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Symmetry reduction and exact solutions of the generalized Nizhnik–Novikov–Veselov equation

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Combining the generalized direct method with the classical Lie method, we investigate the generalized Nizhnik–Novikov–Veselov (GNNV) equation. First, by means of the generalized direct method, a relationship is constructed between the new solutions and the old ones of the equation. At the same time, we obtain the full symmetry group of the GNNV equation, which includes the Lie point symmetry group $\mathcal{S}$ and the discrete groups $\mathcal{D}$. By using the symmetry, the $(2+1)$-dimensional GNNV equations are reduced to $(1+1)$-dimensional reduced equations. Second, the symmetry of the reduced equation is obtained via the classical Lie method. Then we reduce the reduced equation to the ordinary differential equations (ODEs) using the symmetry. Solving the ODEs and based on the relationship obtained above, new solutions of the GNNV equation are obtained. At last, the more general solutions are obtained, which recover one of the solutions by Radha and Lakshmanan [R. Radha, M. Lakshmanan, J. Math. Phys. 35 (9) (1994) 4746].

1. Introduction

Due to the existence of infinitely many symmetries, symmetry group techniques provide one method for obtaining such solutions of partial differential equations. These have many mathematical and physical applications, and are usually obtained either by seeking a solution in a special form or, more generally, by exploiting symmetries of the equation [1–9]. To find the Lie point symmetry group of a nonlinear system and based on the famous first fundamental theorem of Lie [1] advocated by Sophus Lie during the nineteenth century, a standard method had been widely used to find Lie point symmetry algebras and groups for almost all the known integrable systems. Recently, there have been several generalizations of the classical Lie group method for symmetry reductions. Ovsiannikov [2] developed the method of partially invariant solutions. Bluman and Cole [3] proposed the method of conditional symmetries; since all solutions of the classical determining equations necessarily satisfy the nonclassical determining equations, the solution set may be larger in the nonclassical case. From the theory, these methods were further generalized by Olver and Rosenau [4] to include weak symmetries and, even more generally, side conditions or differential constraints. On the other hand, since there exist close interrelations of symmetries to integrability, many authors study κ and τ [5,6]. More recently, motivated by the fact that symmetry reductions of the Boussinesq equation are known that are not obtained using the classical Lie group method, Clarkson and Kruskal [7,8] (CK) introduced a simple direct method to find all the possible similarity reductions of a nonlinear system without using any group theory. Lou and Ma [9] modify CK's direct method to find the generalized Lie and non-Lie symmetry groups for the well-known nonlinear equation. The expressions of the exact finite transformations of the Lie groups are much simpler than those obtained via the standard approaches.

In this paper, combing the generalized direct method with the classical Lie method, we investigate the generalized Nizhnik–Novikov–Veselov (GNNV) equation.

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$$\begin{aligned} &u_t + au_{xxx} + bu_{yyy} + cu_x + du_y - 3a(uv)_x - 3b(uw)_y = 0,\\ &u_x = v_y,\\ &u_y = w_x, \end{aligned} \tag{1}$$

where a, b, c and d are arbitrary constants. Some types of the solutions of the GNNV equation have been studied by many authors. For instance, Radha and Lakshmanan [10] use Painlevé analysis and the Hirota method to obtained its multidromion solutions of the special case $a = b = 1$. The nonlinear super-position formula of GNNV equation was also given by Hu [11]. Xia et al. [12] and Ren et al. [13] obtained the Jacobi Elliptic Function solution of the GNNV equation by sinh-cosh method. Peng [14] obtained a general solution including two arbitrary functions for the GNNV equation by means of WTC truncation method. Starting from an extended mapping approach and a linear variable separation method, Zhu et al. [15] obtained new families of exact solutions with arbitrary functions of the GNNV equation. Yusufoğlu and Bekir [16] employed the tanh method for traveling wave solutions of the GNNV equation. Kumar et al. [17] generated two broad classes of localized coherent structures and elliptic function periodic wave solutions of Eq. (1) by using the Painlevé truncation approach. In Ref. [18], authors investigated the Lie point symmetries and the reductions of Eq. (1) when $a = b = c = d = 1$.

This paper is arranged as follows: In Section 2, by using the generalized direct method, we get a relationship between the new solutions and the old ones of the GNNV equation. Taking special case, the symmetry of the GNNV equation is obtained. Then the GNNV equation is reduced using the symmetry. In Section 3, the symmetry of the reduced equation is obtained via the classical Lie method. Then the reduced equation is reduced to ODEs using the symmetry. Solving the ODEs, new explicit solutions are obtained. Finally, some conclusions and discussions are given in Section 4.

2. Symmetry and reduction of the GNNV equation

To Eq. (1), suppose that the solution has the following form by using the generalized direct method [9]

$$\begin{aligned} &u(x, y, t) = \alpha + \beta U(\xi, \eta, \tau),\\ &v(x, y, t) = \gamma + \theta V(\xi, \eta, \tau),\\ &w(x, y, t) = \zeta + \phi W(\xi, \eta, \tau), \end{aligned} \tag{2}$$

where $\alpha = \alpha(x, y, t)$, $\beta = \beta(x, y, t)$, $\gamma = \gamma(x, y, t)$, $\theta = \theta(x, y, t)$, $\zeta = \zeta(x, y, t)$, $\phi = \phi(x, y, t)$, $\xi = \xi(x, y, t)$, $\eta = \eta(x, y, t)$ and $\tau = \tau(x, y, t)$ are functions of (x, y, t) to be determined by requiring $U(\xi, \eta, \tau)$, $V(\xi, \eta, \tau)$ and $W(\xi, \eta, \tau)$ to satisfy the same GNNV equation as $u(x, y, t)$, $v(x, y, t)$ and $w(x, y, t)$ under the transformation $\{u, x, y, t\} \to \{U, \xi, \eta, \tau\}$, $\{v, x, y, t\} \to \{V, \xi, \eta, \tau\}$ and $\{w, x, y, t\} \to \{W, \xi, \eta, \tau\}$.

Restrict U, V and W to satisfy the same equation as Eq. (1), i.e.

$$\begin{aligned} &U_\tau + aU_{\xi\xi\xi} + bU_{\eta\eta\eta} + cU_\xi + dU_\eta - 3a(UV)_\xi - 3b(UW)_\eta = 0,\\ &U_\xi = V_\eta,\\ &U_\eta = W_\xi. \end{aligned} \tag{3}$$

Substituting Eq. (2) with Eq. (3) into Eq. (1) and let the coefficients of U, V, W and their derivatives be zero, we arrive at some equations to be determined. By solving these equations, we have

$$\begin{aligned} &\xi = \delta_1\tau_t^{1/3}x + \xi_0, \qquad \eta = \delta_2\tau_t^{1/3}y + \eta_0, \qquad \alpha = 0, \qquad \beta = \delta_1\delta_2\tau_t^{2/3},\\ &\gamma = \frac{\tau_{tt}}{9a\tau_t}x + \frac{\delta_1^2\xi_{0t}}{3a\tau_t^{1/3}} - \frac{\delta_1^2 c}{3a}\tau_t^{2/3} + \frac{c}{3a}, \qquad \theta = \delta_1^2\tau_t^{2/3},\\ &\zeta = \frac{\tau_{tt}}{9b\tau_t}y + \frac{\delta_2^2\eta_{0t}}{3b\tau_t^{1/3}} - \frac{\delta_2^2 d}{3b}\tau_t^{2/3} + \frac{d}{3b}, \qquad \phi = \delta_2^2\tau_t^{2/3}, \end{aligned} \tag{4}$$

where $\xi_0 \equiv \xi_0(t)$, $\eta_0 \equiv \eta_0(t)$ and $\tau \equiv \tau(t)$ are arbitrary functions, while δ_1 and δ_2 are determined by

$$\delta_1 = 1, -\frac{1}{2}\left(1 \pm \mathrm{i}\sqrt{3}\right), \qquad \delta_2 = 1, -\frac{1}{2}\left(1 \pm \mathrm{i}\sqrt{3}\right).$$

Using Eq. (2), we have

$$\begin{aligned} &u(x, y, t) = \beta U(\xi, \eta, \tau),\\ &v(x, y, t) = \gamma + \theta V(\xi, \eta, \tau),\\ &w(x, y, t) = \zeta + \phi W(\xi, \eta, \tau), \end{aligned} \tag{5}$$

where $\beta, \gamma, \theta, \zeta, \phi, \xi, \eta$ and τ are determined by Eq. (4).

From the symmetry theorem we know that the Lie point symmetry group $\mathcal{S}$ of the GNNV equation corresponds to $\delta_1 = \delta_2 = 1$. For the GNNV equation, the full symmetry group $\mathcal{G}$ is the product of the usual Lie point symmetry group $\mathcal{S}$ and the discrete group $\mathcal{D}$

$$\mathcal{G} = \mathcal{D} \otimes \mathcal{S},\ \mathcal{D} \equiv \{I, R_1, R_2, R_1^2, R_1R_2, R_2^2, R_1^2R_2, R_1R_2^2, R_1^2R_2^2\},$$

where I is the identity transformation and

$$\begin{aligned}
R_1:\quad & u(x,y,t) \to \frac{1}{2}\left(\mathrm{i}\sqrt{3}-1\right)u\left(x,\frac{1}{2}\left(\mathrm{i}\sqrt{3}-1\right)y,t\right),\\
& v(x,y,t) \to v\left(x,\frac{1}{2}\left(\mathrm{i}\sqrt{3}-1\right)y,t\right),\\
& w(x,y,t) \to -\frac{1}{2}\left(\mathrm{i}\sqrt{3}+1\right)w\left(x,\frac{1}{2}\left(\mathrm{i}\sqrt{3}-1\right)y,t\right),\\
R_2:\quad & u(x,y,t) \to \frac{1}{2}\left(\mathrm{i}\sqrt{3}-1\right)u\left(\frac{1}{2}\left(\mathrm{i}\sqrt{3}-1\right)x,y,t\right),\\
& v(x,y,t) \to -\frac{1}{2}\left(\mathrm{i}\sqrt{3}+1\right)v\left(\frac{1}{2}\left(\mathrm{i}\sqrt{3}-1\right)x,y,t\right),\\
& w(x,y,t) \to w\left(\frac{1}{2}\left(\mathrm{i}\sqrt{3}-1\right)x,y,t\right).
\end{aligned}$$

If we set

$$\tau = t+\epsilon f(t), \qquad \xi_0(t)=\epsilon g(t), \qquad \eta_0(t)=\epsilon h(t), \qquad \delta_1=1, \qquad \delta_2=1,$$

with an infinitesimal parameter ϵ, where $f(t)$, $g(t)$ and $h(t)$ are arbitrary functions, then Eq. (5) can be written as

$$\begin{aligned}
u &= U+\varepsilon\sigma(U),\\
v &= V+\varepsilon\sigma(V),\\
w &= W+\varepsilon\sigma(W).
\end{aligned}$$

We also obtain the symmetry

$$\begin{aligned}
\sigma(U) &= \left(\frac{1}{3}f_t x+g(t)\right)U_x+\left(\frac{1}{3}f_t y+h(t)\right)U_y+f(t)U_t+\frac{2}{3}f_t U,\\
\sigma(V) &= \left(\frac{1}{3}f_t x+g(t)\right)V_x+\left(\frac{1}{3}f_t y+h(t)\right)V_y+f(t)V_t+\frac{2}{3}f_t V+\frac{f_{tt}}{9a}x+\frac{g_t}{3a}-\frac{2cf_t}{3a},\\
\sigma(W) &= \left(\frac{1}{3}f_t x+g(t)\right)W_x+\left(\frac{1}{3}f_t y+h(t)\right)W_y+f(t)W_t+\frac{2}{3}f_t W+\frac{f_{tt}}{9b}y+\frac{h_t}{3b}-\frac{2df_t}{3b}.
\end{aligned} \tag{6}$$

The equivalent vector expression of the above symmetry is

$$\begin{aligned}
\mathcal{V}_1 = {} & \left(\frac{1}{3}f_t x+g(t)\right)\frac{\partial}{\partial x}+\left(\frac{1}{3}f_t y+h(t)\right)\frac{\partial}{\partial y}+f(t)\frac{\partial}{\partial t}-\frac{2}{3}f_t U\frac{\partial}{\partial U}\\
& -\left(\frac{2}{3}f_t V+\frac{f_{tt}}{9a}x+\frac{g_t}{3a}-\frac{2cf_t}{3a}\right)\frac{\partial}{\partial V}-\left(\frac{2}{3}f_t W+\frac{f_{tt}}{9b}y+\frac{h_t}{3b}-\frac{2df_t}{3b}\right)\frac{\partial}{\partial W},
\end{aligned} \tag{7}$$

which can also be obtained by the classical Lie method [8].

Now, we use the symmetry (6) to reduce Eq. (1) by taking the special case

$$g(t)=\frac{C_2}{3}f_t, \tag{8}$$

and

$$h(t)=\frac{C_3}{3}f_t, \tag{9}$$

where C_2 and C_3 are arbitrary constants.

Solving $\sigma(u)=0, \sigma(v)=0, \sigma(w)=0$ with Eqs. (8) and (9), one can get

$$\begin{aligned}
u &= \frac{P\,(l,m)}{f(t)^{\frac{2}{3}}},\\
v &= -\frac{f_t}{9af(t)}(x-C_2)+\frac{c}{3a}+\frac{Q\,(l,m)}{f(t)^{\frac{2}{3}}},\\
w &= -\frac{f_t}{9bf(t)}(y-C_3)+\frac{d}{3b}+\frac{R\,(l,m)}{f(t)^{\frac{2}{3}}},
\end{aligned} \tag{10}$$

where P, Q and R are arbitrary functions of the corresponding variables and $l=\frac{x+C_2}{f(t)^{\frac{1}{3}}}$, $m=\frac{y+C_3}{f(t)^{\frac{1}{3}}}$.

By substituting Eq. (10) into Eq. (1), we obtain the reduction of Eq. (1)

$$\begin{aligned}
&aP_{lll} + bP_{mmm} - 3aPQ_l - 3aP_lQ - 3bP_mR - 3bPR_m = 0,\\
&P_l = Q_m,\\
&P_m = R_l,
\end{aligned} \tag{11}$$

where $P = P(l, m)$, $Q = Q(l, m)$ and $R = R(l, m)$.

3. Symmetry reduction of the reduction of the GNNV equation and solutions of the GNNV equation

In this section, we will discuss the symmetry reduction and solutions of the reduction of the GNNV equation. First, we look for the symmetry of Eq. (11) by using the classical Lie method. To the Eq. (11), we can set

$$\begin{aligned}
&F_1 = aP_{lll} + bP_{mmm} - 3aPQ_l - 3aP_lQ - 3bP_mR - 3bPR_m = 0,\\
&F_2 = P_l - Q_m = 0,\\
&F_3 = P_m - R_l = 0,
\end{aligned} \tag{12}$$

and the corresponding vector field to Eq. (11) as follows

$$\begin{aligned}
\mathcal{V}_2 = {} & \rho(l, m, P, Q, R)\frac{\partial}{\partial l} + \lambda(l, m, P, Q, R)\frac{\partial}{\partial m} + \phi_1(l, m, P, Q, R)\frac{\partial}{\partial P}\\
& + \phi_2(l, m, P, Q, R)\frac{\partial}{\partial Q} + \phi_3(l, m, P, Q, R)\frac{\partial}{\partial R}.
\end{aligned} \tag{13}$$

The third prolongation of the vector field (13) is

$$pr^{(3)}\mathcal{V}_2 = \mathcal{V}_2 + \phi_1^l\frac{\partial}{\partial P_l} + \phi_1^m\frac{\partial}{\partial P_m} + \phi_1^{lll}\frac{\partial}{\partial P_{lll}} + \phi_1^{mmm}\frac{\partial}{\partial P_{mmm}} + \phi_2^l\frac{\partial}{\partial Q_l} + \phi_2^m\frac{\partial}{\partial Q_m} + \phi_3^l\frac{\partial}{\partial R_l} + \phi_3^m\frac{\partial}{\partial R_m},$$

where

$$\begin{aligned}
&\phi_1^l = D_l(\phi_1 - \rho P_l - \lambda P_m) + \rho P_{ll} + \lambda P_{lm},\\
&\phi_1^m = D_m(\phi_1 - \rho P_l - \lambda P_m) + \rho P_{lm} + \lambda P_{mm},\\
&\phi_1^{lll} = D_{lll}(\phi_1 - \rho P_l - \lambda P_m) + \rho P_{llll} + \lambda P_{lllm},\\
&\phi_1^{mmm} = D_{mmm}(\phi_1 - \rho P_l - \lambda P_m) + \rho P_{lmmm} + \lambda P_{mmmm},\\
&\phi_2^l = D_l(\phi_2 - \rho Q_l - \lambda Q_m) + \rho Q_{ll} + \lambda Q_{lm},\\
&\phi_2^m = D_m(\phi_2 - \rho Q_l - \lambda Q_m) + \rho Q_{lm} + \lambda Q_{mm},\\
&\phi_3^l = D_l(\phi_3 - \rho R_l - \lambda R_m) + \rho R_{ll} + \lambda R_{lm},\\
&\phi_3^m = D_m(\phi_3 - \rho R_l - \lambda R_m) + \rho R_{lm} + \lambda R_{mm},
\end{aligned}$$

and $\rho = \rho(l, m, P, Q, R)$, $\lambda = \lambda(l, m, P, Q, R)$, $\phi_1 = \phi_1(l, m, P, Q, R)$, $\phi_2 = \phi_2(l, m, P, Q, R)$, $\phi_3 = \phi_3(l, m, P, Q, R)$.

Let $pr^{(3)}\mathcal{V}_2F_1 = 0$, $pr^{(3)}\mathcal{V}_2F_2 = 0$, $pr^{(3)}\mathcal{V}_2F_3 = 0$ and the coefficients of P, Q, R and their derivatives be zero, we get some equations to be determined. Solving these equations, one can have

$$\rho = C_4l + C_5, \qquad \lambda = C_4m + C_6, \qquad \phi_1 = -2C_4P, \qquad \phi_2 = -2C_4Q, \qquad \phi_3 = -2C_4R,$$

where C_i $(i = 4, 5, 6)$ are arbitrary constants. The vector field can be written as

$$\mathcal{V}_2 = (C_4l + C_5)\frac{\partial}{\partial l} + (C_4m + C_6)\frac{\partial}{\partial m} - 2C_4P\frac{\partial}{\partial P} - 2C_4Q\frac{\partial}{\partial Q} - 2C_4R\frac{\partial}{\partial R}.$$

The corresponding symmetry is

$$\begin{aligned}
&\sigma(P) = (C_4l + C_5)P_l + (C_4m + C_6)P_m + 2C_4P,\\
&\sigma(Q) = (C_4l + C_5)Q_l + (C_4m + C_6)Q_m + 2C_4Q,\\
&\sigma(R) = (C_4l + C_5)R_l + (C_4m + C_6)R_m + 2C_4R.
\end{aligned}$$

Second, we use the symmetry of Eq. (11) to reduce Eq. (11). Solving $\sigma(P) = 0$, $\sigma(Q) = 0$, $\sigma(R) = 0$ respectively, it follows

$$P = \frac{\Phi(Z)}{C_4^2m^2 + 2C_4C_6m + C_6^2}, \qquad Q = \frac{\Psi(Z)}{C_4^2m^2 + 2C_4C_6m + C_6^2}, \qquad R = \frac{\Upsilon(Z)}{C_4^2m^2 + 2C_4C_6m + C_6^2}, \tag{14}$$

where Φ, Ψ and Υ are arbitrary functions of the corresponding variables and $Z = \frac{C_4 l + C_5}{C_4(C_4 m + C_6)}$. By substituting Eq. (14) into Eq. (11), one can get the corresponding reduced equation

$$(a - C_4^3 bZ^3)\Phi'''(Z) - 12C_4^3 bZ^2\Phi''(Z) - 36C_4^3 bZ\Phi'(Z) - 24C_4^3 b\Phi(Z) + 12C_4 b\Phi(Z)\Upsilon(Z) \\ + 3C_4 bZ\left[\Upsilon(Z)\Phi'(Z) + \Phi(Z)\Upsilon'(Z)\right] - 3a\left[\Phi'(Z)\Psi(Z) + \Phi(Z)\Psi'(Z)\right] = 0, \tag{15}$$

$$2C_4\Psi(Z) + C_4 Z\Psi'(Z) + \Phi'(Z) = 0, \tag{16}$$

$$2C_4\Phi(Z) + C_4 Z\Phi'(Z) + \Upsilon'(Z) = 0, \tag{17}$$

where $'$ denotes $\frac{\mathrm{d}}{\mathrm{d}Z}$ and $\Phi = \Phi(Z)$, $\Psi = \Psi(Z)$, $\Upsilon = \Upsilon(Z)$.

From Eqs. (16) and (17), one can easily get

$$\Phi = -\frac{Z\Upsilon - \int \Upsilon \mathrm{d}Z + C_7}{C_4 Z^2}, \qquad \Psi = \frac{Z\Upsilon - 2\int \Upsilon \mathrm{d}Z + C_4 C_8 Z + 2C_7}{C_4^2 Z^3}. \tag{18}$$

And the substitution of Eq. (18) into Eq. (15) yields

$$-60C_7 a\int \Upsilon \mathrm{d}Z + 30C_7^2 a - 24C_4^2 C_7 aZ + 12C_4^2 aZ^3\int \Upsilon \mathrm{d}Z - 24C_4^2 aZ^2\Upsilon + 24C_4^2 aZ\int \Upsilon \mathrm{d}Z \\ - 12C_4^5 bZ^6\Upsilon' + 6C_4^3 bZ^5\Upsilon^2 - 6aZ^3\Upsilon\Upsilon' + 9aZ^2\Upsilon'\int \Upsilon \mathrm{d}Z - 9C_7 aZ^2\Upsilon' - 48aZ\Upsilon\int \Upsilon \mathrm{d}Z \\ + 48C_7 aZ\Upsilon + 12C_4 C_7 C_8 aZ + 18aZ^2\Upsilon^2 + 30a\left(\int \Upsilon \mathrm{d}Z\right)^2 + 6C_4^3 bZ^6\Upsilon'\Upsilon - 6C_4^3 bZ^4\Upsilon\int \Upsilon \mathrm{d}Z \\ + 6C_4^3 C_7 bZ^4\Upsilon - 3C_4^3 bZ^5\Upsilon'\int \Upsilon \mathrm{d}Z + 3C_4^3 C_7 bZ^5\Upsilon' - 3C_4 C_8 aZ^3\Upsilon' + 12C_4 C_8 aZ^2\Upsilon \\ - 12C_4 C_8 aZ\int \Upsilon \mathrm{d}Z - 4C_4^2 aZ^4\Upsilon'' + C_4^2 aZ^5\Upsilon''' - 8C_4^5 bZ^7\Upsilon'' - C_4^5 bZ^8\Upsilon''' = 0. \tag{19}$$

Solving Eq. (19) with Eq. (18), one can find

$$\begin{aligned} \Phi &= -\frac{(4C_4 - 3C_8)\tan\left(\frac{\sqrt{2C_4(4C_4-3C_8)}(\ln(Z)+C_9)}{4C_4}\right)^2}{4Z} - \frac{4C_4 - 3C_8}{4Z}, \\ \Psi &= \frac{-3C_8 + 4C_4}{4C_4 Z^2}\tan\left(\frac{\sqrt{2C_4(4C_4-3C_8)}\,(\ln(Z)+C_9)}{4C_4}\right)^2 \\ &\quad - \frac{\sqrt{2C_4(4C_4-3C_8)}}{2C_4 Z^2}\tan\left(\frac{\sqrt{2C_4(4C_4-3C_8)}\,(\ln(Z)+C_9)}{4C_4}\right) + \frac{4C_4 - C_8}{4C_4 Z^2}, \\ \Upsilon &= C_4\left(C_4 - \frac{3}{4}C_8\right)\tan\left(\frac{\sqrt{2C_4(4C_4-3C_8)}\,(\ln(Z)+C_9)}{4C_4}\right)^2 \\ &\quad + \frac{C_4\sqrt{2C_4(4C_4-3C_8)}}{2}\tan\left(\frac{\sqrt{2C_4(4C_4-3C_8)}\,(\ln(Z)+C_9)}{4C_4}\right) + \frac{C_4(4C_4 - C_8)}{4}, \end{aligned} \tag{20}$$

where C_i $(i = 7, 8, 9)$ are arbitrary constants. So we can obtain a new solution of the GNNV equation by substituting Eq. (20) into Eq. (14) with the help of Eq. (10)

$$u = -\frac{C_4(4C_4 - 3C_8)\left(\tan\left(\frac{\sqrt{2C_4(4C_4-3C_8)}\left(\ln\left(\frac{C_4 x + C_5\sqrt[3]{f(t)} + C_2 C_4}{C_4(C_4 y + C_6\sqrt[3]{f(t)} + C_3 C_4)}\right) + C_9\right)}{4C_4}\right)^2 + 1\right)}{4\left(C_4 x + C_5\sqrt[3]{f(t)} + C_2 C_4\right)\left(C_4 y + C_6\sqrt[3]{f(t)} + C_3 C_4\right)},$$

$$v = \frac{C_4(4C_4-3C_8)\tan\left(\frac{\sqrt{2C_4(4C_4-3C_8)}\left(\ln\left(\frac{C_4x+C_5\sqrt[3]{f(t)}+C_2C_4}{C_4(C_4y+C_6\sqrt[3]{f(t)}+C_3C_4)}\right)+C_9\right)}{4C_4}\right)^2}{4\left(C_4(x+C_2)+C_5\sqrt[3]{f(t)}\right)^2} - \frac{C_4\sqrt{2C_4(4C_4-3C_8)}\tan\left(\frac{\sqrt{2C_4(4C_4-3C_8)}\left(\ln\left(\frac{C_4x+C_5\sqrt[3]{f(t)}+C_2C_4}{C_4(C_4y+C_6\sqrt[3]{f(t)}+C_3C_4)}\right)+C_9\right)}{4C_4}\right)}{2\left(C_4(x+C_2)+C_5\sqrt[3]{f(t)}\right)^2} - \frac{f_t}{9af(t)}(x+C_2) + \frac{C_4(4C_4-C_8)}{4\left(C_4(x+C_2)+C_5\sqrt[3]{f(t)}\right)^2} + \frac{c}{3a},$$

$$w = \frac{C_4\,(4C_4-3C_8)\tan\left(\frac{\sqrt{2C_4(4C_4-3C_8)}\left(\ln\left(\frac{C_4x+C_5\sqrt[3]{f(t)}+C_2C_4}{C_4(C_4y+C_6\sqrt[3]{f(t)}+C_3C_4)}\right)+C_9\right)}{4C_4}\right)^2}{4\left(C_4(y+C_3)+C_6\sqrt[3]{f(t)}\right)^2} + \frac{C_4\sqrt{2C_4(4C_4-3C_8)}\tan\left(\frac{\sqrt{2C_4(4C_4-3C_8)}\left(\ln\left(\frac{C_4x+C_5\sqrt[3]{f(t)}+C_2C_4}{C_4(C_4y+C_6\sqrt[3]{f(t)}+C_3C_4)}\right)+C_9\right)}{4C_4}\right)}{2\left(C_4(y+C_3)+C_6\sqrt[3]{f(t)}\right)^2} - \frac{f_t}{9bf(t)}(y+C_3) + \frac{C_4(4C_4-C_8)}{4\left(C_4(y+C_3)+C_6\sqrt[3]{f(t)}\right)^2} + \frac{d}{3b}.$$

In Section 2, by taking $f(t) = C_1, g(t) = C_2$ and $h(t) = C_3$, the reduction of Eq. (1) will become

$$\begin{aligned} &(C_1c - C_2)P_l + (C_1d - C_3)P_m + C_1(aC_1^2P_{lll} + bC_1^2P_{mmm} - 3aPQ_l - 3aP_lQ - 3bP_mR - 3bPR_m) = 0,\\ &P_l = Q_m,\\ &P_m = R_l, \end{aligned} \tag{21}$$

where $P = P(l, m), Q = Q(l, m), R = R(l, m), l = C_1x - C_2t$ and $m = C_1y - C_3t$.

By using the classical Lie method, one can get the symmetry of Eq. (21)

$$\sigma(P) = (C_4l + C_5)P_l + (C_4m + C_6)P_m + 2C_4P,$$

$$\sigma(Q) = (C_4l + C_5)Q_l + (C_4m + C_6)Q_m + 2C_4Q - \frac{2C_4(C_1c - C_2)}{3C_1a},$$

$$\sigma(R) = (C_4l + C_5)R_l + (C_4m + C_6)R_m + 2C_4R - \frac{C_4(5C_1c - 3C_1d - 2C_3)}{3C_1b}.$$

Solving $\sigma(P) = 0, \sigma(Q) = 0$ and $\sigma(R) = 0$ by choosing $C_4 = 0$, it follows

$$P = \Phi(Z), \qquad Q = \Psi(Z), \qquad R = \Upsilon(Z),$$

where Φ, Ψ and Υ are arbitrary functions of the corresponding variables and $Z = C_6l - C_5m$. Then Eq. (21) will be changed to

$$A\Phi'(Z) + 6C_1\left(\frac{C_6^2a}{C_5} - \frac{C_5^2b}{C_6}\right)\Phi\Phi'(Z) + C_1^3(C_6^3a - C_5^3b)\Phi'''(Z) = 0,$$

$$C_6\Phi'(Z) + C_5\Psi'(Z) = 0,$$

$$C_5\Phi'(Z) + C_6\Upsilon'(Z) = 0,$$

where $A = (C_3C_5 - C_2C_6 - 3C_1C_6C_7a + 3C_1C_5C_8b + C_1C_6c - C_1C_5d)$. The above equations have the solution

$$\Phi = -\frac{C_5C_6}{2C_1(C_6^3a - C_5^3b)}A\,\text{sech}^2\left(\frac{\sqrt{-\frac{A}{C_1^3(C_6^3a-C_5^3b)}}}{2}Z + C_0\right),$$

$$\Psi = \frac{C_6^2}{2C_1(C_6^3a - C_5^3b)}A\,\text{sech}^2\left(\frac{\sqrt{-\frac{A}{C_1^3(C_6^3a-C_5^3b)}}}{2}Z + C_0\right) + C_7,$$

$$\Upsilon = \frac{C_5^2}{2C_1(C_6^3a - C_5^3b)}A\,\text{sech}^2\left(\frac{\sqrt{-\frac{A}{C_1^3(C_6^3a-C_5^3b)}}}{2}Z + C_0\right) + C_8.$$

And one can get the solution of Eq. (1)

$$\begin{aligned} u &= -\frac{C_5C_6}{2C_1(C_6^3a-C_5^3b)}A\,\mathrm{sech}^2\left(\frac{\sqrt{-\frac{A}{C_1^3(C_6^3a-C_5^3b)}}}{2}Z+C_0\right),\\ v &= \frac{C_6^2}{2C_1(C_6^3a-C_5^3b)}A\,\mathrm{sech}^2\left(\frac{\sqrt{-\frac{A}{C_1^3(C_6^3a-C_5^3b)}}}{2}Z+C_0\right)+C_7,\\ w &= \frac{C_5^2}{2C_1(C_6^3a-C_5^3b)}A\,\mathrm{sech}^2\left(\frac{\sqrt{-\frac{A}{C_1^3(C_6^3a-C_5^3b)}}}{2}Z+C_0\right)+C_8, \end{aligned} \tag{22}$$

where $Z = C_1C_6x - C_1C_5y - (C_2C_6 - C_3C_5)t$. If we take $a = b = 1$, $C_2 = \frac{C_1k_1c+C_1l_1d-C_3l_1+C_1k_1^3+C_1l_1^3}{k_1}$, $C_5 = -\frac{C_6l_1}{k_1}$, $C_7 = C_8 = 0$ and $C_0 = \chi_1^{(0)}$, the solution (22) is equivalent to the solution (52) and (53) in Ref. [10].

Taking the solution (22) as a seed solution, via Eq. (5) one can get another solution of Eq. (1)

$$\begin{aligned} u_1 &= -\frac{C_5C_6\delta_1\delta_2\tau_t^{2/3}}{2C_1(C_6^3a-C_5^3b)}A\,\mathrm{sech}^2\left(\frac{\sqrt{-\frac{A}{C_1^3(C_6^3a-C_5^3b)}}}{2}(C_1C_6\xi-C_1C_5\eta-C_2C_6\tau+C_3C_5\tau)+C_0\right),\\ v_1 &= \frac{\tau_{tt}}{9a\tau_t}x+\frac{\delta_1^2\xi_{0t}}{3a\tau_t^{1/3}}-\frac{\delta_1^2c}{3a}\tau_t^{2/3}+\frac{c}{3a}+C_7\delta_1^2\tau_t^{2/3}\\ &\quad+\frac{C_6^2\delta_1^2\tau_t^{2/3}}{2C_1(C_6^3a-C_5^3b)}A\,\mathrm{sech}^2\left(\frac{\sqrt{-\frac{A}{C_1^3(C_6^3a-C_5^3b)}}}{2}(C_1C_6\xi-C_1C_5\eta-C_2C_6\tau+C_3C_5\tau)+C_0\right),\\ w_1 &= \frac{\tau_{tt}}{9b\tau_t}y+\frac{\delta_2^2\eta_{0t}}{3b\tau_t^{1/3}}-\frac{\delta_2^2d}{3b}\tau_t^{2/3}+\frac{d}{3b}+C_8\delta_2^2\tau_t^{2/3}\\ &\quad+\frac{C_5^2\delta_2^2\tau_t^{2/3}}{2C_1(C_6^3a-C_5^3b)}A\,\mathrm{sech}^2\left(\frac{\sqrt{-\frac{A}{C_1^3(C_6^3a-C_5^3b)}}}{2}(C_1C_6\xi-C_1C_5\eta-C_2C_6\tau+C_3C_5\tau)+C_0\right) \end{aligned}$$

where ξ, η, τ are determined by Eq. (4).

4. Conclusion

In summary, the relationship is set up between the new solutions and the old ones of the GNNV equation, and the symmetry of the GNNV equation is obtained by means of the generalized direct method, both the Lie point symmetry groups and the non-Lie symmetry groups are obtained without using any group theory. The Lie symmetry groups obtained via traditional Lie approaches are only special cases. Furthermore, the expressions of the exact finite transformations of the Lie groups are much simpler than those obtained via the standard approaches. Using the symmetry, we get the reduction of the GNNV equation. The symmetry and reduction of the reduced equation are also obtained via the classical Lie method. Solving the second time reduced equation, we obtain a new solution of the GNNV equation. By taking the special case, we recover one of the solutions in Ref. [10]. Based on a given solution, one can construct another new one of the GNNV equation with the help of the obtained relationship.

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