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Finite Symmetry Transformation Groups and Exact Solutions of Konopelchenko–Dubrovsky Equation*

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Abstract *Based on the general direct method developed by Lou et al. [J. Phys. A: Math. Gen.* **38** *(2005) L129], the symmetry group theorem is obtained, from that both the Lie point groups and the non-Lie symmetry groups of the Konopelchenko–Dubrovsky (KD) equation are obtained. From the theorem, some exact solutions of KD equation are derived from a simple travelling wave solution and a multi-soliton solution.*

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1 Introduction

In recent years, the study of symmetries, symmetry groups, symmetry reductions, and group invariant solutions of nonlinear partial differential equations (PDEs) have become one of the most exciting and extremely active areas of research.[1−13] Some powerful methods to obtain the similarity reductions of a given PDE have been developed by mathematicians and physicist, such as, the Lie approach and the direct method. Most recently, Lou *et al.* have developed a general direct method in a series of papers.[14−16] By the general direct method, both the Lie point symmetry groups and the non-Lie symmetry groups can be obtained for some PDEs. Furthermore, the expressions of the exact finite transformations of the Lie groups are much simpler than those obtained via the standard approaches for some nonlinear PDEs.

In this paper, we would like to use general direct method to search for finite symmetry transformation group and some exact solutions of the (2+1)-dimensional Konopelchenko–Dubrovsky (KD) equation:[17]

$$u_t - u_{xxx} - 6buu_x + \frac{3}{2}a^2u^2u_x - 3v_y + 3au_xv = 0\,, \quad (1)$$

$$u_y = v_x\,, \quad (2)$$

where $u = u(x, y, t)$, $v = v(x, y, t)$, the subscripts denote partial differential, a, b are the real parameters. Equations (1) and (2) are nonlinear integrable evolution equations on two spatial dimensions and one temporal. For $u_y = 0$, Eq. (1) becomes the Gardner equation (combined KdV and modified KdV equation). For $a = 0$, Eq. (1) is the well-known Kadomtsev–Petviashvili (KP) equation. At $b = 0$, Eq. (1) is the modified KP equation. Various methods for obtaining exact solution to KD have been proposed. By using the standard truncated Painlevé analysis, Lin *et al.* obtained some new types of multi-soliton solutions of KD equation.[18] Wazwaz used the Exp-function method to obtain some explicit and exact solutions of KD equation.[19] Wang *et al.* investigated the KD equation by the F-expansion method.[20]

2 Transformation Group and Some Solutions of KD Equation

In order to obtain the finite symmetry transformation group of the KD equation, we let

$$u = \alpha_1 + \beta_1 U(\xi, \eta, \tau) + \gamma_1 V(\xi, \eta, \tau)\,, \quad (3)$$

$$v = \alpha_2 + \beta_2 U(\xi, \eta, \tau) + \gamma_2 V(\xi, \eta, \tau)\,, \quad (4)$$

where α_1, α_2, β_1, β_2, γ_1, γ_2, ξ, η, and τ are functions of $\{x, y, t\}$. Restricting $U(\xi, \eta, \tau) \equiv U$, $V(\xi, \eta, \tau) \equiv V$, and it satisfies the same form as the KD equations (1) and (2) but with new independent variables, i.e

$$U_\tau = U_{\xi\xi\xi} + 6bUU_\xi - \frac{3}{2}a^2U^2U_\xi + 3V_\eta - 3aVU_\xi\,, \quad (5)$$

$$U_\eta = V_\xi\,. \quad (6)$$

Substituting Eqs. (3) and (4) into Eqs. (1) and (2), then eliminate U_τ and U_η by using Eqs. (5) and (6), from that, the remained determining equations of the functions ξ, η, τ, α_1, α_2, β_1, β_2, γ_1, γ_2 can be read off by vanishing the coefficients of the polynomials of U, V and its derivatives, then it is straightforward to find out the general

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solution of the determining equations. The result reads

$$\gamma_1 = 0, \quad \tau = \tau(t), \quad \beta_1 = \delta_2\delta_1\tau_t^{1/3}, \quad \beta_2 = {\delta_2}^2\tau_t^{2/3},$$

$$\gamma_2 = \frac{1}{9}\frac{\delta_2\delta_1\tau_{tt}y}{\tau_t^{2/3}} + \frac{1}{6}\frac{{\delta_2}^2\eta_{0t}}{\tau_t^{1/3}}, \tag{7}$$

$$\eta = {\delta_2}^2\delta_1\tau_t^{2/3}y + \eta_0,$$

$$\xi = \delta_2\tau_t^{1/3}x + \frac{1}{18}\frac{\delta_2\tau_{tt}y^2}{\tau_t^{2/3}} + \frac{1}{6}\frac{\delta_1{\delta_2}^2\eta_{0t}y}{\tau_t^{1/3}} + \xi_0, \tag{8}$$

$$\alpha_1 = -\frac{2\tau_t^{1/3}\delta_2\delta_1 b}{a^2} + \frac{2b}{a^2} - \frac{1}{9}\frac{\tau_{tt}y}{a\tau_t} - \frac{1}{6}\frac{\delta_2\delta_1\eta_{0t}}{a\tau_t^{2/3}}, \tag{9}$$

$$\begin{aligned}\alpha_2 = {} & \Big(\frac{1}{54}\frac{\tau_{tt}^2}{a\tau_t^2} - \frac{1}{54}\frac{\tau_{ttt}}{a\tau_t}\Big)y^2 \\ & + \Big(-\frac{1}{18}\frac{\delta_2\delta_1\eta_{0tt}}{a\tau_t^{2/3}} - \frac{2}{9}\frac{b\delta_2\delta_1\tau_{tt}}{a^2\tau_t^{2/3}} + \frac{1}{27}\frac{\delta_2\delta_1\eta_{0t}\tau_{tt}}{a\tau_t^{5/3}}\Big)y \\ & - \frac{1}{9}\frac{\tau_{tt}x}{a\tau_t} - \frac{1}{3}\frac{\xi_{0t}{\delta_2}^2}{a\tau_t^{1/3}} - \frac{2\tau_t^{2/3}b^2{\delta_2}^2}{a^3} + \frac{2b^2}{a^3} \\ & + \frac{1}{72}\frac{\eta_{0t}^2{\delta_2}^2}{a\tau_t^{4/3}} - \frac{1}{3}\frac{b\eta_{0t}{\delta_2}^2}{a^2\tau_t^{1/3}},\end{aligned} \tag{10}$$

where $\xi_0 \equiv \xi_0(t)$, $\eta_0 \equiv \eta_0(t)$, $\tau(t)$ are arbitrary functions of time t while the constants δ_1 and δ_2 possess discrete values determined by

$$\delta_1 = \pm 1, \quad \delta_2 = 1, \quad \frac{1}{2}(i\sqrt{3} - 1), \quad -\frac{1}{2}(i\sqrt{3} + 1), \quad (i = \sqrt{-1}). \tag{11}$$

In summary, the following theorem holds:

Theorem If $U \equiv U(x,y,t)$, $V \equiv V(x,y,t)$ is a solution of the KD equation (1) and (2), then so is

$$u = -\frac{2\tau_t^{1/3}\delta_2\delta_1 b}{a^2} + \frac{2b}{a^2} - \frac{1}{9}\frac{\tau_{tt}y}{a\tau_t} - \frac{1}{6}\frac{\delta_2\delta_1\eta_{0t}}{a\tau_t^{2/3}} + \delta_2\delta_1\tau_t^{1/3}U(\xi,\eta,\tau), \tag{12}$$

$$\begin{aligned}v = {} & \Big(\frac{1}{54}\frac{\tau_{tt}^2}{a\tau_t^2} - \frac{1}{54}\frac{\tau_{ttt}}{a\tau_t}\Big)y^2 \\ & + \Big(-\frac{1}{18}\frac{\delta_2\delta_1\eta_{0tt}}{a\tau_t^{2/3}} - \frac{2}{9}\frac{b\delta_2\delta_1\tau_{tt}}{a^2\tau_t^{2/3}} + \frac{1}{27}\frac{\delta_2\delta_1\eta_{0t}\tau_{tt}}{a\tau_t^{5/3}}\Big)y \\ & - \frac{1}{9}\frac{\tau_{tt}x}{a\tau_t} - \frac{1}{3}\frac{\xi_{0t}{\delta_2}^2}{a\tau_t^{1/3}} - \frac{2\tau_t^{2/3}b^2{\delta_2}^2}{a^3} + \frac{2b^2}{a^3} \\ & + \frac{1}{72}\frac{\eta_{0t}^2{\delta_2}^2}{a\tau_t^{4/3}} - \frac{1}{3}\frac{b\eta_{0t}{\delta_2}^2}{a^2\tau_t^{1/3}} + {\delta_2}^2\tau_t^{2/3}V(\xi,\eta,\tau) \\ & + \Big(\frac{1}{9}\frac{\delta_2\delta_1\tau_{tt}y}{\tau_t^{2/3}} + \frac{1}{6}\frac{{\delta_2}^2\eta_{0t}}{\tau_t^{1/3}}\Big)U(\xi,\eta,\tau),\end{aligned} \tag{13}$$

with Eqs. (7) and (8), where $a \neq 0$, ξ_0, η_0, and τ are arbitrary functions of t and discrete value of the δ_1 and δ_2 are given by Eq. (11).

Remark It is necessary to point out that: (i) when $a = 0$, the finite symmetry transformation group of the KP equation have been obtained by Lou *et al.* in Ref. [14]; (ii) when $b = 0$, Eqs. (12) and (13) become the solution of modified KP equation.

From the new symmetry group theorem, we know that for the real KD equation, the symmetry group is divided into two sectors: the Lie point symmetry group which corresponds to

$$\delta_1 = \delta_2 = 1, \tag{14}$$

and a coset of the Lie group which is related to

$$\delta_1 = -1, \quad \delta_2 = 1. \tag{15}$$

The coset is equivalent to the reflected transformation of, i.e. $y \to -y$ company with the usual Lie point symmetry transformation.

In other words, if we denote by S the Lie point symmetry group of the real KD equation, by σ^y the reflection of y, by I the identity transformation, and by $\mathcal{C} \equiv \{I, \sigma^y\}$ the discrete reflection group, then the full Lie symmetry group $\mathcal{G}_{\rm RKD}$ of the real KD equation given by Theorem can be expressed as

$$\mathcal{G}_{\rm RKD} = \mathcal{C} \otimes \mathcal{S}. \tag{16}$$

For the complex KD equation, the symmetry group is divided into six sectors, which correspond to

$$\delta_1 = 1, \qquad \delta_2 = 1, \tag{17}$$

$$\delta_1 = 1, \qquad \delta_2 = \frac{1}{2}(i\sqrt{3} - 1), \tag{18}$$

$$\delta_1 = 1, \qquad \delta_2 = -\frac{1}{2}(i\sqrt{3} + 1), \tag{19}$$

$$\delta_1 = -1, \qquad \delta_2 = 1, \tag{20}$$

$$\delta_1 = -1, \qquad \delta_2 = \frac{1}{2}(i\sqrt{3} - 1), \tag{21}$$

and

$$\delta_1 = -1, \qquad \delta_2 = -\frac{1}{2}(i\sqrt{3} + 1) \tag{22}$$

of Theorem respectively. That is to say, the full symmetry group, $\mathcal{G}_{CKD}$, expressed by Theorem for the complex KD equation is the product of the usual Lie point symmetry group $\mathcal{S}$ (Theorem with Eqs. (12) and (13)) and the discrete group $\mathcal{D}_3$

$$\mathcal{G}_{\rm CKD} = \mathcal{D}_3 \otimes \mathcal{S}, \tag{23}$$

$$\mathcal{D}_3 \equiv \{I,\ \sigma^y,\ R_1,\ R_2,\ \sigma^y R_1,\ \sigma^y R_2\}, \tag{24}$$

where I is the identity transformation, σ^y is the reflection of y and

$$R_1 : \{u(x,y,t),\ v(x,y,t)\} \to \Big\{\frac{({\rm i}\sqrt{3} - 1)}{2}u\Big(\frac{(i\sqrt{3} - 1)}{2}x,\ -\frac{(i\sqrt{3} + 1)}{2}y,\ t\Big),$$

$$
-\frac{(i\sqrt{3}+1)}{2}v\Big(\frac{(i\sqrt{3}-1)}{2}x,\ -\frac{(i\sqrt{3}+1)}{2}y,\ t\Big)\Big\}\,, \tag{25}
$$

$$
\begin{aligned}
R_2:\{u(x,y,t),\ v(x,y,t)\}\to\Big\{&-\frac{(i\sqrt{3}+1)}{2}u\Big(-\frac{(i\sqrt{3}+1)}{2}x,\ \frac{(i\sqrt{3}-1)}{2}y,\ t\Big)\,,\\
&\frac{(i\sqrt{3}-1)}{2}v\Big(-\frac{(i\sqrt{3}+1)}{2}x,\ \frac{(i\sqrt{3}-1)}{2}y,\ t\Big)\Big\}\,.
\end{aligned} \tag{26}
$$

From Theorem, by restricting ($f\equiv f(t)$, $g\equiv g(t)$, $h\equiv h(t)$)

$$
\tau=t+\epsilon f,\qquad \xi_0=\epsilon g,\qquad \eta_0=\epsilon h\,,
$$

we can obtain the general Lie point symmetries of KD equation which are linear combinations of the following forms,

$$
\begin{aligned}
\sigma&\equiv\sigma_1(f)+\sigma_2(g)+\sigma_3(h)\\
&\equiv\begin{pmatrix} v_tf+\Big(\frac{1}{3}xf_t+\frac{1}{18}f_{tt}y^2\Big)v_x+\frac{2}{3}v_yyf_t+\Big(\frac{2}{3}v-\frac{4}{3}\frac{b^2}{a^3}\Big)f_t+P(x,y,t)-\frac{1}{54}\frac{y^2f_{ttt}}{a}\\ \Big(\frac{1}{3}xf_t+\frac{1}{18}f_{tt}y^2\Big)u_x+\frac{2}{3}u_yyf_t+\Big(\frac{1}{3}u-\frac{2}{3}\frac{b}{a^2}\Big)f_t+u_tf-\frac{1}{9}\frac{f_{tt}y}{a}\end{pmatrix}\\
&\quad+\begin{pmatrix}\frac{1}{6}v_xg_ty+v_yg+\Big(-\frac{1}{3}\frac{b}{a^2}+\frac{1}{6}u\Big)g_t-\frac{1}{18}\frac{yg_{tt}}{a}\\ \frac{1}{6}u_xg_ty+u_yg-\frac{1}{6}\frac{g_t}{a}\end{pmatrix}+\begin{pmatrix}v_xh-\frac{1}{3}\frac{h_t}{a}\\ u_xh\end{pmatrix},
\end{aligned}
$$

where

$$
P(x,y,t)=\Big(-\frac{1}{9}\frac{x}{a}+\frac{1}{9}uy-\frac{2}{9}\frac{yb}{a^2}\Big)f_{tt}\,.
$$

The commutation relations for the Kac-Moody–Virasoro algebra among $\sigma_1(f)$, $\sigma_2(g)$, and $\sigma_3(h)$ are as follows:

$$
\begin{aligned}
&[\sigma_1(f_1),\sigma_1(f_2)]=\sigma_1(f_1f_{2t}-f_2f_{1t}),\qquad [\sigma_1(f),\sigma_2(g)]=\sigma_2\Big(fg_t-\frac{2}{3}gf_t\Big)\,,\\
&[\sigma_1(f),\sigma_3(h)]=\sigma_3\Big(\frac{1}{3}hf_t-fh_t\Big),\qquad [\sigma_2(g_1),\sigma_2(g_2)]=\frac{1}{6}\sigma_3(g_1g_{2t}-g_2g_{1t})\,,\\
&[\sigma_2(g),\sigma_3(h)]=0,\qquad [\sigma_3(h_1),\sigma_3(h_2)]=0\,.
\end{aligned}
$$

From the above concrete analysis, It is seen that the symmetry group is a product of a discrete $\mathcal{C}$ group and an infinite dimensional Kac-Moody–Virasoro type Lie group with three arbitrary functions for the real KP equation. Furthermore, for the complex KD equation, its full known Lie group in form Eqs. (3) and (4) is a product of two discrete groups ($\mathcal{C}$ and $\mathcal{D}_3$) and the infinite dimensional Kac-Moody–Virasoro type Lie group.

By means of Theorem given in this paper, we can obtain many kinds of new exact solutions starting from some simple solution for KD equation.

We can easily obtain one travelling wave solution for KD equation as follows

$$
u=\frac{2kb-la}{ka^2}+\delta_3\frac{2k}{a}\tanh(kx+ly+t\omega)\,, \tag{27}
$$

$$
v=\frac{-2k\omega a^2+12k^2b^2-4k^4a^2+3l^2a^2}{6k^2a^3}+\delta_3\frac{2l}{a}\tanh(kx+ly+t\omega)\,, \tag{28}
$$

where $\delta_3=\pm1$, k, l, ω are arbitrary constants. Then the application of symmetry group Theorem on the solution Eqs. (27) and (28) the group invariant solution

$$
u=-\frac{1}{6}\frac{(-12kb+\delta_2\delta_1\eta_{0t}ak+6\delta_2\delta_1la)}{ka^2}+\frac{2\delta_2\delta_1\delta_3k}{a}\tanh(k\xi+l\eta+\omega t)\,, \tag{29}
$$

$$
\begin{aligned}
v=\frac{{\delta_2}^2}{72k^2a^3}[&-4y{\delta_2}^2\delta_1\eta_{0tt}a^2k^2-24\xi_{0t}a^2k^2+144b^2\delta_2k^2+\eta_{0t}^2a^2k^2\\
&-24k\omega a^2-48k^4a^2+36l^2a^2-144l\delta_3\tanh(k\xi+l\eta+\omega t)k^2a^2\\
&-12\eta_{0t}a^2kl-24\eta_{0t}a^2k^3\delta_3\tanh(k\xi+l\eta+\omega t)]\,,
\end{aligned} \tag{30}
$$

with

$$
\eta={\delta_2}^2\delta_1y+\eta_0,\qquad \xi=\delta_2x+\frac{1}{6}{\delta_2}^2\eta_{0t}y\delta_1+\xi_0\,,
$$

where $\xi_0\equiv\xi_0(t)$, $\eta_0\equiv\eta_0(t)$, δ_2, and δ_1 determined by Eq. (11).

By selecting the arbitrary functions ξ_0,η_0, introduced by the transformation group, one can obtain complicated and abundant structures of KD equation.

At same time, from one of multi-soliton solution[16] of KD equation, we can obtain a multi-soliton-like of KD equation by Eqs. (12) and (13), in which

$$U(\xi,\eta,\tau) = \frac{2}{a(1+\sum_{i=1}^{n}(\cosh(H)\exp(E)))}\sum_{i=1}^{n}((\sinh(H)K_i + \cosh(H)k_i)\exp(E))\,,$$

$$V(\xi,\eta,\tau) = -\frac{2}{a^2(1+\sum_{i=1}^{n}(\cosh(H)\exp(E)))}\sum_{i=1}^{n}[(-2\sinh(H)K_i b + 2\sinh(H)K_i a k_i - 2\cosh(H) b k_i + \cosh(H) a K_i^2 + \cosh(H) a k_i^2)\exp(E)]\,,$$

where

$$H = \frac{1}{a^2}(K_i\xi a^2 + 2K_i\eta ab - 2K_i\eta a^2 k_i + 12K_i\tau b^2 - 24K_i\tau ab k_i + 12K_i\tau a^2 k_i^2 + 4K_i^3\tau a^2 + \omega_2 a^2)\,,$$

$$E = \frac{1}{a^2}(k_i\xi a^2 + 2\eta ab k_i - \eta a^2 K_i^2 - \eta a^2 k_i^2 + 12\tau a^2 k_i K_i^2 - 12\tau ab k_i^2 - 12\tau ab K_i^2 + 12\tau b^2 k_i + 4\tau a^2 k_i^3 + \omega_1 a^2)\,,$$

and ξ, η, τ are determined by Eqs. (7) and (8), ω_1, ω_2, k_i, and K_i are arbitrary constants.

3 Conclusions

In summary, making use of the powerful general direct method proposed by Lou *et al.* and symbolic computation, the finite symmetry transformation groups theorem for the KD equation are given. According to the theorem, Lie point symmetry group can be recovered from the the full symmetry group, i.e. the Lie symmetry groups obtained via traditional Lie approaches are only special cases. We study relationships of the symmetry group for the real (complex) KD equation and the corresponding infinite dimensional Kac-Moody–Virasoro type Lie group with three arbitrary functions, respectively. It is necessary to point out that we can obtain all group invariable solutions of a known solution by the theorem. Here, we take a travelling wave solution and a multi-soliton solution of KD equation as two examples to illustrate how to construct a new solution from the symmetry groups theorem.

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