aligned} \tag{54}$$

So the r-term approximate solutions are evaluated as follows:

$$\phi_{n,r} = \sum_{m=0}^{r-1} u_{n,m}, \tag{55}$$

$$\psi_{n,r} = \sum_{m=0}^{r-1} v_{n,m}. \tag{56}$$

Now, we will give the numerical solution of the self-dual network equations. We select $\gamma = -1$ and the exact solutions of the problem is

$$\begin{aligned}
u_n &= \tanh\left(\frac{1}{4}\right) \tanh\left(\frac{1}{2}n - \omega t\right),\\
v_n &= -\tanh\left(\frac{1}{4}\right) \tanh\left[\frac{1}{2}(n-1) - \omega t\right],
\end{aligned} \tag{57}$$

where $\omega = -2\tanh(\frac{1}{4})$. We suppose the solutions at $t = 0$ as the initial conditions of the problem. So we get

$$\begin{aligned} u_{n,0} &= u_n(0), \\ v_{n,0} &= v_n(0). \end{aligned} \tag{58}$$

The remaining components $u_{n,m}$ and $v_{n,m}$, $m \geqslant 1$ can be obtained using Maple. Therefore, we can get nine-term approximate solutions $\phi_{n,9} = \sum_{m=0}^{8} u_{n,m}$ and $\psi_{n,9} = \sum_{m=0}^{8} v_{n,m}$.

Using ADM-Padé technique at $n = 1$, we obtain the following rational approximations [2/2] and [4/4], respectively, are

$$\begin{aligned} u[2/2]_1 &= \frac{0.1131811160 + 0.1253337358t + 0.1132231877 \times 10^{-2}t^2}{1 + 0.1201831747t + 0.8182221609 \times 10^{-1}t^2}, \\ v[4/4]_1 &= \frac{-0.1131811160t + 0.2812270026 \times 10^{-2}t^2 - 0.4487519899 \times 10^{-2}t^3 + 0.3025435885 \times 10^{-3}t^4}{1 - 0.1844253733 \times 10^{-1}t + 0.1094232516t^2 - 0.6345435228 \times 10^{-2}t^3 + 0.1677135554 \times 10^{-2}t^4}. \end{aligned} \tag{59}$$

Figs. 6 and 7 show the truncated series solutions $(\phi_{n,5},\ \psi_{n,9})$ of ADM give a good approximant in small interval of convergence $(-2 \leqslant t \leqslant 2)$ and outside it, high error is obtained, while the ADM-Padé solutions $(u[2/2]_1, v[4/4]_1)$ enlarge the convergence domain $(-6 \leqslant t \leqslant 6)$. We find that the ADM-Padé solutions greatly improve the accuracy of the truncated series solution of ADM at $n = 0, 1, 2$. Figs. 8 and 9 show the ADM solutions $(\phi_{n,5}, \psi_{n,9})$ and the ADM-Padé solutions $(u[2/2]_n, v[4/4]_n)$ in the interval of convergence at $n = 0, 1, 2$, respectively. Fig. 10 shows the exact solution (u_n, v_n) in the corresponding interval at $n = 0, 1, 2$. In Table 2, we compare between the absolute error of the ADM-Padé solutions

$$|u_n - u[2/2]_n|, \quad |v_n - v[4/4]_n|,$$

with the absolute error of the ADM solutions

$$|u_n - \phi_{n,5}|, \quad |v_n - \psi_{n,9}|,$$

at some points for $n = 1$. the graphic and numerical illustrations show the ADM-Padé technique enlarges the convergence domain of the solutions.

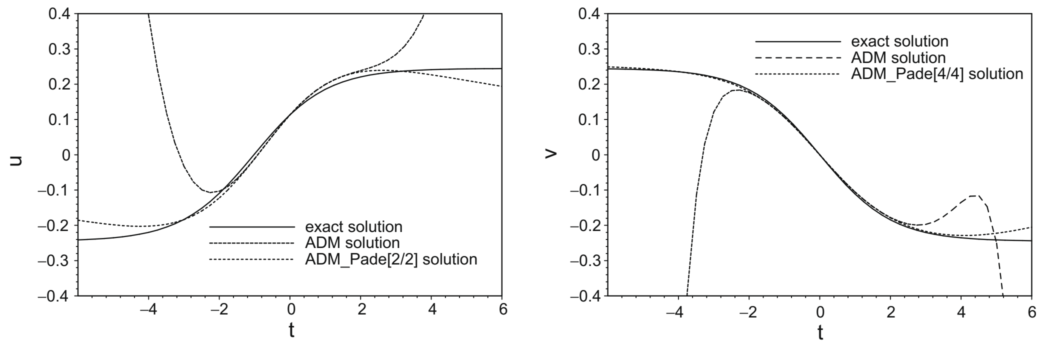

Fig. 6. The comparison among the ADM solutions $(\phi_{n,5}, \psi_{n,9})$, the ADM-Padé solutions $(u[2/2]_n, v[4/4]_n)$ and the exact solutions (u_n, v_n) of the nonlinear self-dual network equations at $n = 1$.

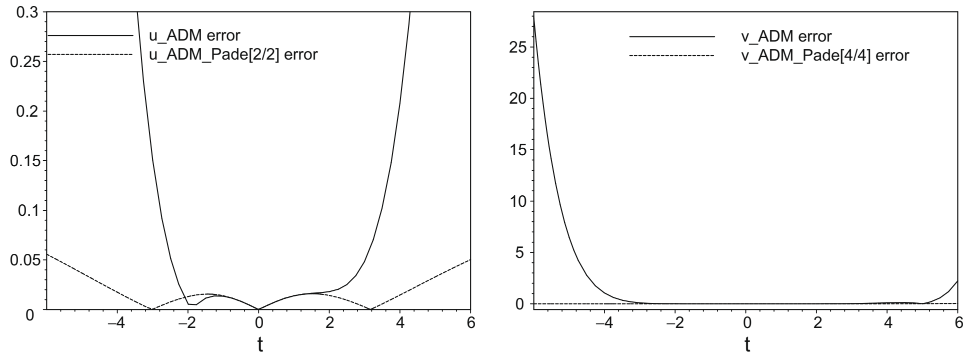

Fig. 7. The comparison of the absolute error between the ADM solutions $(\phi_{n,5}, \psi_{n,9})$ and the ADM-Padé solutions $(u[2/2]_n, v[4/4]_n)$ of the nonlinear self-dual network equations at $n = 1$.

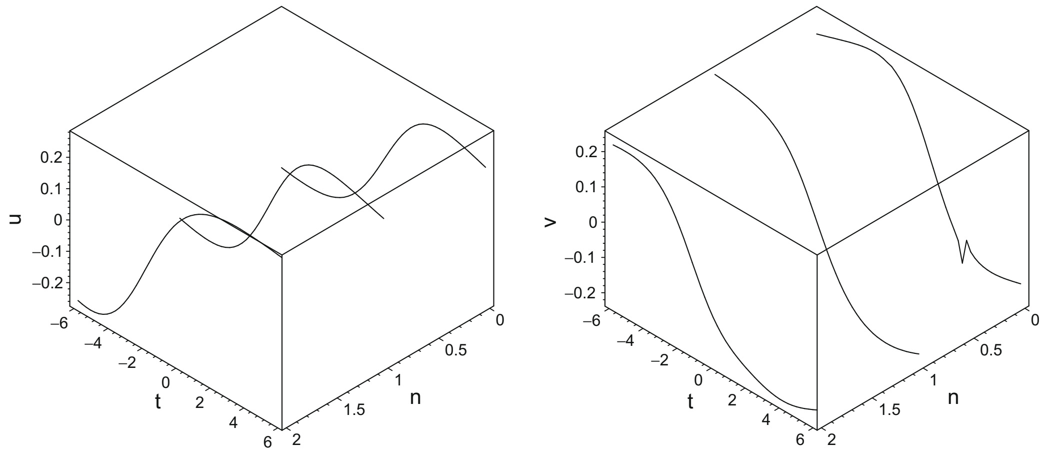

Fig. 8. The ADM-Padé solutions $(u[2/2]_n, v[4/4]_n)$ of the nonlinear self-dual network equations in the interval of convergence at $n = 0, 1, 2$.

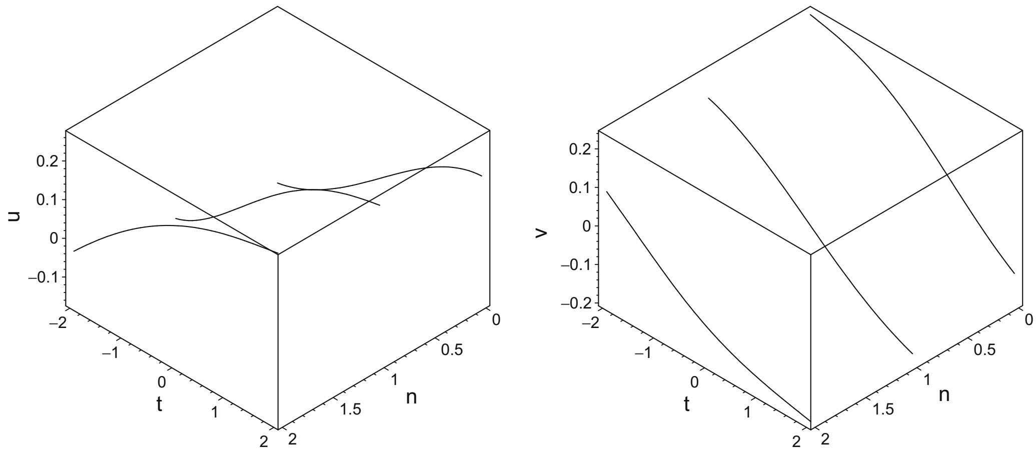

Fig. 9. The ADM solutions $(\phi_{n,5}, \psi_{n,9})$ of the nonlinear self-dual network equations in the interval of convergence at $n = 0, 1, 2$.

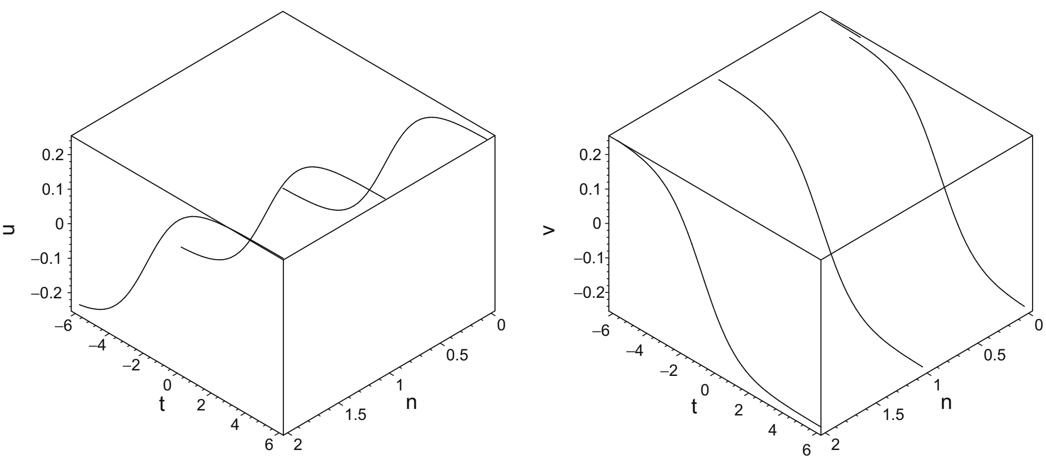

Fig. 10. The exact solutions (u_n, v_n) of the nonlinear self-dual network equations in the corresponding interval at $n = 0, 1, 2$.

Table 2
The comparison of the absolute error between the ADM solutions $(\phi_{n,5}, \psi_{n,9})$ and the ADM-Padé solutions $(u[2/2]_n, v[4/4]_n)$ of the nonlinear self-dual network equations at $n = 1$.

t	u		v	
	ADM error	ADM-Padé[2/2] error	ADM error	ADM-Padé[4/4] error
−6	3.629389617	$5.57442867 \times 10^{-2}$	2.785788834×10	5.6591762×10^{-3}
−4	$6.135576195 \times 10^{-1}$	$1.74398533 \times 10^{-2}$	1.057038734	2.446404×10^{-4}
−2	5.950913×10^{-3}	$1.30950984 \times 10^{-2}$	7.9942487×10^{-3}	5.0750722×10^{-3}
0	0	0	0	0
2	$1.80535025 \times 10^{-2}$	$1.42149103 \times 10^{-2}$	6.6635999×10^{-3}	5.6150383×10^{-3}
4	$2.076500585 \times 10^{-1}$	$1.46057674 \times 10^{-2}$	$1.005059431 \times 10^{-1}$	6.7014041×10^{-3}
6	1.654350787	$5.02850819 \times 10^{-2}$	2.234225068	$3.84464148 \times 10^{-2}$

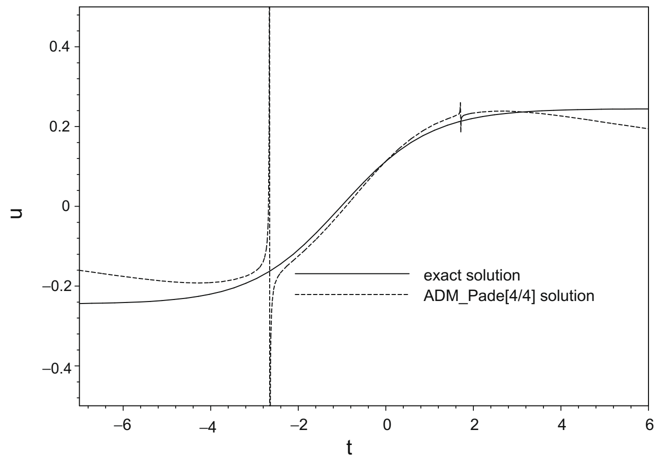

Fig. 11. The comparison between the exact solution u_n and the ADM-Padé solution $u[4/4]_n$ of the nonlinear self-dual network equations at $n = 1$.

Table 3
The locations of spurious poles of padé approximants $[L/M]$ to u_n of the nonlinear self-dual network equations at $n = 1$.

(L, M)	Poles
(1,1)	−5.183130254
(1,2)	None
(2,1)	3.290425852
(2,2)	None
(2,3)	2.719873569
(3,2)	None
(3,3)	−114.3392243
(3,4)	3.944018800
(4,3)	None
(4,4)	−2.652941440, 1.709003568

Remarks. Since the Padé approximants to the function f are generated by an algorithmic method, it can happen that poles of the approximants cluster in regions, where convergence could be expected. These poles which locally make uniform convergence impossible are spurious [32] because they do not correspond to analytic properties of the function f. In Ref. [32], Stahl gives the theoretical analysis of the possibility of spurious poles from orthogonality relations. Fig. 11 shows two spurious poles of padé approximant [4/4] to the exact solution $u_n = \tanh(\frac{1}{4})\tanh(\frac{1}{2}n - \omega t)$ where $\omega = -2\tanh(\frac{1}{4})$ at $n = 1$,which has no singularity except at infinity. Because of the huge size of calculations, we obtained at most 9 terms ADM series solution $\phi_{n,9} = \sum_{m=0}^{8} u_{n,m}$ of the nonlinear self-dual network equations at $n = 1$. From Table 3 which lists the locations of spurious poles of padé approximants $[L/M]$ of the nonlinear self-dual network equations, spurious poles can be avoided by choosing padé approximants of appropriate orders. We conclude that the diagonal approximants work well here intuitively and choose the padé approximant [2/2] to u_n at $n = 1$ shown in Fig. 12. The basic goal of this work has been to find good approximate solutions of the nonlinear lattice equations and has been achieved.

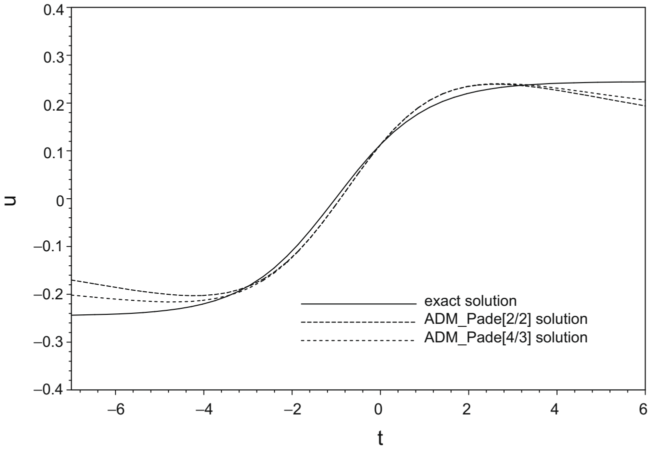

Fig. 12. The comparison among the exact solution u_n, the ADM-Padé solution $u[2/2]_n$ and $u[4/3]_n$ of the nonlinear self-dual network equations at $n = 1$.

5. Discussion and conclusions

In this paper, we solve Belov–Chaltikian lattice and the nonlinear self-dual network equations with appropriate initial conditions using ADM-Padé technique which gives the approximate solution with higher accuracy and faster convergence rate than using ADM [25–31] alone. From the good results, we can conclude that the application of Padé approximants to Adomian's series solution greatly improve the convergence domain and accuracy of the solution. Comparisons are made between approximate solutions and exact solutions to illustrate the validity and the great potential of the technique. Although the ADM-Padé technique works very well in the paper, it should also be noted that the ADM-Padé solutions we presented converging to available exact solutions are one set of solutions of DDEs and the series solution of the nonlinear problem with unknown solution should be looked into carefully.

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