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A Note on Nonclassical Symmetries of a Class of Nonlinear Partial Differential Equations and Compatibility*

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Abstract *The nonclassical symmetries of a class of nonlinear partial differential equations obtained by the compatibility method is investigated. We show the nonclassical symmetries obtained in [J. Math. Anal. Appl.* **289** *(2004) 55, J. Math. Anal. Appl.* **311** *(2005) 479] are not all the nonclassical symmetries. Based on a new assume on the form of invariant surface condition, all the nonclassical symmetries for a class of nonlinear partial differential equations can be obtained through the compatibility method. The nonlinear Klein–Gordon equation and the Cahn–Hilliard equations all serve as examples showing the compatibility method leads quickly and easily to the determining equations for their all nonclassical symmetries for two equations.*

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1 Introduction

The nonclassical method of reduction was devised originally by Bluman and Cole, in 1969, to find new exact solutions of the heat equation.[1] The nonclassical method could be used for an arbitrary system of differential equations, for the purposes of this paper, we restrict ourselves to one nth-order PDE of (1+1)-dimension as follows:

$$\Delta(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0\,. \tag{1}$$

Suppose the form of Eq. (1) is invariant under a group action on (x, t, u) space given by its infinitesimals

$$\begin{aligned} x^* &= x + X(x, t, u)\epsilon + O(\epsilon^2)\,,\\ t^* &= t + T(x, t, u)\epsilon + O(\epsilon^2)\,,\\ u^* &= u + U(x, t, u)\epsilon + O(\epsilon^2)\,. \end{aligned} \tag{2}$$

The invariance requirement is

$$\Gamma^{(n)}\Delta|_{\Delta=0} = 0\,, \tag{3}$$

where $\Gamma^{(n)}$ is the n-th extension of the infinitesimal generator

$$\Gamma = T\frac{\partial}{\partial t} + X\frac{\partial}{\partial x} + U\frac{\partial}{\partial u}\,. \tag{4}$$

Solving Eq. (3) leads to the infinitesimals X, T, and U for the classical Lie point symmetry. The nonclassical method seeks the invariance of the original Eq. (1) augmented with the invariant surface condition

$$\Delta_0 = Xu_x + Tu_t - U = 0\,. \tag{5}$$

The nonclassical symmetries[2] are determined by

$$\Gamma^{(n)}\Delta|_{\Delta=0,\Delta_0=0} = 0\,, \quad \Gamma^{(1)}\Delta_0|_{\Delta=0,\Delta_0=0} = 0\,. \tag{6}$$

It is easily show that

$$\Gamma^{(1)}\Delta_0|_{\Delta_0=0} = -(T_u u_t + X_u u_x - U_u)\Delta_0|_{\Delta_0=0} = 0\,. \tag{7}$$

So the nonclassical symmetries are determined by the governing equation

$$\Gamma^{(n)}\Delta|_{\Delta=0,\Delta_0=0} = 0\,. \tag{8}$$

Solving this governing equation leads to a set of the determining equations for the infinitesimals X, T, and U. When the determining equations are solved, that gives rise to the nonclassical symmetries of Eq. (1).

Now we consider the two classes of nonlinear partial differential equations:

$$u_t = F(t, x, u, u_x, u_{xx}, \ldots, u_{x(n-1)})u_{x(n)} + G(t, x, u, u_x, u_{xx}, \ldots, u_{x(n-1)})\,, \tag{9}$$

$$u_{tt} = F(t, x, u, u_t, u_{tx}, \ldots, u_{tx(m)}, u_x, u_{xx}, \ldots, u_{x(n-1)})u_{x(n)} + G(t, x, u, u_t, u_{tx}, \ldots, u_{tx(m)}, u_x, u_{xx}, \ldots, u_{x(n-1)})\,. \tag{10}$$

where $u_{x(n)} = \partial_x^n u$, $u_{tx(m)} = \partial_x^m\partial_t u$ and F, G are smooth functions of their arguments.

We note that Arrigo *et al.*[3] show that the determining equations for the nonclassical symmetries of Eq. (9) can be obtained through compatibility with the invariant surface condition

$$u_t = U - Xu_x\,. \tag{11}$$

In Ref. [4], Niu *et al.* show that the determining equations for the nonclassical symmetries of Eq. (10) can be

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obtained through compatibility with Eq. (11).

However, they all assume the infinitesimals $T \neq 0$ to obtain the determining equations for the nonclassical symmetries. Here, we prove that the determining equations for the nonclassical symmetries of Eq. (9) and Eq. (10) can also be obtained through compatibility in the case $T = 0$. First, we present the derivation of nonclassical symmetries for the nonlinear Klein–Gordon equation via compatibility with the invariant surface condition in the two cases. Second, we prove that, in the case $T = 0$, the compatibility with the invariant surface condition can also lead to the governing equation of the nonclassical symmetries for two classes of nonlinear PDEs with arbitrary order. Third, we consider the nonclassical symmetries of the Cahn–Hilliard equations illustrating this method.

2 Derivation of Nonclassical Symmetries for a Nonlinear Klein–Gordon Equation by Compatibility Method

In this section, we obtain that the governing equation for the nonclassical symmetries of the nonlinear Klein–Gordon equation by compatibility method. The following result shows the governing equation obtained by compatibility method, in two cases $T \neq 0$ and $T = 0$, are as same as the governing equation using the vector fields and their prolongations.

The nonlinear Klein–Gordon equation:

$$u_{tt} = c_0 u_{xx} - c_1 \sin u \,, \tag{12}$$

where c_0 and c_1 is constant.

In the case $T \neq 0$, without loss of generality, we may set $T = 1$, we denote Eq. (9) by Δ_1, and the invariant surface condition Eq. (5) by Δ_2 then

$$\Delta_1 = u_{tt} + c_0 u_{xx} - c_1 \sin u \,, \tag{13}$$

$$\Delta_2 = X u_x + T u_t - U \,. \tag{14}$$

The determining equations of the nonlinear Klein–Gordon equation are obtained by requiring the governing equation as follows:

$$\Gamma^{(2)} \Delta_1 |_{\Delta_1=0,\Delta_2=0} = 0 \,, \tag{15}$$

where the infinitesimal generator Γ is given by

$$\Gamma = \frac{\partial}{\partial t} + X \frac{\partial}{\partial x} + U \frac{\partial}{\partial u} \tag{16}$$

with the first and second extensions as

$$\Gamma^{(1)} = \Gamma + U_{[t]} \frac{\partial}{\partial u_t} + U_{[x]} \frac{\partial}{\partial u_x} \,, \tag{17}$$

$$\Gamma^{(2)} = \Gamma^{(1)} + U_{[tt]} \frac{\partial}{\partial u_{tt}} + U_{[tx]} \frac{\partial}{\partial u_{tx}} + U_{[xx]} \frac{\partial}{\partial u_{xx}} \,. \tag{18}$$

The coefficients of the operators in Eqs. (17) and (18) are given by

$$\begin{aligned} U_{[t]} &= D_t(U - X u_x - T u_t) + X u_{tx} + T u_{tt} \\ &= D_t(U - X u_x) + X u_{tx} \,, \end{aligned} \tag{19}$$

$$\begin{aligned} U_{[x]} &= D_x(U - X u_x - T u_t) + X u_{xx} + T u_{xt} \\ &= D_t(U - X u_x) + X u_{xx} \,, \end{aligned} \tag{20}$$

$$\begin{aligned} U_{[tt]} &= D_{tt}(U - X u_x - T u_t) + X u_{ttx} + T u_{ttt} \\ &= D_{tt}(U - X u_x) + X u_{ttx} \,, \end{aligned} \tag{21}$$

$$\begin{aligned} U_{[tx]} &= D_{tx}(U - X u_x - T u_t) + X u_{txx} + T u_{txt} \\ &= D_{tx}(U - X u_x) + X u_{txx} \,, \end{aligned} \tag{22}$$

$$\begin{aligned} U_{[xx]} &= D_{xx}(U - X u_x - T u_t) + X u_{xxx} + T u_{xxt} \\ &= D_{xx}(U - X u_x) + X u_{xxx} \,. \end{aligned} \tag{23}$$

In the case $T = 0$, without loss of generality, we may set $X = 1$, then the vector fields and their prolongations are:

$$\Gamma = \frac{\partial}{\partial x} + U \frac{\partial}{\partial u} \,, \tag{24}$$

$$U_{[t]} = D_t(U - X u_x - T u_t) + X u_{tx} + T u_{tt} = D_t(U) \,, \tag{25}$$

$$U_{[x]} = D_x(U - X u_x - T u_t) + X u_{xx} + T u_{xt} = D_x(U), \tag{26}$$

$$U_{[tt]} = D_{tt}(U - X u_x - T u_t) + X u_{ttx} + T u_{ttt} = D_{tt}(U), \tag{27}$$

$$U_{[tx]} = D_{tx}(U - X u_x - T u_t) + X u_{txx} + T u_{txt} = D_{tx}(U), \tag{28}$$

$$U_{[xx]} = D_{xx}(U - X u_x - T u_t) + X u_{xxx} + T u_{xxt} = D_{xx}(U), \tag{29}$$

where the total differential operators D_t and D_x are given by

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \cdots \,, \tag{30}$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + \cdots \,. \tag{31}$$

Invariance of the nonlinear Klein–Gordon equation is given by Eq. (15), which by Eqs. (17) and (18), gives

$$\Gamma^{(2)} \Delta_1 |_{\Delta_1=0,\Delta_2=0} = U_{[tt]} - c_0 U_{[xx]} + c_1 U \cos u = 0 \,. \tag{32}$$

In the case $T = 1$, substituting Eqs. (19)–(23) into Eq. (32) gives the governing equation for the infinitesimals X, T, and U. In the case $T = 0$, substituting Eqs. (25)–(29) into Eq. (32) gives the governing equation for the infinitesimals X, T, and U, solving this governing equation leads to a set of the determining equations for X, T, and U. Next we will make use of the compatibility between the nonlinear Klein–Gordon equation and the invariant surface condition to derive Eq. (32).

In the case $T = 1$, total differentiation D_t of the nonlinear Klein–Gordon equation Eq. (12) gives

$$D_t(u_{tt}) = D_t(c_0 u_{xx} - c_1 \sin u) \,. \tag{33}$$

Through the compatibility substituting $u_t = U - X u_x$ into Eq. (33) gives

$$\begin{aligned} D_t(u_{tt}) = D_{tt}(u_t) &= D_{tt}(U - X u_x) \\ &= D_t(c_0 u_{xx} - c_1 \sin u) \\ &= c_0 u_{txx} - c_1 u_t \cos u \\ &= c_0 D_{xx}(U - X u_x) - c_1 \cos u (U - X u_x) \,. \end{aligned} \tag{34}$$

Adding $X u_{ttx} + c_0 X u_{xxx} - c_0 X u_{xxx}$ to both sides and regrouping give

$$\begin{aligned} & D_{tt}(u_t) + X u_{ttx} + c_0 X u_{xxx} - c_0 X u_{xxx} \\ &= D_{tt}(u_t) + X u_{ttx} = U_{[tt]} = c_0 D_{xx}(U - X u_x) + c_0 X u_{xxx} \\ &\quad - c_0 X u_{xxx} + X u_{ttx} - c_1 \cos u (U - X u_x) \\ &= c_0 U_{[xx]} - c_1 U \cos u + X(u_{ttx} - c_0 u_{xxx} + c_1 u_x \cos u) \,. \end{aligned}$$

By virtue of $D_x(u_{tt}) = D_x(c_0 u_{xx} - c_1 \sin u)$ gives

$$u_{ttx} - c_0 u_{xxx} + c_1 u_x \cos u = 0\,.$$

So it gives the governing Eq. (32)

$$U_{[tt]} - c_0 U_{[xx]} + c_1 U \cos u = 0\,.$$

Following Eqs. (33) and (34) and using $u_{ttx} = c_0 u_{xxx} - c_1 u_x \cos u$, $\Delta_2 = 0$, we can obtain the governing equation, then the determining equations of the nonlinear Klein–Gordon equation are:

$$3c_0 X_u - 3X_u X^2 = 0\,, \tag{35}$$

$$\begin{aligned}
&-c_0 U_u + 2X_t X + U_u X^2 + 2c_0 X_x + 2X_u X U = 0\,,\\
&-4X_u^2 X - X_{uu} X^2 + c_0 X_{uu} = 0\,,\\
&3X_u^2 U + 2X_{uu} U X + 6X_u U_u X - 3X_u X_x X + 2c_0 X_{ux}\\
&\qquad + 2X_{ut} X + U_{uu} X^2 + 3X_u X_t - c_0 U_{uu} = 0\,,
\end{aligned} \tag{36}$$

$$\begin{aligned}
&-X_u U_t - 2U_{ut} X + 2X_t X_x + c_0 X_{xx} - X_{tt}\\
&\qquad + 3X_u U_x X - 4X_u U_u U + 2X_u X_x U + U_u X X_x\\
&\qquad - 2U_{uu} U X - 2U_u^2 X - X_{uu} U^2 - 2X_{ut} U\\
&\qquad - 2c_0 U_{ux} - 3U_u X_t = 0\,,\\
&-2X_u U_x U + c_1 U \cos u - U_u X U_x - c_0 U_{xx} + 2U_{ut} U\\
&\qquad + U_u U_t + U_{uu}^2 U + U_{uu} U^2 - 2X_t U_x + U_{tt} = 0\,.
\end{aligned} \tag{37}$$

Through Eq. (35), we obtain

$$X_u = 0\,. \tag{38}$$

Substituting Eq. (38) into Eq. (36), we can obtain

$$U_{uu} = 0\,. \tag{39}$$

Substituting Eqs. (38) and (39) into the other determining equations, we can obtain the determining equations:

$$\begin{aligned}
&X_u = 0\,, \quad U_{uu} = 0\,,\\
&-c_0 U_u + 2X_t X + U_u X^2 + 2c_0 X_x = 0\,,\\
&-2U_{ut} X + 2X_t X_x + c_0 X_{xx} - X_{tt} + U_u X X_x\\
&\qquad - 2U_u^2 X - 2c_0 U_{xu} - 3U_u X_t = 0\,,\\
&c_1 U \cos u - U_u X U_x - c_0 U_{xx} + 2U_{ut} U\\
&\qquad + U_u U_t - 2X_t U_x + U_{tt} = 0\,.
\end{aligned}$$

In the case $T = 0$

$$\begin{aligned}
D_x(u_{tt}) &= D_{tt}(u_x) = D_{tt}(U)\\
&= c_0 u_{xxx} - c_1 u_x \cos u\\
&= c_0 D_{xx}(U) - c_1 U \cos u\,.
\end{aligned} \tag{40}$$

So it gives the governing Eq. (32)

$$U_{[tt]} - c_0 U_{[xx]} + c_1 U \cos u = 0\,.$$

Following Eq. (40) and using $u_{tt} = c_0 u_{xx} - c_1 \sin u$, $u_x = U$, we can obtain the governing equation, then the determining equations of the nonlinear Klein–Gordon equation are:

$$U_{uu} = 0\,, \tag{41}$$

$$U_{tu} = 0\,, \tag{42}$$

$$\begin{aligned}
&U_{tt} - c_1 U_u \sin u - c_0 U_{xx} - 2c_0 U_{xu} U\\
&\qquad - c_0 U_{uu} U^2 + c_1 U \cos u = 0\,.
\end{aligned} \tag{43}$$

Substituting Eqs. (41) and (42) into Eq. (43) we can obtain the determining equations:

$$\begin{aligned}
&U_{uu} = 0\,, \quad U_{tu} = 0\,,\\
&U_{tt} - c_1 U_u \sin u - c_0 U_{xx} + c_1 U \cos u = 0\,.
\end{aligned}$$

Then the determining equations for the nonclassical symmetries of the nonlinear Klein–Gordon equation are derived through the compatibility.

3 Derivation of Nonclassical Symmetries for a Class of Nonlinear PDEs by Compatibility Method in Case $T = 0$

If we denote Eq. (9) by Δ_1, Eq. (10) by Δ_2 and the invariant surface condition Eq. (5) with $T = 0$ by Δ_3, then

$$\Delta_1 = u_t - F u_{x(n)} - G\,, \tag{44}$$

$$\Delta_2 = u_{tt} - F u_{x(n)} - G\,, \tag{45}$$

$$\Delta_3 = u_x - U\,. \tag{46}$$

The governing equations for the nonclassical symmetries of Eqs. (9) and (10) are obtained by requiring that

$$\Gamma^{(k)} \Delta_1 |_{\Delta_1 = 0, \Delta_3 = 0} = 0\,, \tag{47}$$

$$\Gamma^{(k)} \Delta_2 |_{\Delta_2 = 0, \Delta_3 = 0} = 0\,, \tag{48}$$

where $k = \max\{m+1, n\}$, the infinitesimal generator Γ is given in Eq. (4) and its k-th extension is given recursively as

$$\Gamma^{(k)} = \Gamma^{(k-1)} + \sum_{i=0}^{k} U_{[t(k-i)x(i)]} \frac{\partial}{\partial u_{t(k-i)x(i)}}\,, \tag{49}$$

where $u_{t(k-i)x(i)} = \partial_t^{k-i} \partial_x^i u$, the coefficients of the operators in Eq. (49) are given by

$$U_{[t(k-i)x(i)]} = D_t^{k-i} D_x^i (U)\,.$$

Invariance of Eq. (9) is given by Eq. (47) from which we obtain

$$U_{[t]} = F U_{[x(n)]} + \Gamma^{(k)} F u_{x(n)} + \Gamma^{(k)} G\,. \tag{50}$$

Invariance of Eq. (10) is given by Eq. (48) from which we obtain

$$U_{[tt]} = F U_{[x(n)]} + \Gamma^{(k)} F u_{x(n)} + \Gamma^{(k)} G\,. \tag{51}$$

Solving Eq. (50) leads to a set of the determining equations of Eq. (9). Solving the Eq. (51) leads to a set of the determining equations of Eq. (10). Next we give and prove an important relationship between the extended infinitesimal generator $\Gamma^{(k)}$ and the total derivative operators D_x and D_t.

Lemma If $\Gamma^{(k)}$ is the extended infinitesimal generator, and D_x and D_t are total derivative operators, then for any smooth function

$$F(t, x, u, u_t, u_{tx}, \ldots, u_{tx(m)}, u_x, u_{xx}, \ldots, u_{x(n)})\,,$$

$\Gamma^{(k)} F = D_x(F)$, provided $u_x = U$.

Proof From the definition of $\Gamma^{(k)}$, D_x, and D_t, it is clear that

$$\begin{aligned}
\Gamma^{(k)} F &= F_x + U F_u + U_{[t]} F_{u_t}\\
&\qquad + \sum_{j=0}^{k} U_{[tx(j)]F_{u_{tx(j)}}} + \sum_{i=0}^{k} U_{[x(i)]F_{u_{x(i)}}}
\end{aligned}$$

$$= F_x + F_u u_x + F_{u_t} u_{tx} + \sum_{j=0}^{k} F_{u_{tx(j)}} u_{[tx(j+1)]}$$

$$+ \sum_{i=0}^{k} F_{u_{x(i)}} u_{[x(i+1)]} = D_x(F) .$$

Theorem 1 If the infinitesimal $T = 0$, the determining equations for the nonclassical symmetries of Eq. (9) can be obtained through compatibility with the invariant surface condition $u_x = U$, where $U = U(x, t, u)$ are smooth functions.

Proof Total differentiation D_x of Eq. (9) gives

$$D_x(u_t) = D_x(F)u_{x(n)} + Fu_{x(n+1)} + D_x(G) . \qquad (52)$$

Substituting $u_x = U$ into Eq. (52), we can obtain

$$D_t(U) = D_x(F)u_{x(n)} + FD_x^n(U) + D_x(G) .$$

From the definition of $U_{[t]}$ and $U_{[x(n)]}$, it is clear that

$$U_{[t]} = D_x(F)u_{x(n)} + FU_{[x(n)]} + D_x(G) .$$

Through the above Lemma this equation becomes

$$U_{[t]} = \Gamma^{(k)}(F)u_{x(n)} + FU_{[x(n)]} + \Gamma^{(k)}G .$$

Theorem 2 If the infinitesimal $T = 0$, the determining equations for the nonclassical symmetries of Eq. (10) can be obtained through compatibility with the invariant surface condition $u_x = U$, where $U = U(x, t, u)$ are smooth functions.

Proof Suppose that the two equations are compatible, total differentiation D_x of Eq. (10) gives

$$D_{tt}(u_x) = Fu_{x(n+1)} + D_x(F)u_{x(n)} + D_x(G) . \qquad (53)$$

Substituting $u_x = U$ into Eq. (53), we can obtain

$$D_{tt}(U) = FD_x^n(U) + D_x(F)u_{x(n)} + D_x(G) ,$$

from the definition of $U_{[tt]}$, $U_{[x(n)]}$, it is clear that

$$U_{[tt]} = FU_{[x(n)]} + D_x(F)u_{x(n)} + D_x(G) .$$

Through the above Lemma this equation becomes

$$U_{[tt]} = FU_{[x(n)]} + \Gamma^{(k)}Fu_{x(n)} + \Gamma^{(k)}G .$$

4 Examples

Arrigo *et al.* have considerd the KdV equation and their generalizations showing that compatibility leads to the determining equation for their nonclassical symmetries. Now we further generalizes to equations of a family of Cahn–Hilliard equations[5] and their generalizations.

The Cahn–Hilliard equation describing diffusion for decomposition of a one-dimensional binary solution can be written as

$$u_t + (ku_{xxx} - f(u)u_x)_x = 0 ,$$

without loss of generality, we denote $k = 1$.

In the case $T = 1$, if the Cahn–Hilliard equation and the invariant surface condition are rewritten as

$$u_{xxxx} = -u_t + f(u)u_{xx} + f'(u)u_x^2 , \qquad (54)$$

$$u_t = U - Xu_x , \qquad (55)$$

then requiring the compatibility condition, gives

$$D_t(u_{xxxx}) - D_{xxxx}(u_t) = 0 .$$

Using

$$D_t(-(U - Xu_x) + f(u)u_{xx} + f'(u)u_x^2) - D_{xxxx}(U - Xu_x) = 0 ,$$

expanding and using Eq. (54) to eliminate u_{xxxx}, then using Eq. (55) to eliminate u_t and differential consequences gives rise to

$$\begin{aligned}
&X_{uuuu}u_x^5 + (-U_{uuuu} + 4X_{xuuu})u_x^4 + 10X_{uuu}u_{xx}u_x^3 \\
&\quad + (3X_uf'(u) - 4U_{xuuu} - f(u)X_{uu} + 6X_{xxuu})u_x^3 \\
&\quad + (f(u)U_{uu} - 6U_{xxuu} - 2f(u)X_{xu} + f'(u)U_u \\
&\quad + 4X_uX + 2X_xf'(u) + f''(u)U + 4X_{xxxu})u_x^2 \\
&\quad + 10X_{uu}u_{xxx}u_x^2 + (24X_{xuu} - 6U_{uuu})u_{xx}u_x^2 \\
&\quad + (-4X_uU + 2f(u)U_{xu} - f(u)X_{xx} + 2f'(u)U_x \\
&\quad - 4U_{xxxu} + X_t + X_{xxxx} + 4X_xX)u_x(16X_{xu} \\
&\quad - 4U_{uu})u_{xxx}u_x + (-12U_{xuu} + 18X_{xxu} \\
&\quad + 2X_uf(u))u_{xx}u_x + 15X_{uu}u_{xx}^2u_x + 10X_uu_{xx}u_{xxx} \\
&\quad + (f'(u)U - 6U_{xxu} + 2X_xf(u) + 4X_{xxx})u_{xx} \\
&\quad + (-4U_{xu} + 6X_{xx})u_{xxx} + (-3U_{uu} + 12X_{xu})u_{xx}^2 \\
&\quad + (-U_{xxxx} - U_t - 4X_xU + f(u)U_{xx}) .
\end{aligned}$$

Then we can obtain the determining equations

$$\begin{aligned}
&X_{uuuu} = 0 , \quad -U_{uuuu} + 4X_{xuuu} = 0 , \ X_{uuu} = 0 , \\
&3X_uf'(u) - 4U_{xuuu} - f(u)X_{uu} + 6X_{xxuu} = 0 , \\
&f(u)U_{uu} - 6U_{xxuu} - 2f(u)X_{xu} + f'(u)U_u + 4X_uX \\
&\qquad + 2X_xf'(u) + f''(u)U + 4X_{xxxu} = 0 , \\
&X_{uu} = 0 , \quad 4X_{xuu} - U_{uuu} = 0 , \\
&-4X_uU + 2f(u)U_{xu} - f(u)X_{xx} + 2f'(u)U_x \\
&\qquad - 4U_{xxxu} + X_t + X_{xxxx} + 4X_xX = 0 , \\
&4X_{xu} - U_{uu} = 0 , \quad -12U_{xuu} + 18X_{xxu} + 2X_uf(u) = 0 , \\
&X_u = 0 , \qquad (56) \\
&f'(u)U - 6U_{xxu} + 2X_xf(u) + 4X_{xxx} = 0 , \\
&-2U_{xu} + 3X_{xx} = 0 , \quad -U_{uu} + 4X_{xu} = 0 , \\
&-U_{xxxx} - U_t - 4X_xU + f(u)U_{xx} = 0 . \qquad (57)
\end{aligned}$$

Through Eqs. (56) and (57) we can obtain

$$X_u = 0 , \quad U_{uu} = 0 .$$

So we can obtain the determining equations of the Cahn–Hilliard equation, and it reads as follows:

$$\begin{aligned}
&X_u = 0 , \quad U_{uu} = 0 , \quad -2U_{xu} + 3X_{xx} = 0 , \\
&f'(u)U - 6U_{xxu} + 2X_xf(u) + 4X_{xxx} = 0 , \\
&f'(u)U_u + 2X_xf'(u) + f''(u)U = 0 , \\
&-U_{xxxx} - U_t - 4X_xU + f(u)U_{xx} = 0 , \\
&2f(u)U_{xu} - f(u)X_{xx} + 2f'(u)U_x \\
&\qquad - 4U_{xxxu} + X_t + X_{xxxx} + 4X_xX = 0 .
\end{aligned}$$

In the case $T = 0$, requiring the compatibility condition

$$D_x(u_t) = D_t(u_x) , \quad D_x(u_{xxx} - f(u)u_x)_x) - D_t(u_x) = 0 .$$

Using

$$u_x = U , \quad u_t = (u_{xxx} - f(u)u_x)_x ,$$

it gives rise to

$$\begin{aligned}U_{xxxx} &+ 4UU_{xx}U_{uu} + 4U_{xu}U_{xx} - f(u)U_{xx} + 3U_{uu}(U_x)^2 \\ &+ 6U^2U_{uuu}U_x + 12UU_{uux}U_x + 10UU_uU_{uu}U_x \\ &+ 6U_{uxx}U_x + 4U_uU_{ux}U_x - 3f'(u)UU_x + U^4U_{uuuu} \\ &+ 4U^3U_{uuux} + 6U^3U_uU_{uuu} + 6U^2U_{uuxx} \\ &+ 12U^2U_uU_{uux} + 4U^3(U_{uu})^2 + 12U^2U_{ux}U_{uu} \\ &+ 7U^2(U_u)^2U_{uu} - f(u)U^2U_{uu} + 4UU_{uxxx} \\ &+ 6UU_uU_{uxx} + 8U(U_{ux})^2 + 4U(U_u)^2U_{ux} \\ &- 2f(u)UU_{ux} - 2f'(u)U^2U_u + U_t - f''(u)U^3 = 0\,.\end{aligned}$$

That is same to the governing equation by Gandarias[6] using the vector fields and their prolongations.

This further generalizes to equations of the form

$$u_t = u_{xxxx} + R(u, u_x, u_{xx}, u_{xxx})\,. \qquad (58)$$

In the case $T = 1$, Eq. (58) and the invariant surface condition are rewritten as

$$u_{xxxx} = U - Xu_x - R(u, u_x, u_{xx}, u_{xxx})\,, \qquad (59)$$

$$u_t = U - Xu_x\,. \qquad (60)$$

Then requiring compatibility,

$$D_t(u_{xxxx}) - D_{xxxx}(u_t) = 0\,,$$

leads, by virtue of Eqs. (59) and (60), to

$$D_t(U - Xu_x - R(u, u_x, u_{xx}, u_{xxx})) - D_{xxxx}(U - Xu_x) = 0\,.$$

Expanding and using Eqs. (59) and (60), to eliminate u_t, u_{xxxx} gives rise to the determining equations.

In the case $T = 0$, the invariant surface condition are rewritten as

$$u_x = U\,, \qquad (61)$$

then requiring compatibility,

$$D_t(u_x) - D_x(u_t) = 0\,,$$

leads, by virtue of Eqs. (58) and (61), to

$$D_t(U) - D_x(U_{xxx} + R(u, U, U_x, U_{xx})) = 0\,.$$

Expanding and using Eq. (61) to eliminate u_x gives rise to the determining equations.

5 Conclusion

In this paper, we have considered a method of deriving the determining equations for the nonclassical symmetries of two classes of nonlinear partial differential equations. As a note on Arrigo,[3] Niu *et al.*,[4] we have proved that the determining equations for the nonclassical symmetries of Eqs. (9)and (10) can also be obtained through compatibility in the case $T = 0$. Can the determing equations for the potential symmetries of partial differential equations be derived by imposing a condition of compatibility? This is a topic of future work.

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