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Two-dimensional symmetry reduction of (2 + 1)-dimensional nonlinear Klein–Gordon equation

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ABSTRACT

In this paper, one-dimensional optimal system of group invariant solutions of (2 + 1)-dimensional Klein–Gordon system is constructed. Then the classification of group invariant solutions is given out and the corresponding two-dimensional symmetry reduced equations are obtained. At last some symmetry transformations are gained in detail. Especially, we obtain the most general solution from a given solution by use of six variable one-parameter subgroups transformations.

1. Introduction

As we know, the symmetry study plays an important role in almost all the scientific fields, especially, in the soliton theory. One of the main applications of the Lie theory of symmetry groups for differential equations is to get the group invariant solutions. Via any subgroup of the symmetry group, the original equation can be reduced to an equation with fewer independent variables. For more contents on this topic, you can refer to the excellent books [1–3]. But sometimes the symmetry reductions especially the most general one are hard to carry out even by the computer algebra. Otherwise, a basic problem concerning group invariant solutions is its classification which is involving optimal systems. About the optimal systems, a lot of excellent work has done by many famous experts [1,2,4–6] and some examples of optimal systems can also be found in Ibragimov [7]. Up to now, several methods have been developed to construct optimal systems. Here we will use Olver's method which only depends on fragments of the theory of Lie algebras. In Ref. [2], it is said that the problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras, so we concentrate on the latter.

The generalized $(d+1)$-dimensional nonlinear Klein–Gordon field equation is widely applied in many physical fields. It covers many important physical equations, such as the simple sine-Gordon equation [8], the double sine-Gordon equation [9–11], the $\lambda\phi^4$ equation [12], the $\gamma\phi^6+\lambda\phi^4$ [9–11] and $\gamma\phi^3+\lambda\phi^4$ equations [9–11], etc.

In this paper, we investigate two-dimensional symmetry similarity reductions of a special nonlinear Klein–Gordon equation:

$$\phi_{xx}+\phi_{yy}-\phi_{tt}=\sin(\phi). \tag{1}$$

It is needed to point out that the known similarity reductions of the nonlinear Klein–Gordon equation (1) are usually limited to obtaining one-dimensional similarity reductions. More recently, Tang and Lou [8] break the limitation to obtain two types of two-dimensional similarity reductions for Eq. (1) using the standard classical Lie group approach. Here we construct one-dimensional optimal system of group invariant solutions of Eq. (1) and obtain its 10 two-dimensional symmetry reduced equations. It is remarked that some results obtained by us are not included in the results obtained in Ref. [8]. Lastly, we obtain the most general solution from a given solution by use of six variable one-parameter subgroups transformations.

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The paper is organized as follows. In Section 2, one-dimensional optimal system of Eq. (1) via its symmetry algebra and the adjoint representations tables is constructed. Then the corresponding group invariant solutions are given. In Section 3, the most general solution of Eq. (1) obtainable from a given solution is gained by group transformations. Finally, conclusions are followed.

2. One-dimensional optimal system and group invariant solutions

Firstly, we can easily get classical symmetries of the (2 + 1)-dimensional nonlinear Klein–Gordon equation (1) using the point Lie symmetry method. To Eq. (1), by applying classical method, we consider the one-parameter group of infinitesimal transformations in (x, y, t, ϕ) given by

$$\begin{aligned} x^* &= x + \epsilon X(x, y, t, \phi) + o(\epsilon^2), \\ y^* &= y + \epsilon Y(x, y, t, \phi) + o(\epsilon^2), \\ t^* &= t + \epsilon T(x, y, t, \phi) + o(\epsilon^2), \\ \phi^* &= \phi + \epsilon \Phi(x, y, t, \phi) + o(\epsilon^2), \end{aligned} \tag{2}$$

where ϵ is group parameter. It is required that Eq. (1) be invariant under the transformation (2), and this yields a system of overdetermined, linear equations for the infinitesimals X, Y, T and Φ. Solving these equations, one can have

$$X = -a_1 y + a_3 t + a_5, \quad Y = a_1 x + a_2 t + a_4, \quad T = a_3 x + a_2 y + a_6, \quad \Phi = 0, \tag{3}$$

with arbitrary constants $a_1, a_2, \ldots, a_6$, which are the same as the results in Ref. [8]. Hence, the corresponding vector can be written as

$$V = (-a_1 y + a_3 t + a_5)\frac{\partial}{\partial x} + (a_1 x + a_2 t + a_4)\frac{\partial}{\partial y} + (a_3 x + a_2 y + a_6)\frac{\partial}{\partial t}. \tag{4}$$

Therefore we can say that the symmetry algebra of Eq. (1) is generated by the six vector fields

$$V_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \quad V_2 = t\frac{\partial}{\partial y} + y\frac{\partial}{\partial t}, \quad V_3 = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}, \quad V_4 = \frac{\partial}{\partial y}, \quad V_5 = \frac{\partial}{\partial x}, \quad V_6 = \frac{\partial}{\partial t}. \tag{5}$$

The commutation relations between these vector fields are given by the following table, the entry in row i and the column j representing $[V_i, V_j]$:

	V_1	V_2	V_3	V_4	V_5	V_6
V_1	0	V_3	$-V_2$	V_5	$-V_4$	0
V_2	$-V_3$	0	$-V_1$	$-V_6$	0	$-V_4$
V_3	V_2	V_1	0	0	$-V_6$	$-V_5$
V_4	$-V_5$	V_6	0	0	0	0
V_5	V_4	0	V_6	0	0	0
V_6	0	V_4	V_5	0	0	0

From the above commutator table, one can see that the operators $V_i (i = 1, 2, \ldots, 6)$ form a Lie algebra, which is a six-dimensional symmetry algebra. The Lie algebra spanned by $V_1, V_2, \ldots, V_6$ generates the symmetry group of Eq. (1). To compute the adjoint representation, we use the Lie series in conjunction with the above commutator table.

To go on with the classification of group invariant solutions, we have to compute the adjoint representation using Lie series in conjunction with the above commutator table. For instance, there is

$$Ad(\exp(\epsilon V_1))V_2 = V_2 - \epsilon[V_1, V_2] + \frac{1}{2!}\epsilon^2[V_1, [V_1, V_2]] - \cdots = \cos(\epsilon)V_2 - \sin(\epsilon)V_3.$$

In this manner, we construct the table

Ad	V_1	V_2	V_3
V_1	V_1	$\cos(\epsilon)V_2 - \sin(\epsilon)V_3$	$\cos(\epsilon)V_3 + \sin(\epsilon)V_2$
V_2	$\cosh(\epsilon)V_1 + \sinh(\epsilon)V_3$	V_2	$\cosh(\epsilon)V_3 + \sinh(\epsilon)V_1$
V_3	$\cosh(\epsilon)V_1 - \sinh(\epsilon)V_2$	$\cosh(\epsilon)V_2 - \sinh(\epsilon)V_1$	V_3
V_4	$V_1 + \epsilon V_5$	$V_2 - \epsilon V_6$	V_3
V_5	$V_1 - \epsilon V_4$	V_2	$V_3 - \epsilon V_6$
V_6	V_1	$V_2 - \epsilon V_4$	$V_3 - \epsilon V_5$
Ad	V_4	V_5	V_6
V_1	$\cos(\epsilon)V_4 - \sin(\epsilon)V_5$	$\cos(\epsilon)V_5 + \sin(\epsilon)V_4$	V_6
V_2	$\cosh(\epsilon)V_4 + \sinh(\epsilon)V_6$	V_5	$\cosh(\epsilon)V_6 + \sinh(\epsilon)V_4$
V_3	V_4	$\cosh(\epsilon)V_5 + \sinh(\epsilon)V_6$	$\cosh(\epsilon)V_6 + \sinh(\epsilon)V_5$
V_4	V_4	V_5	V_6
V_5	V_4	V_5	V_6
V_6	V_4	V_5	V_6

with the (i,j)th entry indicating $Ad(\exp(\epsilon V_i))V_j$. Following Ovsiannikov [1], one calls two subalgebras v_2 and v_1 of a given Lie algebra equivalent if one can find an element g in the Lie group so that $Adg(v_1) = v_2$, where Adg is the adjoint representation of g on v. Given a nonzero vector

$$V = a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4 + a_5V_5 + a_6V_6,$$

our task is to simplify as many of the coefficients a_i as possible though judicious applications of adjoint maps to V. In this way, omitting the detailed computation, we obtain an optical system $\mathscr{S}$ of the Lie algebra for Eq. (1):

$$V_1 + V_2 + a_3V_3 + a_4V_4(a3 \neq 0), V_1 + a_3V_3 + a_4V_4(a3 \neq 0), V_2 + a_3V_3 + a_4V_4(a3 \neq 0), V_1 + a_2V_2 + a_6V_6, V_2 + a_5V_5, V_4, V_6, V_4 + V_6, V_5, V_5 + V_6. \tag{6}$$

Then we will do the symmetry reductions for every element in the optical system $\mathscr{S}$ by solving the characteristic equations. We take $V_2 + a_5V_5 = t\frac{\partial}{\partial y} + y\frac{\partial}{\partial t} + a_5\frac{\partial}{\partial x}$ for example. Its characteristic equation reads

$$\frac{dx}{a_5} = \frac{dy}{t} = \frac{dt}{y} = \frac{d\phi}{0}.$$

To solve this equation directly, we can get two invariants:

$$\xi = -y^2 + t^2, \quad \eta = x - a_2\ln(y + t).$$

Substituting $\phi = \varphi(\xi,\eta)$ into Eq. (1), one can obtain its corresponding reduced equation:

$$\varphi_{\eta\eta} + 4a_5\varphi_{\xi\eta} - 4\xi\varphi_{\xi\xi} - 4\varphi_{\xi} = \sin(\varphi).$$

For some elements in $\mathscr{S}$, it is still a little more difficult and tedious to solve their characteristic equations. For this case, thanks to the computer algebra, i.e. Maple or Mathematica, we can successfully decide their invariants and reduced equations. We give out all the results in the following table in which there is $\phi = \varphi(\xi,\eta)$ for all the cases.

Generators	Invariants	Reduced equations
V_4	$\xi = x,\ \eta = t$	$\varphi_{xx} - \varphi_{tt} = \sin(\varphi)$
V_5	$\xi = y,\ \eta = t$	$\varphi_{yy} - \varphi_{tt} = \sin(\varphi)$
V_6	$\xi = x,\ \eta = y$	$\varphi_{xx} + \varphi_{yy} = \sin(\varphi)$
$V_4 + V_6$	$\xi = x,\ \eta = y - t$	$\varphi_{\xi\xi} = \sin(\varphi)$
$V_5 + V_6$	$\xi = y,\ \eta = x - t$	$\varphi_{\xi\xi} = \sin(\varphi)$
$V_2 + a_5V_5$	$\xi = -y^2 + t^2,\ \eta = x - a_2\ln(y + t)$	$\varphi_{\eta\eta} + 4a_5\varphi_{\xi\eta} - 4\xi\varphi_{\xi\xi} - 4\varphi_{\xi} = \sin(\varphi)$
$V_1 + a_2V_2 + a_6V_6$	$a_6 \neq 0 : \xi = x^2 + (1 - a_2^2)y^2 + a_2^2t^2 + 2a_2xt - 2a_2a_6y$, $\eta = [a_2a_6(a_2^2 - 1)x + a_2a_6\sqrt{a_2^2 - 1}(1 - a_2^2)y$ $+a_2^2a_6(a_2^2 - 1)t - a_2^2a_6^2\sqrt{a_2^2 - 1}]\exp\frac{\sqrt{a_2^2-1}(a_2x+t)}{a_6}$	$4[(1 - a_2^2)\xi + a_2^2a_6^2]\varphi_{\xi\xi} - 4(a_2^2 - 1)\eta\varphi_{\xi\eta} + \frac{\eta^2(a_2^2-1)^2}{a_6^2}\varphi_{\eta\eta}$ $+4(1 - a_2^2)\varphi_{\xi} + \frac{(a_2^2-1)^2}{a_6^2}\eta\varphi_{\eta} = \sin(\varphi)$
	$a_6 = 0 : \xi = a_2x + t,\ \eta = (1 + a_2^2)x^2 + 2a_2xt + y^2$	$(a_2^2 - 1)\varphi_{\xi\xi} + 4a_2^2\xi\varphi_{\xi\eta} + 4(\eta + a_2^2\xi^2)\varphi_{\eta\eta}$ $+2(a_2^2 + 2)\varphi_{\eta} = \sin(\varphi)$
$V_1 + a_3V_3$ $+a_4V_4(a_3 \neq 0)$	$a_4 \neq 0 : \xi = (a_3^2 - 1)x^2 - y^2 - a_3^2t^2 - 2a_4x + 2a_3yt$, $\eta = [a_4\sqrt{(a_3^2 - 1)}(a_3^2 - 1)x$ $+a_4(1 - a_3^2)y + a_3a_4(a_3^2 - 1)t - a_4^2\sqrt{a_3^2 - 1}]\exp\frac{\sqrt{a_3^2-1}(t-a_3y)}{a_4a_3}$	$4[(a_3^2 - 1)\xi + a_4^2]\varphi_{\xi\xi} + 4(a_3^2 - 1)\eta\varphi_{\xi\eta}$ $+\frac{(a_3^2-1)^2}{a_4^2a_3^2}\eta^2\varphi_{\eta\eta} + 4(a_3^2 - 1)\varphi_{\xi} + \frac{(a_3^2-1)^2}{a_4^2a_3^2}\eta\varphi_{\eta} = \sin(\varphi)$
	$a_4 = 0 : \xi = t - a_3y,\ \eta = (1 + a_3^2)y^2 + x^2 - 2a_3yt$	$(a_3^2 - 1)\varphi_{\xi\xi} + 4a_3^2\xi\varphi_{\xi\eta} + 4(a_3^2\xi^2 + \eta)\varphi_{\eta\eta}$ $+2(a_3^2 + 2)\phi_{\eta} = \sin(\varphi)$
$V_2 + a_3V_3$ $+a_4V_4(a_3 \neq 0)$	$a_4 \neq 0 : \xi = a_3^2x^2 + y^2 - (a_3^2 + 1)t^2 - 2a_4t + 2a_3xy$, $\eta = [a_3\sqrt{a_3^2 + 1}x$ $+\sqrt{a_3^2 + 1}y + (a_3^2 + 1)t + a4]\exp\frac{\sqrt{a_3^2+1}(x-a_3y)}{a_4a_3}$	$4[(a_3^2 + 1)\xi - a_4^2]\varphi_{\xi\xi} + 4(a_3^2 + 1)\eta\varphi_{\xi\eta}$ $+\frac{(a_3^2+1)^2}{a_4^2a_3^2}\eta^2\varphi_{\eta\eta} + 4(a_3^2 + 1)\varphi_{\xi} + \frac{(a_3^2+1)^2}{a_4^2a_3^2}\eta\varphi_{\eta} = \sin(\varphi)$
	$a_4 = 0 : \xi = x - a_3y,\ \eta = (1 - a_3^2)x^2 + a_3^2t^2 - 2a_3xy$	$(a_3^2 + 1)\varphi_{\xi\xi} + 4\xi\varphi_{\xi\eta} + 4(\xi^2 - a_3^2\eta)\varphi_{\eta\eta}$ $+2(-2a_3^2 + 1)\varphi_{\eta} = \sin(\varphi)$
$V_1 + V_2 + a_3V_3$ $+a_4V_4(a_3 \neq 0)$	$a_4 \neq 0 : \xi = (1 - a_3^2)x^2 + (1 + a_3^2)t^2 + 2xt - 2a_3xy$ $-2a_3yt + +2a_4x + 2a_4t,\ \eta = (x + t)\exp\frac{x-a_3y+t}{a_4}$	$-4a_3^2\xi\varphi_{\xi\xi} - 4a_3^2\eta\varphi_{\xi\eta} + \frac{a_3^2}{a_4^2}\eta^2\varphi_{\eta\eta} - 4a_3^2\varphi_{\xi}$ $+\frac{a_3^2}{a_4^2}\eta\varphi_{\eta} = \sin(\varphi)$
	$a_4 = 0 : \xi = (x + t)[(1 - a_3^2)x - 2a_3y + (1 + a_3^2)t]$, $\eta = x + t - a_3y + t$	$-4a_3^2\xi\varphi_{\xi\xi} + a_3^2\varphi_{\eta\eta} - 4a_3^2\varphi_{\xi} = \sin(\varphi)$

One can find that the case $V_1 + a_2V_2 + a_6V_6 (a_6 \neq 0)$ is not included in the two types of similarity reductions which Tang and Lou had given out in Ref. [8]. That is because if we substituted $a_3 = 0$ into type one in Ref. [8], we only get $\eta = 0$.

3. Group transformations

In Ref. [13], Clarkson and Kruskal (CK) introduced a direct method to derive symmetry reductions of a nonlinear system without using any group theory. For many types of nonlinear systems, the method can be used to find all the possible similarity reductions. In Refs. [14–16], Lou and Ma modified CK's direct method to find out the generalized Lie and non-Lie symmetry groups of differential equations by an ansatz reading

$$u(x,y,t) = \alpha(x,y,t) + \beta(x,y,t)U(\xi,\eta,\tau), \tag{7}$$

where ξ, η, τ are all functions of x, y, t. Eq. (7) also points that if $U(x,y,t)$ is a solution of the original differential equation, so is $u(x,y,t)$. In fact, the most general solution from a given solution can also be gained by the general group transformation without the ansatz (7).

The one-parameter groups G_i generated by V_i are given in the following table. The entries give the transformed point $exp(\epsilon V_i)(x,y,t,\phi) = (\tilde{x},\tilde{y},\tilde{t},\tilde{\phi})$:

$$\begin{aligned}
&G_1 : (x,y,t,\phi) \mapsto (x\cos(\epsilon) - y\sin(\epsilon), x\sin(\epsilon) + y\cos(\epsilon), t, \phi),\\
&G_2 : (x,y,t,\phi) \mapsto (x, y\cosh(\epsilon) + t\sinh(\epsilon), y\sinh(\epsilon) + t\cosh(\epsilon), \phi),\\
&G_3 : (x,y,t,\phi) \mapsto (x\cosh(\epsilon) + t\sinh(\epsilon), y, x\sinh(\epsilon) + t\cosh(\epsilon), \phi),\\
&G_4 : (x,y,t,\phi) \mapsto (x, y+\epsilon, t, \phi),\\
&G_5 : (x,y,t,\phi) \mapsto (x+\epsilon, y, t, \phi),\\
&G_6 : (x,y,t,\phi) \mapsto (x, y, t+\epsilon, \phi).
\end{aligned} \tag{8}$$

Since each group G_i is a symmetry group, transformations (8) imply that if $\phi = f(x,y,t)$ is a solution of Eq. (1), so are functions

$$\begin{aligned}
&\phi^{(1)} = f(x\cos(\epsilon) + y\sin(\epsilon), -x\sin(\epsilon) + y\cos(\epsilon), t),\\
&\phi^{(2)} = f(x, y\cosh(\epsilon) - t\sinh(\epsilon), -y\sinh(\epsilon) + t\cosh(\epsilon)),\\
&\phi^{(3)} = f(x\cosh(\epsilon) - t\sinh(\epsilon), y, -x\sinh(\epsilon) + t\cosh(\epsilon)),\\
&\phi^{(4)} = f(x, y-\epsilon, t),\\
&\phi^{(5)} = f(x-\epsilon, y, t),\\
&\phi^{(6)} = f(x, y, t-\epsilon).
\end{aligned}$$

The general one-parameter group of symmetries is obtained by considering linear combination $a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4 + a_5V_5 + a_6V_6$ of the given vector fields. Yet, the explicit formulae for the transformations are very complicated. Factually, it can be represented uniquely in the form

$$g = \exp(\epsilon_1 V_1) \cdot \exp(\epsilon_2 V_2) \cdot \exp(\epsilon_3 V_3) \cdot \exp(\epsilon_4 V_4) \cdot \exp(\epsilon_5 V_5) \cdot \exp(\epsilon_6 V_6). \tag{9}$$

Thus the most general solution $\tilde{\phi}$ obtainable from a given solution $\phi = p(x,y,t)$ by group transformations (9) is in the form (for simplicity, one can do it by the computer algebra):

$$\begin{aligned}
&\tilde{\phi} = p(\xi,\eta,\tau),\\
&\xi = (c\alpha + a\beta\gamma)x + (-ac + \alpha\beta\gamma)y + b\gamma t + (-ce + d\beta\gamma)a + bf\gamma + (cd + e\beta\gamma)\alpha,\\
&\eta = abx + b\alpha y + \beta t + abd + be\alpha + \beta f,\\
&\tau = (\alpha\gamma + ac\beta)x + (-a\gamma + c\alpha\beta)y + bct + (-e\gamma + cd\beta)a + bcf + (d\gamma + ce\beta)\alpha,
\end{aligned} \tag{10}$$

where $a, d, e, f, \beta, \gamma$ are arbitrary constants and there are $b = \sqrt{1+\beta^2}, c = \sqrt{1+\gamma^2}, \alpha = \sqrt{1-a^2}$.

In fact, if we take the infinitesimal form

$$a = \epsilon a_1, \quad \beta = \epsilon a_2, \quad \gamma = \epsilon a_3, \quad e = \epsilon a_4, \quad d = \epsilon a_5, \quad f = \epsilon a_6,$$

with infinitesimal parameter ϵ for the general group (10) under the transformation $\phi \mapsto \phi + \epsilon\sigma$, we can get the results

$$\sigma = (-a_1 y + a_3 t + a_5)\phi_x + (a_1 x + a_2 t + a_4)\phi_y + (a_3 x + a_2 y + a_6)\phi_t.$$

4. Conclusions

In summary, the problem of classifying group invariant solutions reduces to the problem of classifying subgroups of the full symmetry group under conjugation. And the problem of finding an optimal of subgroups is equivalent to that of finding an optimal system of subalgebras. In this paper, for one-dimensional subalgebras of a (2 + 1)-dimensional Klein–Gordon equation, its one-parameter optimal system is constructed. Hence the classification of group invariant solutions of the (2 + 1)-dimensional Klein–Gordon equation is obtained. The most general one-parameter group of symmetries is lastly given out as the composition of transforms in the various one-subgroups $G_1, G_2, \ldots, G_6$ and the most general solution obtainable from a given solution is constructed.

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