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Finite Symmetry Transformation Groups and Some Exact Solutions to (2+1)-Dimensional Cubic Nonlinear Schrödinger Equantion*

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Abstract *Making use of the direct method proposed by Lou et al. and symbolic computation, finite symmetry transformation groups for a (2+1)-dimensional cubic nonlinear Schrödinger (NLS) equation and its corresponding cylindrical NLS equations are presented. Nine related linear independent infinitesimal generators can be obtained from the finite symmetry transformation groups by restricting the arbitrary constants in infinitesimal forms. Some exact solutions are derived from a simple travelling wave solution.*

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1 Introduction

In recent years, the study of symmetries, symmetry groups, symmetry reductions, and group invariant solutions of nonlinear partial differential equations (PDEs) has become one of the most exciting and extremely active areas of research.[1−15] Some powerful methods to obtain the similarity reductions of a given PDE have been developed by mathematicians and physicists, such as the Lie approach and the direct method. In 1974, Bluman and Cole[1] and later Olver and Rosenau[2] generalized the Lie approach to encompass symmetry transformations so that the invariant becomes a subset of the possible solutions of PDEs. The direct method is developed by Clarkson and Kruskal[3] in 1989. It is an easy way to get the similarity reductions of PDEs without using group theory. More recently, some authors have further developed the method in the study of the generalized conditional symmetries of PDEs. They started from the ansatz of the solution forms (the direct, modified direct methods, etc.) without touching symmetry theories.[5] Most recently, Lou *et al.* have developed a new symmetry group method in a series of papers.[13−15] By the new symmetry group method, both the Lie point symmetry groups and the non-Lie symmetry groups can be obtained for some nonlinear partial differential equations (PDEs). Furthermore, the expressions of the exact finite transformations of the Lie groups are much simpler than those obtained via the standard approaches for some nonlinear PDEs.

In this paper, we would like to apply the new symmetry group method proposed by Lou *et al.* to construct finite symmetry transformation groups for some PDEs and seek for some exact solutions from the transformation groups. In Sec. 2, the finite symmetry transformation groups of a (2 + 1)-dimensional cubic nonlinear Schrödinger (NLS) equation and the corresponding cylindrical NLS equations are obtained by the new symmetry group method. From the finite symmetry transformation, the NLS's infinitesimal generators by classical Lie approach can be recovered. In Sec. 3, some exact solutions obtained from the transformation theorems are given. The last section is a short summary and discussion.

2 Finite Transformation Groups of a (2+1)-Dimensional Cubic NLS Equantion and Related Cylindrical NLS Equation

The (2 + 1)-dimensional cubic nonlinear Schrödinger (NLS) equation[16−18] may be write as

$$i\psi_t + c(\psi_{xx} + \psi_{yy}) + a|\psi|^2\psi + b\psi = 0\,, \tag{1}$$

where a, b, and c are constants, $\psi(x, y, z)$ is a complex function, and subscripts x, y, and t represent partial derivatives. Equation (1) is an extension of the following two-dimensional cubic nonlinear NLS equation

$$i\psi_t + c(\psi_{xx} + \psi_{yy}) + a|\psi|^2\psi = 0\,. \tag{2}$$

In Eq. (1), constant b plays the role of absorption coefficient, acts as a defocusing mechanism, and depends on some physical parameters. The complex function $\psi(x, y, t)$ in the mathematical model of (2 + 1)-dimensional cubic NLS equation (1) arises in many physical applications and

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also $\psi(x,y,t)$ has different physical meanings in different branches of physics. It may be an electromagnetic potential and the NLS equation then describes, for instance, the collapse of Langmuir waves with collisional damping. In other applications, $\psi(x,y,t)$ can be a complex order parameter, describing various physical phenomena close to critical stability, in the context of the complex Ginzburg-Landau equation where b plays the role of the instability parameter.

In Ref. [16], symmetry reductions for Eq. (1) to complex ordinary differential equations are presented by means of Lies method of infinitesimal transformation groups. According to the results of Ref. [16], the Lie algebra of infinitesimal symmetries of the (2+1)-dimensional cubic NLS equation is spanned by the nine vector fields,

$$v_1 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\,, \tag{3}$$

$$\begin{aligned} v_2 &= \frac{1}{2}xt\frac{\partial}{\partial x} + \frac{1}{2}yt\frac{\partial}{\partial y} + \frac{1}{2}t^2\frac{\partial}{\partial t} + \Big[i\Big(\frac{x^2+y^2}{8c} + \frac{bt^2}{2}\Big) - \frac{t}{2}\Big]\psi\frac{\partial}{\partial \psi}\,,\\ v_3 &= \frac{1}{2}x\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} + \Big(ibt - \frac{1}{2}\Big)\psi\frac{\partial}{\partial \psi}\,,\\ v_4 &= \frac{\partial}{\partial t} + ib\psi\frac{\partial}{\partial \psi}\,, \quad v_5 = t\frac{\partial}{\partial y} + \frac{i}{2c}x\psi\frac{\partial}{\partial \psi}\,,\\ v_6 &= \frac{\partial}{\partial y}\,, \quad v_7 = t\frac{\partial}{\partial x} + \frac{i}{2c}x\psi\frac{\partial}{\partial \psi}\,,\\ v_8 &= \frac{\partial}{\partial x}\,, \quad v_9 = -\frac{i}{2c}\psi\frac{\partial}{\partial \psi}\,. \end{aligned} \tag{7}$$

Six families of exact analytical solutions for Eq. (1) are reported by the extended projective Riccati equations method in Ref. [17]. In Ref. [18], three families of exact analytical solitons of Eq. (1) are obtained by an extended subequation rational expansion method.

In order to obtain finite transformation group of Eq. (1), we write it as the system

$$\begin{aligned} &i\psi_t + c(\psi_{xx} + \psi_{yy}) + a\psi^2\phi + b\psi = 0\,,\\ &-i\phi_t + c(\phi_{xx} + \phi_{yy}) + a\psi\phi^2 + b\phi = 0\,, \end{aligned} \tag{8}$$

where ϕ is the conjugate function of ψ.

First, let

$$\begin{aligned} \psi &= \alpha_1 + \beta_1\Psi(\xi,\eta,\tau) + \delta_1\Phi(\xi,\eta,\tau)\,,\\ \phi &= \alpha_2 + \beta_2\Psi(\xi,\eta,\tau) + \delta_2\Phi(\xi,\eta,\tau)\,, \end{aligned} \tag{9}$$

where α_1, α_2, β_1, β_2, δ_1, δ_2, ξ, η, and τ are functions of x, y, t, while Ψ and Φ satisfy the same equations as Eq. (8),

$$\begin{aligned} &i\Psi_\tau + c(\Psi_{\xi\xi} + \Psi_{\eta\eta}) + a\Psi^2\Phi + b\Psi = 0\,,\\ &i\Phi_\tau + c(\Phi_{\xi\xi} + \Phi_{\eta\eta}) + a\Psi\Phi^2 + b\Phi = 0\,. \end{aligned} \tag{10}$$

Substituting Eq. (9) into Eq. (8) and eliminating all terms including Ψ_τ and Φ_τ by Eq. (10), we obtain two polynomial differential equations with respect to $\{\Psi,\Phi\}$ and their derivatives. Then collecting the coefficients of $\{\Psi,\Phi\}$ and their derivatives, we obtain a set of over-determined PDEs with respect to differentiable functions: $\{\alpha_1,\alpha_2,\beta_1,\beta_2,\delta_1,\delta_2,\xi,\eta,\tau\}$. Because the set of PDEs include 112 differential equations, so we omit it in the paper.

With the help of symbolic computation, after finishing some quite tedious calculations, the general solutions of the set of PDEs are as follows:

$$\alpha_1 = \alpha_2 = \delta_1 = \beta_2 = 0\,, \quad \delta_2 = \frac{\tau_t}{\beta_1}\,, \tag{11}$$

$$\xi = 2\omega_1 k\frac{\sqrt{A_1}}{t+A_2}x - 2\omega_2\frac{\sqrt{A_1(1-k^2)}}{t+A_2}y + \frac{C_4}{t+A_2} + C_3\,, \tag{12}$$

$$\eta = 2\omega_1\frac{\sqrt{(1-k^2)A_1}}{t+A_2}x + 2\omega_2 k\frac{\sqrt{A_1}}{t+A_2}y + \frac{C_2}{t+A_2} + C_1\,, \quad \tau = -\frac{4A_1}{t+A_2} + A_3\,, \tag{13}$$

$$\beta_1 = \frac{C_5}{t+A_2}\exp\Big[\frac{i}{(t+A_2)}\Big(\frac{x^2+y^2}{4c} + \frac{C_4k + C_2\sqrt{1-k^2}}{4c\omega_1\sqrt{A_1}}x + \frac{C_2k - C_4\sqrt{1-k^2}}{4c\omega_2\sqrt{A_1}}y + bt^2 + bA_2t + \frac{64A_1^2bc + C_2^2 + C_4^2}{16A_1c}\Big)\Big], \tag{14}$$

where A_1, A_2, A_3, k, C_1, C_2, C_3, C_4, and C_5 are arbitrary complex constants and

$$\omega_1 = \pm 1\,, \quad \omega_2 = \pm 1\,. \tag{15}$$

From the above results, one can get the following symmetry group theorem for Eq. (1).

Theorem 1 If $\Psi = \Psi(x,y,t)$ is a solution of (2+1)-dimensional cubic NLS equation then so is

$$\psi = \beta_1\Psi(\xi,\eta,\tau)\,, \tag{16}$$

with Eqs. (11) $\sim$ (14).

In order to reproduce the nine vector fields $v_1, v_2, \ldots, v_9$ in Eqs. (3) $\sim$ (7) from classical Lie group method, we need take the arbitrary complex constants A_1, A_2, A_3, k, C_1, C_2, C_3, C_4, and C_5 to be different forms with respect to an infinitesimal parameter ϵ. Here we take v_2 as an example to describe the reproduce procedure.

Set $A_1 = 1/2$, $A_3 = \epsilon$, $C_5 = 1 - ib\epsilon$ and $A_2 = k = C_1 = C_2 = C_3 = C_4 = 0$ then Eq. (16) is changed into

$$\psi^{(2)} = \frac{1 - ib\epsilon}{t}\exp\Big[\frac{i + b\epsilon}{t}\Big(\frac{x^2+y^2}{4c} + bt^2 + 2b\Big)\Big]\Psi\Big(-\frac{\omega_2\sqrt{2}}{t}y, \frac{\omega_1\sqrt{2}}{t}x, -\frac{2}{t} + \epsilon\Big)\,. \tag{17}$$

From Eq. (17), the corresponding symmetry group G_2 is as follows:

$$G_2: \quad \Big(\frac{-\sqrt{2}y}{\omega_1(t-\epsilon)}, \frac{\sqrt{2}x}{\omega_2(t-\epsilon)}, \frac{-2}{t-\epsilon}, -\frac{(1-ib\epsilon)(t-\epsilon)}{2}\exp\Big[\frac{-i(x^2+y^2+8bc}{4c(t-\epsilon)} - b(t-\epsilon)\Big]\psi\Big)\,. \tag{18}$$

Thus from Eq. (18), the vector field v_2 in Eq. (4) can be derived.

It is well-known when set $\psi(x,y,t) = u(\rho,t)\exp(i\kappa\theta)$ with $\rho = \sqrt{x^2+y^2}$ and $\theta = \arctan(x/y)$, Eq. (1) changes to the so-called cylindrical nonlinear Schrödinger (CNLS) equation,

$$iu_t + c(u_{\rho\rho} + \rho^{-1}u_\rho) + a|u|^2u + bu = 0\,, \tag{19}$$

which plays an important role in the theory of light wave envelopes in dispersive media with nonlinear refractive index.[19]

Due to $\tilde\rho = \sqrt{\xi^2+\eta^2}$ with no respect to $\tilde t = \tau$, we only set $C_1 = C_2 = C_3 = C_4 = 0$ in Theorem 1, then the finite symmetry transformation group of the CNLS equation (19) can be derived as follows.

Theorem 2 If $U = U(\rho,t)$ is a solution of the CNLS equation (19) then so is

$$u = \frac{C_5}{t+A_2}\exp\Big[i\Big(\frac{\rho^2}{4c(t+A_2)} + 4bt + \frac{4A_1b}{t+A_2}\Big)\Big] \times U\Big(\pm\frac{2\sqrt{A_1}}{t+A_2}\rho, -\frac{4A_1}{t+A_2} + A_3\Big)\,, \tag{20}$$

where A_1, A_2, and A_3 are arbitrary constants and the arbitrary constant C_5 has been redefined.

Remark

(i) When solving the over-determined PDEs, we can obtain another family of solutions in which ψ is a function of $\Phi(\xi,\eta,\tau)$ and ϕ is a function of $\Psi(\xi,\eta,\tau)$. For brevity, we omit it in the paper.

(ii) It is necessary to point out that Theorem 2, i.e., the finite symmetry transformation group can also be derived by new symmetry group method acting on Eq. (19) directly.

3 Some Exact Solutions From Symmetry Transformation Theorem

To find some types of exact solutions in high dimensions is one of the most important and difficult work. In this section, we just write down some types of exact solutions with help of the group transformation theorems and the known trivial solutions for the (2+1)-dimensional cubic NLS equaiton.

It is easy to verify that Eq. (1) possesses the following traveling wave solution,

$$\psi(x,y,t) = \pm\sqrt{\frac{2c(k_1^2+l_1^2)}{a}}\,\mathrm{sech}(k_1x + l_1y + \lambda_1t) \times \exp[i(k_2x + l_2y + \lambda_2t)]\,, \tag{21}$$

where

$$\lambda_1 = -2c(k_1k_2 + l_1l_2)\,, \quad \lambda_2 = c(k_1^2 - k_2^2 + l_1^2 - l_2^2) + b\,, \tag{22}$$

with k_1, k_2, l_1, and l_2 being arbitrary constants.

According to the symmetry group transformation Theorem 1, the travelling wave solution Eq. (19) is changed to

$$\psi(x,y,t) = \pm\beta_1\sqrt{\frac{2c(k_1^2+l_1^2)}{a}}\,\mathrm{sech}(k_1\xi + l_1\eta + \lambda_1\tau) \times \exp[i(k_2\xi + l_2\eta + \lambda_2\tau)]\,, \tag{23}$$

where $\{\beta_1,\xi,\eta,\tau\}$ and $\{\lambda_1,\lambda_2\}$ are determined by Eq. (14) and Eq. (22), respectively.

It is easy to verify that the solution (23) includes some results by some authors via different methods.[16–18] For example, when $A_2 = 0$, $k = 1 = \omega_1 = \omega_2 = 1$, $k_1 = l_1 = C_5 = 1/\sqrt{2}$, $A_1 = 1/8$, $l_2 = -1/\sqrt{2} - C_2/c$, $k_2 = 1/\sqrt{2} - C_4/c$, $A_3 = -[2(C_1+C_3) - 2\sqrt{2}\gamma]/4(C_2+C_4)$, and $\{C_1,C_2,C_3,C_4,\gamma,\alpha\}$ satisfy a complex constraint condition, the solution Eq. (23) reduces to

$$\psi(x,y,t) = \frac{1}{t}\sqrt{\frac{c}{a}}\Big[\frac{1}{2}\Big(-\frac{x}{t} + \frac{y}{t} + \gamma\Big)\Big] \times \exp\Big[i\Big(bt + \frac{x^2+y^2}{4ct} + \frac{x+y}{2t} + \alpha\Big)\Big]\,, \tag{24}$$

which is the solution (26) obtained in Ref. [16].

4 Summary and Discussion

Making use of the new symmetry group method proposed by Lou *et al.* and symbolic computation, finite symmetry transformation groups for a (2+1)-dimensional cubic nonlinear Schrödinger (NLS) equation and corresponding cylindrical NLS equations are given. The Lie symmetry groups obtained via traditional Lie approaches can be recovered by restricting the arbitrary constants in infinitesimal forms. Some exact solutions are derived from a simple seed solutions.

References

[1] G.W. Bluman and J.D. Cole, *Similarity Methods for Differential Equations*, Springer, New York (1974).

[2] P. Olver and P. Rosenau, Phys. Lett. A **114** (1986) 107.

[3] G.W. Bluman and S. Kumei, *Symmetries and Differential Equation*, *Appl Math Sci.*, Springer, Berlin (1989).

[4] P.J. Olver, *Graduate Texts Math*, *Application of Lie Groups to Differential Equation*, 2nd. ed., Springer, New York (1993).

[5] P.A. Clarkson and M.D. Kruskal, J. Math. Phys. **30** (1989) 2201.

[6] S.Y. Lou, X.Y. Tang, and J. Lin, J. Math. Phys. **41** (2000) 8286; X.Y. Tang, X.M. Qian, J. Lin, and S.Y. Lou, J. Phys. Soc. Jpn. **73** (2004) 1464.

[7] R. Radha and M. Lakshmanan, J. Math. Phys. **35** (1994) 4746; S.Y. Lou, J. Phys. A: Math. Gen. **29** (1996) 5989.

[8] D. David, N. Kamran, D. Levi, and P. Winternitz, J. Math. Phys. **27** (1986) 1225; K.M. Tamizhmani, A. Ramani, and B. Grammaticos, J. Math. Phys. **32** (1991) 2635; S.Y. Lou, Phys. Rev. Lett. **71** (1993) 4099; F. Güngör and P. Winternitz, Math. Anal. Appl. **276** (2002) 314.

[9] C.Z. Qu, Commun. Theor. Phys. (Beijing, China) **29** (1998) 153; S.L. Zhang, S.Y. Lou, and C.Z. Qu, Chin. Phys. **15** (2006) 2765; S.L. Zhang, S.Y. Lou, C.Z. Qu, and R.H. Yue, Commun. Theor. Phys. (Beijing, China) **44** (2005) 589.

[10] Z.Y. Yan, Commun. Theor. Phys. (Beijing, China) **35** (2001) 647; Commun. Theor. Phys. (Beijing, China) **37** (2002) 27; Commun. Theor. Phys. (Beijing, China) **37** (2002) 269.

[11] C. Tian, *Lie Group and Its Applications in Differential Equations*, Science Press, Beijing (2001) (in Chinese).

[12] X.Y. Tang and S.Y. Lou, Commun. Theor. Phys. (Beijing, China) **37** (2002) 139; Chin. Phys. Lett. **19** (2002) 1; X.Y. Tang and P.K. Shukla, J. Phys. A: Math. Theor. **40** (2007) 3729.

[13] S.Y. Lou and H.C. Ma, J. Phys. A: Math. Gen. **38** (2005) L129; S.Y. Lou and H.C. Ma, Chaos, Solitons and Fractals **30** (2006) 804.

[14] S.Y. Lou, M. Jia, X.Y. Tang, and F. Huang, Phys. Rev. E **75** (2007) 056318; S.Y. Lou and X.Y. Tang, J. Math. Phys. **45** (2004) 1020.

[15] H.C. Ma and S.Y. Lou, Commun. Theor. Phys. (Beijing, China) **44** (2005) 193; Chin. Phys. **14** (2005) 1495; H.C. Ma, Commun. Theor. Phys. (Beijing, China) **43** (2005) 1047.

[16] E.A. Saied, R.G.A. EI-Rahman, and M.I. Ghonamy, J. Phys. A: Math. Gen. **36** (2003) 6751.

[17] B. Li and H.Q. Zhang, Int. J. Mod. Phys. C **15** (2004) 741.

[18] W.M. Guo, B. Li, and Y. Chen, Commun. Theor. Phys. (Beijing, China) **48** (2007) 987.

[19] A. Hasegawa, *Optical Solitons in Fibers*, 2nd. ed., Springer, Berlin (1990).