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Symmetry Analysis of Two Types of (2+1)-Dimensional Nonlinear Klein–Gorden Equation*

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Abstract *By means of the classical symmetry method, we investigate two types of the (2+1)-dimensional nonlinear Klein–Gorden equation. For the wave equation, we give out its symmetry group analysis in detail. For the second type of the (2+1)-dimensional nonlinear Klein–Gorden equation, an optimal system of its one-dimensional subalgebras is constructed and some corresponding two-dimensional symmetry reductions are obtained.*

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1 Introduction

Symmetry group techniques provide one method for obtaining solutions of partial differential equations.[1−4] Since Sophus Lie[1] set up the theory of Lie point symmetry group, a standard method had been widely used to find Lie point symmetry algebras and groups for almost all the known differential systems. The adjoint representation of a Lie group on its Lie algebra was known to Lie. Its use in classifying group-invariant solutions appeared in Refs. [5] and [6] which are written by Ovsiannikov and Olver, respectively. The latter reference contains more details on how to perform the classification of subgroup under the adjoint action. The method has received extensive development by Patera, Winternitz, and Zassenhaus: see [1] and the references therein for many examples of optimal systems of subgroups for the important Lie group of mathematical physics. As is known, the symmetry method allows one to determine special class of exact solutions of the given partial differential equation(s). Furthermore, from the knowledge of symmetry group one obtains the Lie algebra of the equation under consideration. The main use of these generators is to obtain a reduction of variables in a given equation, which can be obtained by solving the characteristic equation. Reductions of equation may be obtained from any linear combination. However, it is usually not feasible to list all possible similarity reductions for there is almost an infinite number of such combinations. The fact is important because by means of adjoint representation of symmetry group on its Lie algebra one finds invariant solutions, which are not related by a transformation in the symmetry group. Therefore, it is sufficient to consider only linear combinations, which lead to reductions that are inequivalent with respect to symmetry transformations; this set of solutions is called an optimal system. Precisely, by introducing the adjoint representation of the Lie algebra, we obtain the following basic fields of an optimal system, from which every other solutions can be derived.

In this paper, we will use the symmetry method and the optimal system theory to investigate the group invariant solutions of two types of (2+1)-dimensional nonlinear Klein–Gordon field equations. It is known that the generalized (2+1)-dimensional nonlinear Klein–Gordon field equation reads

$$\phi_{xx} + \phi_{yy} - \phi_{tt} = F(\phi)\,, \qquad (1)$$

where $F(\phi)$ is an arbitrary smooth function. Various special nonlinear Klein–Gordon equations, such as the simple sine-Gordon equation,[7] the double sine-Gordon equation,[8] the $\lambda\phi^4$ equation,[9] the $\gamma\phi^6 + \lambda\phi^4$,[8] and $\gamma\phi^3 + \lambda\phi^4$ equations[8] etc., are widely applied in many physical fields.

One can consider three types of the (2+1)-dimensional Klein–Gordon equations:
(i) $F(\phi) = 0$. Equation (1) reads: $\phi_{xx} + \phi_{yy} - \phi_{tt} = 0$;
(ii) $F'(\phi) \neq F(\phi)$. For example, there is $F(\phi) = \sin(\phi)$;
(iii) $F'(\phi) = F(\phi)$ ($F(\phi) \neq 0$). Equation (1) reads: $F(\phi) = \exp(\phi)$.

The cause for the consideration depends on the dimensions of their symmetry group. The symmetry group of case (i) spanned by ten vector fields and two additional vector fields is infinitesimal, while both the symmetry groups of case (ii) and case (iii) are finite which are

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spanned by six and seven vector fields, respectively.

For case (ii), Tang and Lou[7] have obtained two types of two-dimensional similarity reductions for this equation using the standard classical Lie group approach. In our another paper, we also investigate its symmetry group transformation in detail. Furthermore we construct one-parameter optimal system of the group invariant solutions for case (ii) and give out all the corresponding symmetry reductions. Here, we do not investigate this case any more. In this paper, we conduct a detailed discussion for another two types. The symmetry and group invariant solutions for case (i) and case (iii) are investigated by us in Secs. 2 and 3, respectively. Section 4 is our conclusions.

2 Symmetry Group Transformation of Wave Equation

In this section, we investigate the wave equation

$$\phi_{xx}+\phi_{yy}-\phi_{tt}=0\,. \tag{2}$$

To Eq. (2), by applying the classical method, we consider the one-parameter group of infinitesimal transformations in (x,y,t,ψ) given by

$$\begin{aligned}
x^* &= x+\epsilon\xi(x,y,t,\psi)+o(\epsilon^2)\,,\\
y^* &= y+\epsilon\eta(x,y,t,\psi)+o(\epsilon^2)\,,\\
t^* &= t+\epsilon\tau(x,y,t,\psi)+o(\epsilon^2)\,,\\
\phi^* &= \phi+\epsilon\Phi(x,y,t,\psi)+o(\epsilon^2)\,,
\end{aligned} \tag{3}$$

where ϵ is group parameter. It is required that the set of Eq. (2) be invariant under the transformation (3), and this yields a system of over determined, linear equations for the infinitesimals ξ, η, τ, and Φ. By solving these equations, one can get

$$\begin{aligned}
\xi &= a_7x-a_4y+a_5t+2a_9xy+2a_{10}xt\\
&\quad +a_8(x^2-y^2+t^2)+a_1\,,\\
\eta &= a_4x+a_7y+a_6t+2a_8xy+2a_{10}yt\\
&\quad +a_9(-x^2-y^2+t^2)+a_2\,,\\
\tau &= a_5x+a_6y+a_7t+2a_8xt+2a_9yt\\
&\quad +a_{10}(x^2-y^2+t^2)+a_3\,,\\
\Phi &= -(a_8x+a_9y+a_{10}t-a_{11})\phi+\alpha(x,y,t)/2\,,
\end{aligned}$$

where a_i $(i=1,2,\ldots,11)$ are arbitrary constants and $\alpha(x,y,t)$ is any solution of Eq. (2) reflecting the linearity of the equation. In Ref. [6], Olver has also reproved the well-known result that the infinitesimal symmetry group of the wave equation is spanned by the ten vector fields

$$\begin{aligned}
&v_1=\frac{\partial}{\partial x},\quad v_2=\frac{\partial}{\partial y},\quad v_3=\frac{\partial}{\partial t},\\
&v_4=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y},\quad v_5=t\frac{\partial}{\partial x}+x\frac{\partial}{\partial t},\\
&v_6=t\frac{\partial}{\partial y}+y\frac{\partial}{\partial t},\quad v_7=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+t\frac{\partial}{\partial t},\\
&v_8=(x^2-y^2+t^2)\frac{\partial}{\partial x}+2xy\frac{\partial}{\partial y}+2xt\frac{\partial}{\partial t}-x\phi\frac{\partial}{\partial \phi},\\
&v_9=2xy\frac{\partial}{\partial x}+(-x^2+y^2+t^2)\frac{\partial}{\partial y}+2yt\frac{\partial}{\partial t}-y\phi\frac{\partial}{\partial \phi},\\
&v_{10}=2xt\frac{\partial}{\partial x}+2yt\frac{\partial}{\partial y}+(x^2+y^2+t^2)\frac{\partial}{\partial t}-t\phi\frac{\partial}{\partial \phi},
\end{aligned} \tag{4}$$

which generate the conformal algebra for $\mathcal{R}^3$ with the given Lorentz metric, and the additional vector fields

$$v_{11}=\phi\frac{\partial}{\partial\phi},\quad v_\alpha=\alpha(x,y,t)\frac{\partial}{\partial\phi}\,. \tag{5}$$

The commutation relations between these vector fields is given by the following Table 1, the entry in row i and the column j representing $[v_i,v_j]=v_iv_j-v_jv_i$:

Table 1 Commutation relations between vector fields.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_α
v_1	0	0	0	v_2	v_3	0	v_1	$2v_7-2v_{11}$	$-v_4$	$2v_5$	0	v_{α_1}
v_2	0	0	0	$-v_1$	0	v_3	v_2	$2v_4$	$2v_7-2v_{11}$	$2v_6$	0	v_{α_2}
v_3	0	0	0	0	v_1	v_2	v_3	v_5	v_6	$2v_7-2v_{11}$	0	v_{α_3}
v_4	$-v_2$	v_1	0	0	$-v_6$	v_5	0	$-v_9$	v_8	0	0	v_{α_4}
v_5	$-v_3$	0	$-v_1$	v_6	0	v_4	0	v_{10}	0	v_8	0	v_{α_5}
v_6	0	$-v_3$	$-v_2$	$-v_5$	$-v_4$	0	0	0	v_{10}	v_9	0	v_{α_6}
v_7	$-v_1$	$-v_2$	$-v_3$	0	0	0	0	v_8	v_9	v_{10}	0	v_{α_7}
v_8	$2v_{11}-2v_7$	$-2v_4$	$-v_5$	v_9	$-v_{10}$	0	$-v_9$	0	0	0	0	v_{α_8}
v_9	v_4	$2v_{11}-2v_7$	$-v_6$	$-v_8$	0	$-v_{10}$	$-v_9$	0	0	0	0	v_{α_9}
v_{10}	$-2v_5$	$-2v_6$	$2v_{11}-2v_7$	0	$-v_8$	$-v_9$	$-v_{10}$	0	0	0	0	$v_{\alpha_{10}}$
v_{11}	0	0	0	0	0	0	0	0	0	0	0	$-v_\alpha$
v_α	$-v_{\alpha_1}$	$-v_{\alpha_2}$	$-v_{\alpha_3}$	$-v_{\alpha_4}$	$-v_{\alpha_5}$	$-v_{\alpha_6}$	$-v_{\alpha_7}$	$-v_{\alpha_8}$	$-v_{\alpha_9}$	$-v_{\alpha_{10}}$	v_α	0

where

$$\begin{aligned}
&\alpha_1 = \alpha_x, \quad \alpha_2 = \alpha_y, \quad \alpha_3 = \alpha_t, \quad \alpha_4 = -y\alpha_x + x\alpha_y,\\
&\alpha_5 = t\alpha_x + x\alpha_t, \quad \alpha_6 = t\alpha_y + y\alpha_t, \quad \alpha_7 = x\alpha_x + y\alpha_y + t\alpha_t,\\
&\alpha_8 = (x^2 - y^2 + t^2)\alpha_x + 2xy\alpha_y + 2xt\alpha_t + 2x\alpha,\\
&\alpha_9 = -2xy\alpha_x + (x^2 - y^2 - t^2)\alpha_y - 2yt\alpha_t - 2y\alpha,\\
&\alpha_{10} = 2xt\alpha_x + 2yt\alpha_y + (x^2 + y^2 + t^2)\alpha_t + 2t\alpha.
\end{aligned}$$

You can conclude that if $\alpha(x,y,t)$ is any solution of the wave equation, so are $\alpha_1, \alpha_2, \ldots, \alpha_{10}$ as given above. Therefore, one can see that the totality of infinitesimal symmetries must be a Lie algebra from the above commutator table.

The one-parameter groups G_i generated by the v_i are given by the following table. The entries give the transformed point $\exp(\epsilon v_i)(x,y,t,\phi) = (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\phi})$:

$$\begin{aligned}
G_1 &: (x,y,t,\phi) \mapsto (x+\epsilon, y, t, \phi),\\
G_2 &: (x,y,t,\phi) \mapsto (x, y+\epsilon, t, \phi),\\
G_3 &: (x,y,t,\phi) \mapsto (x, y, t+\epsilon, \phi),\\
G_4 &: (x,y,t,\phi) \mapsto (x\cos(\epsilon) - y\sin(\epsilon), x\sin(\epsilon) + y\cos(\epsilon), t, \phi),\\
G_5 &: (x,y,t,\phi) \mapsto (x\cosh(\epsilon) + t\sinh(\epsilon), y, x\sinh(\epsilon) + t\cosh(\epsilon), \phi),\\
G_6 &: (x,y,t,\phi) \mapsto (x, y\cosh(\epsilon) + t\sinh(\epsilon), y\sinh(\epsilon) + t\cosh(\epsilon), \phi),\\
G_7 &: (x,y,t,\phi) \mapsto (x\exp(\epsilon), y\exp(\epsilon), t\exp(\epsilon), \phi),\\
G_8 &: (x,y,t,\phi) \mapsto \Big(\frac{x+\epsilon(t^2-x^2-y^2)}{1-2\epsilon x-\epsilon^2(t^2-x^2-y^2)}, \frac{y}{1-2\epsilon x-\epsilon^2(t^2-x^2-y^2)}, \frac{t}{1-2\epsilon x-\epsilon^2(t^2-x^2-y^2)},\\
&\qquad \phi\sqrt{1-2\epsilon x-\epsilon^2(t^2-x^2-y^2)}\Big),\\
G_9 &: (x,y,t,\phi) \mapsto \Big(\frac{x}{1-2\epsilon y-\epsilon^2(t^2-x^2-y^2)}, \frac{y+\epsilon(t^2-x^2-y^2)}{1-2\epsilon y-\epsilon^2(t^2-x^2-y^2)}, \frac{t}{1-2\epsilon y-\epsilon^2(t^2-x^2-y^2)},\\
&\qquad \phi\sqrt{1-2\epsilon y-\epsilon^2(t^2-x^2-y^2)}\Big),\\
G_{10} &: (x,y,t,\phi) \mapsto \Big(\frac{x}{1-2\epsilon t+\epsilon^2(t^2-x^2-y^2)}, \frac{y}{1-2\epsilon t+\epsilon^2(t^2-x^2-y^2)}, \frac{t-\epsilon(t^2-x^2-y^2)}{1-2\epsilon t+\epsilon^2(t^2-x^2-y^2)},\\
&\qquad \phi\sqrt{-1+2\epsilon t-\epsilon^2(t^2-x^2-y^2)}\Big),\\
G_{11} &: (x,y,t,\phi) \mapsto (x, y, t, \phi\exp(\epsilon)),\\
G_\alpha &: (x,y,t,\phi) \mapsto (x, y, t, \phi + \alpha(x,y,t)\epsilon).
\end{aligned}$$

Since each G_i is a symmetry group, the above transformations imply that if $\phi = f(x,y,t)$ is a solution of Eq. (2), so are the functions:

$$\begin{aligned}
\phi^{(1)} &= f(x-\epsilon, y, t),\\
\phi^{(2)} &= f(x, y-\epsilon, t),\\
\phi^{(3)} &= f(x, y, t-\epsilon),\\
\phi^{(4)} &= f(x\cos(\epsilon) + y\sin(\epsilon), -x\sin(\epsilon) + y\cos(\epsilon), t, \phi),\\
\phi^{(5)} &= f(x\cosh(\epsilon) - t\sinh(\epsilon), y, -x\sinh(\epsilon) + t\cosh(\epsilon), \phi),\\
\phi^{(6)} &= f(x, y\cosh(\epsilon) - t\sinh(\epsilon), -y\sinh(\epsilon) + t\cosh(\epsilon), \phi),\\
\phi^{(7)} &= f(x\exp(-\epsilon), y\exp(-\epsilon), t\exp(-\epsilon), \phi),\\
\phi^{(8)} &= f\Big(\frac{x-\epsilon(t^2-x^2-y^2)}{1+2\epsilon x-\epsilon^2(t^2-x^2-y^2)}, \frac{y}{1+2\epsilon x-\epsilon^2(t^2-x^2-y^2)}, \frac{t}{1+2\epsilon x-\epsilon^2(t^2-x^2-y^2)},\\
&\qquad \phi\sqrt{1+2\epsilon x-\epsilon^2(t^2-x^2-y^2)}\Big),
\end{aligned}$$

$$\begin{aligned}
\phi^{(9)} &= f\Big(\frac{x}{1+2\epsilon y-\epsilon^2(t^2-x^2-y^2)}, \frac{y-\epsilon(t^2-x^2-y^2)}{1+2\epsilon y-\epsilon^2(t^2-x^2-y^2)}, \frac{t}{1+2\epsilon y-\epsilon^2(t^2-x^2-y^2)},\\
&\qquad \phi\sqrt{1+2\epsilon y-\epsilon^2(t^2-x^2-y^2)}\Big),\\
\phi^{(10)} &= f\Big(\frac{x}{1+2\epsilon t+\epsilon^2(t^2-x^2-y^2)}, \frac{y}{1+2\epsilon t+\epsilon^2(t^2-x^2-y^2)}, \frac{t+\epsilon(t^2-x^2-y^2)}{1+2\epsilon t+\epsilon^2(t^2-x^2-y^2)},\\
&\qquad \phi\sqrt{-1-2\epsilon t-\epsilon^2(t^2-x^2-y^2)}\Big),\\
\phi^{(11)} &= f(x,y,t,\phi\exp(-\epsilon)), \quad \phi^{(12)} = f(x,y,t,\phi-\epsilon\alpha(x,y,t)).
\end{aligned}$$

The general one-parameter group of symmetries is obtained by considering linear combination $a_1v_1+a_2v_2+a_3v_3+a_4v_4+a_5v_5+a_6v_6+a_7v_7+a_8v_8+a_9v_9+a_{10}v_{10}+a_{11}v_{11}+v_\alpha$ of the given vector fields; the explicit formulae for the transformations are very complicated. Factually, It can be represented uniquely in the form

$$\begin{aligned}
g &= \exp(\epsilon_1 v_1)\exp(\epsilon_2 v_2)\exp(\epsilon_3 v_3)\exp(\epsilon_4 v_4)\exp(\epsilon_5 v_5)\exp(\epsilon_6 v_6)\\
&\quad \times \exp(\epsilon_7 v_7)\exp(\epsilon_8 v_8)\exp(\epsilon_9 v_9)\exp(\epsilon_{10} v_{10})\exp(\epsilon_{11} v_{11})\exp(v_\alpha)\,.
\end{aligned}\tag{6}$$

Thus the most general solution obtainable from a given solution $\Psi(x,y,t)$ by group transformations (6) is in the form (for simplicity, one can do it by computer algebra):

$$\begin{aligned}
\phi &= \frac{c_{12}\beta(x,y,t)}{c_{11}} + \frac{\Psi(\xi,\eta,\tau)}{c_{11}\sqrt{1-2c_8x-2c_9y-2c_{10}t-({c_8}^2+{c_9}^2-{c_{10}}^2)}},\\
\xi &= \{[c_1({c_8}^2+{c_9}^2-{c_{10}}^2)+c_4c_7(d_6c_9-c_6c_{10})+d_4c_7(c_5d_6c_{10}-c_5c_6c_9-d_5c_8)](x^2+y^2-t^2)\\
&\quad +(c_4d_5c_7-2c_2c_8)x+(c_4c_5c_6c_7+d_4d_6c_7-2c_2c_9)y\\
&\quad +(d_4c_5d_6c_7-c_4c_6c_7-2c_1c_{10})t+c_1\}/(1-2c_8x-2c_9y-2c_{10}t-({c_8}^2+{c_9}^2-{c_{10}}^2)),\\
\eta &= \{[c_2({c_8}^2+{c_9}^2-{c_{10}}^2)+d_4c_7(c_6c_{10}-d_6c_9)+c_4c_7(c_5d_6c_{10}-c_5c_6c_9-d_5c_8)](x^2+y^2-t^2)\\
&\quad +(d_4d_5c_7-2c_1c_8)x+(d_4c_5c_6c_7-c_4d_6c_7-2c_1c_9)y\\
&\quad +(c_4c_5d_6c_7-d_4c_6c_7-2c_2c_{10})t+c_2\}/(1-2c_8x-2c_9y-2c_{10}t-({c_8}^2+{c_9}^2-{c_{10}}^2)),\\
\tau &= \{[c_3({c_8}^2+{c_9}^2-{c_{10}}^2)+c_7(d_5d_6c_{10}-d_5c_6c_9-c_5c_8)](x^2+y^2-t^2)+(c_5c_7-2c_3c_8)x+(d_5c_6c_7-2c_3c_9)y\\
&\quad +(d_5d_6c_7-2c_3c_{10})t+c_3\}/(1-2c_8x-2c_9y-2c_{10}t-({c_8}^2+{c_9}^2-{c_{10}}^2)),
\end{aligned}$$

where $\beta(x,\,y,\,t)$ is any solution of Eq. (2), $c_1,\,c_2,\,\ldots,c_{11}$ are arbitrary constants and there are $d_4=\sqrt{1-{c_4}^2}$, $d_5=\sqrt{1+{c_5}^2}$, $d_6=\sqrt{1+{c_6}^2}$.

3 Reductions and Symmetry Group Transformation for case (iii)

3.1 *An Optimal System and Reductions by One-Dimensional Subalgebras*

Case (iii) reads

$$\phi_{xx}+\phi_{xx}-\phi_{tt}=\exp(\phi)\,. \tag{7}$$

Using the same standard point Lie group approach in Sec. 2, we should find all the possible invariances of Eq. (7) under the transformation

$$(x,y,t,\phi)\longrightarrow(x,y,t,\phi)+\epsilon(X,Y,T,\Phi)\,, \tag{8}$$

where X, Y, T and Φ are functions of x, y, t, and ϕ, respectively, and ϵ is an infinitesimal parameter. Substituting Eq. (8) into Eq. (7), expanding it to the first order of ϵ and removing the coefficients of different derivatives of function ϕ, one can get the results as follows:

$$\begin{aligned}
X &= c_2x-c_5y+c_1t+c_7,\\
Y &= c_5x+c_2y+c_4t+c_6,\\
T &= c_1x+c_4y+c_2t+c_3, \quad \Phi=-2c_2,
\end{aligned}$$

with arbitrary constants $c_1,\,c_2,\,\ldots,\,c_7$. Hence the corresponding vector field can be written as

$$\begin{aligned}
v &= (c_2x-c_5y+c_1t+c_7)\frac{\partial}{\partial x}+(c_5x+c_2y+c_4t+c_6)\frac{\partial}{\partial y}\\
&\quad +(c_1x+c_4y+c_2t+c_3)\frac{\partial}{\partial t}-2c_2\frac{\partial}{\partial\phi}.
\end{aligned}$$

Being different from including six arbitrary constants in case (ii), there are seven constants in the vector field of case (iii). Therefore, we can say that the symmetry algebra of Eq. (7) is generated by the seven vector fields:

$$\begin{aligned}
&v_1 = t\frac{\partial}{\partial x}+x\frac{\partial}{\partial t}, \quad v_2 = x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+t\frac{\partial}{\partial t}-2\frac{\partial}{\partial\phi},\\
&v_3 = \frac{\partial}{\partial t}, \quad v_4 = t\frac{\partial}{\partial y}+y\frac{\partial}{\partial t},\\
&v_5 = -y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}, \quad v_6=\frac{\partial}{\partial y}, \quad v_7=\frac{\partial}{\partial x}.
\end{aligned}$$

The commutator relation $[v_i, v_j] = v_i v_j - v_j v_i$ is as in Table 2.

Table 2 Commutation relations.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	0	0	$-v_7$	v_5	v_4	0	$-v_3$
v_2	0	0	$-v_3$	0	0	$-v_6$	$-v_7$
v_3	v_7	v_3	0	v_6	0	0	0
v_4	$-v_5$	0	$-v_6$	0	$-v_1$	$-v_3$	0
v_5	$-v_4$	0	0	v_1	0	v_7	$-v_6$
v_6	0	v_6	0	v_3	$-v_7$	0	0
v_7	v_3	v_7	0	0	v_6	0	0

About the optimal systems, a lot of excellent work has done by many famous experts.[5−6,10−12] Some examples of optimal systems can also be found in Ibragimov.[13] Several methods have been developed to construct optimal systems. Here we will use Olver's method, which only depends on fragments of the theory of Lie algebras. In Ref. [6], it is said that the problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras and so we concentrate on the latter.

The Lie algebra spanned by $v_1, v_2, \ldots, v_7$ generates the symmetry group of Eq. (7). To compute the adjoint representation, we use the Lie series in conjunction with the above commutator table. For instance

$$\mathrm{Ad}(\exp(\epsilon v_1))v_2 = v_2 - \epsilon[v_1, v_2] + \frac{1}{2!}[v_1, [v_1, v_2]] - \cdots$$

In this manner, we construct Table 3:

Table 3 Adjoint representation.

Ad	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	v_1	v_2	$\cosh(\epsilon)v_3 + \sinh(\epsilon)v_7$	$\cosh(\epsilon)v_4 - \sinh(\epsilon)v_5$	$\cosh(\epsilon)v_5 - \sinh(\epsilon)v_4$	v_6	$\cosh(\epsilon)v_7 + \sinh(\epsilon)v_3$
v_2	v_1	v_2	$\exp(\epsilon)v_3$	v_4	v_5	$\exp(\epsilon)v_6$	$\exp(\epsilon)v_7$
v_3	$v_1 - \epsilon v_7$	$v_2 - \epsilon v_3$	v_3	$v_4 - \epsilon v_6$	v_5	v_6	v_7
v_4	$\cosh(\epsilon)v_1 + \sinh(\epsilon)v_5$	v_2	$\cosh(\epsilon)v_3 + \sinh(\epsilon)v_6$	v_4	$\cosh(\epsilon)v_5 + \sinh(\epsilon)v_1$	$\cosh(\epsilon)v_6 + \sinh(\epsilon)v_3$	v_7
v_5	$\cosh(\epsilon)v_1 + \sinh(\epsilon)v_4$	v_2	v_3	$\cosh(\epsilon)v_4 - \sinh(\epsilon)v_1$	v_5	$\cosh(\epsilon)v_6 - \sinh(\epsilon)v_7$	$\cosh(\epsilon)v_7 + \sinh(\epsilon)v_6$
v_6	v_1	$v_2 - \epsilon v_6$	v_3	$v_4 - \epsilon v_3$	$v_5 + \epsilon v_7$	v_6	v_7
v_7	$v_1 - \epsilon v_3$	$v_2 - \epsilon v_7$	v_3	v_4	$v_5 - \epsilon v_6$	v_6	v_7

with the (i, j)-th entry indicating $\mathrm{Ad}(\exp(\epsilon v_i))v_j$. Following Ovsiannikov,[5] one calls two subalgebras v_2 and v_1 of a given Lie algebra equivalent if one can fine an element g in the Lie group so that $\mathrm{Ad}g(v_1) = v_2$, where $\mathrm{Ad}g$ is the adjoint representation of g on v. Given a nonzero vector

$$v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 + a_6v_6 + a_7v_7,$$

our task is to simplify as many of the coefficients a_i as possible though judicious applications of adjoint maps to v. In this way, omitting the detailed complex computation, we obtain an optical system $\mathcal{S}$ of the Lie algebra for Eq. (7):

$$\begin{aligned}
&v_1, v_3, v_4, v_5, v_6, v_7, v_1 + v_5, v_3 + v_6, v_3 + v_7, v_4 + v_7, v_4 + a_5v_5 \quad (a_5 \neq 0), \quad v_1 + a_4v_4,\\
&v_2 + a_4v_4 \quad (a_4 \neq 0), \quad v_2 + a_1v_1 \quad (a_1 \neq 0), \quad v_2 + a_5v_5 \quad (a_5 \neq 0), \quad v_2 + a_1(v_1 + v_5) \quad (a_1 \neq 0),\\
&v_2 + a_4(v_4 + v_5), \quad v_3 + v_4 + a_5v_5 \quad (a_5 \neq 0), \quad a_1v_1 + v_3 + a_5v_5 \quad (a_5 \neq 0), \quad v_1 + a_4v_4 + v_6,\\
&v_2 + a_4v_4 + \sqrt{{a_4}^2 - 1}v_5 + v_7, \quad \sqrt{1 - {a_4}^2}v_1 + v_2 + a_4v_4 + v_7, \quad a_1v_1 + v_2 + a_4v_4 + v_6 \quad (a_4 \neq 0, {a_1}^2 \neq 1),\\
&a_1v_1 + v_2 + a_5v_5 + v_6 \quad (a_5 \neq 0, {a_1}^2 \neq 1), \quad v_1 + v_2 + v_4 + v_5, \quad v_1 + v_2 + v_4 + v_5 + v_6.
\end{aligned}$$

Then we will do the one-parameter symmetry reductions for the most elements in the optical system $\mathcal{S}$ by solving the characteristic equations. For some elements in $\mathcal{S}$, it is still more difficult and tedious to solve their characteristic equations even by the computer algebra, i.e. *Maple* or *Mathematica*. In that cases, we can not deduce their reduced equations. This need our further study to investigate them. We give out the results we obtained in Table 4:

Table 4 Results of characteristic equations.

Generators	Invariant patameters	Invariant solutions	Reduced equations
v_1	$\xi = x^2 - t^2, \eta = y$	$\phi = \varphi(\xi, \eta)$	$4\xi\varphi_{\xi\xi} + \varphi_{\eta\eta} + 4\varphi_\xi - \exp(\varphi) = 0$
v_3	$\xi = x, \eta = y$	$\phi = \varphi(x, y)$	$\varphi_{xx} + \varphi_{yy} - \exp(\varphi) = 0$
v_4	$\xi = y^2 - t^2, \eta = x$	$\phi = \varphi(\xi, \eta)$	$4\xi\varphi_{\xi\xi} + \varphi_{\eta\eta} + 4\varphi_\xi - \exp(\varphi) = 0$
v_5	$\xi = x^2 + y^2, \eta = t$	$\phi = \varphi(\xi, \eta)$	$4\xi\varphi_{\xi\xi} - \varphi_{\eta\eta} + 4\varphi_\xi - \exp(\varphi) = 0$
v_6	$\xi = x, \eta = t$	$\phi = \varphi(x, t)$	$\varphi_{xx} - \varphi_{tt} - \exp(\varphi) = 0$
v_7	$\xi = y, \eta = t$	$\phi = \varphi(y, t)$	$\varphi_{yy} - \varphi_{tt} - \exp(\varphi) = 0$

$v_1 + v_5$	$\xi = t - y, \eta = x^2 + 2y^2 - 2yt$	$\phi = \varphi(\xi, \eta)$	$4(\xi^2 + \eta)\varphi_{\eta\eta} + 4\xi\varphi_{\xi\eta} + 6\varphi_\eta - \exp(\varphi) = 0$
$v_3 + v_6$	$\xi = y - t, \eta = x$	$\phi = \varphi(\xi, \eta)$	$\varphi_{\eta\eta} - \exp(\varphi) = 0$
$v_3 + v_7$	$\xi = x - t, \eta = y$	$\phi = \varphi(\xi, \eta)$	$\varphi_{\eta\eta} - \exp(\varphi) = 0$
$v_4 + v_7$	$\xi = \frac{y}{x+1}, \eta = \frac{t}{x+1}$	$\phi = -2\ln(x+1)$ $+\varphi(\xi, \eta)$	$(\xi^2 + 1)\varphi_{\xi\xi} + (\eta^2 - 1)\varphi_{\eta\eta} + 2\xi\eta\varphi_{\xi\eta} + 2\xi\varphi_\xi + 2\eta\varphi_\eta + 2 - \exp(\varphi) = 0$
$v_4 + a_5 v_5 (a_5 \neq 0)$	$\xi = (t + x)y^{-1-a_5}, \eta = (t - x)y^{-1+a_5}$	$\phi = -2\ln(y) + \varphi(\xi, \eta)$	$\xi^2(a_5 + 1)^2\varphi_{\xi\xi} + \eta^2(a_5 - 1)^2\varphi_{\eta\eta} + 2[(1 - a_5^2)\xi\eta + 2]\varphi_{\xi\eta} + (a_5 + 1)(a_5 + 2)\eta\varphi_\xi + (a_5 - 1)(a_5 - 2)\xi\varphi_\eta + 2 - \exp(\varphi) = 0$
$v_1 + a_4 v_4$	$\xi = (a_4^2 - 1)x^2 + t^2 - 2a_4 xy, \eta = y - a_4 x$	$\phi = \varphi(\xi, \eta)$	$4(a_4^2\eta^2 - \xi)\varphi_{\xi\xi} + (a_4^2 + 1)\varphi_{\eta\eta} + 4a_4^2\eta\varphi_{\xi\eta} + 2(a_4^2 -)\varphi_\xi - \exp(\varphi) = 0$
$v_2 + a_4 v_4 (a_4 \neq 0)$	$\xi = (y + t)x^{-1-a_4}, \eta = (y - t)x^{-1+a_4}$	$\phi = -a\ln(x) + \varphi(\xi, \eta)$	$\xi^2(a_4 + 1)^2\varphi_{\xi\xi} + \eta^2(a_4 - 1)^2\varphi_{\eta\eta} + 2[(1 - a_4^2)\xi\eta + 2]\varphi_{\xi\eta} + (a_4 + 1)(a_4 + 2)\xi\varphi_\xi + (a_4 - 1)(a_4 - 2)\eta\varphi_\eta + 2 - \exp(\varphi) = 0$
$v_2 + a_1 v_1 (a_1 \neq 0)$	$\xi = (x + t)y^{-1-a_1}$, $\eta = (-t + x)y^{-1+a_1}$	$\phi = -2\ln(y) + \varphi(\xi, \eta)$	$\xi^2(a_1 + 1)^2\varphi_{\xi\xi} + \eta^2(a_1 - 1)^2\varphi_{\eta\eta} + 2[(1 - a_1^2)\xi\eta + 2]\varphi_{\xi\eta} + (a_1 + 1)(a_1 + 2)\xi\varphi_\xi + (a_1 - 1)(a_1 - 2)\eta\varphi_\eta + 2 - \exp(\varphi) = 0$
$v_2 + a_5 v_5 (a_5 \neq 0)$	$\xi = \frac{\cos(a_5\ln(t))x + \sin(a_5\ln(t))y}{t}$, $\eta = \frac{\cos(a_5\ln(t))y - \sin(a_5\ln(t))x}{t}$	$\phi = -2\ln(t) + \varphi(\xi\eta)$	$[1 - (\xi - a_5\eta)^2]\varphi_{\xi\xi} + [1 - (a_5\xi + \eta)^2]\varphi_{\eta\eta} + 2[a_5(\eta^2 - \xi^2) + (a_5^2 - 1)\xi\eta]\varphi_{\xi\eta} + [(a_5^2 - 2)\xi + 3a_5\eta]\varphi_\xi + [(a_5^2 - 2)\eta - 3a_5\xi]\varphi_\eta - 2 - \exp(\varphi) = 0$
$v_3 + v_4 + a_5 v_5$ $(a_5 \neq 0)$	$\xi = a_5^2 x^2 + (a_5^2 - 1)y^2 + t^2 + 2a_5 xt - 2y$, $\eta = \sqrt{a_5^2 - 1}x + a_5\sqrt{a_5^2 - 1}t$ $+a_5 \arctan\left(\frac{\sqrt{a_5^2 - 1}(t + a_5 x)}{(a_5^2 - 1)y - 1}\right)$	$\phi = \varphi(\xi, \eta)$	$4((a_5^2 - 1)\xi + 1)^2\varphi_{\xi\xi} - (a_5^2 - 1)^3(\xi - 1)\varphi_{\eta\eta}$ $+4(a_5^2 - 1)[(a_5^2 - 1)\xi + 1]\varphi_\xi + [(a_5^2 - 1)\xi$ $+1]\exp(\varphi) = 0$
$a_1 v_1 + v_3 + a_5 v_5$ $(a_5 \neq 0)$	$\xi = (a_5^2 - a_1^2)x^2 + a_5^2 y^2 + a_1^2 t^2$ $-2a_1 x - 2a_1 a_5 yt$, $\eta = [(a_1^2 - a_5^2)x + a_1 - \sqrt{(}a_1^2 - a_5^2)$ $(a_5 y - a_1 t)]\exp\left(\frac{\sqrt{a_1^2 - a_5^2}(a_1 y - a_5 t)}{a_5}\right)$	$\phi = \varphi(\xi, \eta)$	$4a_1^2 a_5^2\varphi_{\xi\xi} + (a_1^2 - a_5^2)\eta^2\varphi_{\eta\eta} - 4(a_1^2 - a_5^2)$ $a_5^2\eta\varphi_{\xi\eta} + 4a_5^4(1 - a_1^2)\varphi_\xi + (a_1^2 + a_5^2)^2$ $\eta\varphi_\eta - a_5^2\exp(\varphi) = 0$
$v_1 + v_6 + a_4 v_4$	$\xi = x^2 + a_4^2 y^2 - (a_4^2 + 1)t^2 - 2a_4 t + 2a_4 xy, \quad \eta = [(x + a_4 y)\sqrt{1 + a_4^2} + (1 + a_4^2)t + a_4]\exp((a_4 x - y)\sqrt{1 + a_4^2})$	$\phi = \varphi(\xi, \eta)$	$4[(1 + a_4^2)\xi - a_4^2]\varphi_{\xi\xi} + (1 + a_4^2)^2\eta^2\varphi_{\eta\eta} + 4(1 + a_4^2)\eta\varphi_{\xi\eta} + 4(1 + a_4^2)\varphi_\xi + (1 + a_4^2)^2\eta\varphi_\eta - \exp(\varphi) = 0$
$v_1 + v_2 + v_4 + v_5$	$\xi = t - y, \eta = \frac{x + t}{(x + t - y)^2}$	$\phi = -\ln(y)$ $-2\text{arctanh}\left(\frac{x + t + y}{x + t - y}\right)$ $+\varphi(\xi, \eta)$	$4\eta^3\varphi_{\eta\eta} - 2\eta\varphi_{\xi\eta} + 6\eta^2\varphi_\eta + \exp(\varphi) = 0$
$v_1 + v_2 + v_4 + v_5 + v_6$	$\xi = \frac{x + t}{(x - y + t - 1)^2}$, $\eta = \frac{x^2 + y^2 - t^2 + 2y + 1 - (x + t)\ln(x + t)}{x + t}$	$\phi = -\ln(x + t) + \varphi(\xi, \eta)$	$2\xi^3\varphi_{\xi\xi} - 2\varphi_{\eta\eta} - 2\xi\varphi_{\xi\eta} - 3\xi^2\varphi_\xi + \varphi_\eta$ $-\frac{1}{2}\exp(\varphi) = 0$

3.2 *Group Transformation*

In [2], Clarkson and Kruskal (CK) introduced a direct method to derive symmetry reductions of a nonlinear system without using any group theory. For many types of nonlinear systems, the method can be used to find all the possible similarity reductions. In Ref. [4], Lou and Ma modified CK's direct method to find out the generalized Lie and non-Lie symmetry groups of differential equations by an ansatz reading

$$u(x, y, t) = \alpha(x, y, t) + \beta(x, y, t)U(\xi, \eta, \tau)\,, \tag{9}$$

where ξ, η, τ are all functions of x, y, t, Eq. (9) also points that if $U(x, y, t)$ is a solution of the original differential equation, so is $u(x, y, t)$. The most general solution obtainable from a given solution can also be construed by the general group transformation.

The one-parameter groups G_i generated by v_i are given in the following table. The entries give the transformed point $\exp(\epsilon V_i)(x, y, t, \phi) = (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\phi})$:

$$\begin{aligned}
G_1 &: (x, y, t, \phi) \mapsto (x\cosh(\epsilon) + t\sinh(\epsilon), y, x\sinh(\epsilon) \\
&\qquad + t\cosh(\epsilon), \phi), \\
G_2 &: (x, y, t, \phi) \mapsto (x\exp(\epsilon), y\exp(\epsilon), t\exp(\epsilon), \phi - 2\epsilon), \\
G_3 &: (x, y, t, \phi) \mapsto (x, y, t + \epsilon, \phi), \\
G_4 &: (x, y, t, \phi) \mapsto (x, y\cosh(\epsilon) + t\sinh(\epsilon), y\sinh(\epsilon) \\
&\qquad + t\cosh(\epsilon), \phi), \\
G_5 &: (x, y, t, \phi) \mapsto (x\cos(\epsilon) - y\sin(\epsilon), x\sin(\epsilon) \\
&\qquad + y\cos(\epsilon), t, \phi), \\
G_6 &: (x, y, t, \phi) \mapsto (x, y + \epsilon, t, \phi), \\
G_7 &: (x, y, t, \phi) \mapsto (x + \epsilon, y, t, \phi)\,.
\end{aligned} \tag{10}$$

Since each group G_i is a symmetry group, transformations (10) imply that if $\phi = f(x, y, t)$ is a solution of Eq. (7), so are functions

$$\begin{aligned}
&\phi^{(1)} = f(x\cosh(\epsilon) - t\sinh(\epsilon), y, -x\sinh(\epsilon) + t\cosh(\epsilon)),\\
&\phi^{(2)} = f(x\exp(-\epsilon), y\exp(-\epsilon), t\exp(-\epsilon)) + 2\epsilon,\\
&\phi^{(3)} = f(x, y, t-\epsilon),\\
&\phi^{(4)} = f(x, y\cosh(\epsilon) - t\sinh(\epsilon), -y\sinh(\epsilon) + t\cosh(\epsilon)),\\
&\phi^{(5)} = f(x\cos(\epsilon) + y\sin(\epsilon), -x\sin(\epsilon) + y\cos(\epsilon), t),\\
&\phi^{(6)} = f(x, y-\epsilon, t),\\
&\phi^{(7)} = f(x-\epsilon, y, t).
\end{aligned}$$

The general one-parameter group of symmetries is obtained by considering linear combination $a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 + a_6v_6 + a_7v_7$ of the given vector fields; the explicit formulae for the transformations are very complicated. Factually, It can be represented uniquely in the form

$$\begin{aligned}
g = {}&\exp(\epsilon_1 v_1)\exp(\epsilon_2 v_2)\exp(\epsilon_3 v_3)\exp(\epsilon_4 v_4)\\
&\times\exp(\epsilon_5 v_5)\exp(\epsilon_6 v_6)\exp(\epsilon_7 v_7)\,.
\end{aligned}\tag{11}$$

Thus the most general solution obtainable from a given solution $\Phi(x, y, t)$ by group transformations (11) is in the form (for simplicity, one can do it by computer algebra):

$$\begin{aligned}
\phi &= 2a_2 + \Phi(X, Y, T),\\
X &= a_2[(x + a_7)(b_1b_5 + a_1a_4a_5)\\
&\quad - (y + a_6)(a_5b_1 - a_1a_4b_5) + (tb_4 + a_3)a_1],\\
Y &= a_2[(x + a_7)a_5b_4 + (y + a_6)b_4b_5 + ta_4],\\
T &= a_2[(x + a_7)(a_1b_5 + b_1a_4a_5)\\
&\quad - (y + a_6)(a_5a_1 - b_1a_4b_5) + (tb_4 + a_3)b_1],
\end{aligned}$$

where $a_1, a_2, \ldots, a_7$ are arbitrary constants and there are $b_1 = \sqrt{1 + a_1^2}$, $b_4 = \sqrt{1 + a_4^2}$, $b_5 = \sqrt{1 - a_5^2}$.

4 Conclusions

In summary, in this paper we investigate two types of the (2+1)-dimensional nonlinear Klein–Gordon field equations including $F(\phi) = 0$ called wave equation and $F(\phi) = \exp(\phi)$.

For the wave equation, we give out its symmetry group analysis in detail. One can see that its symmetry group is more complicate than that of case (iii) for which we can not obtain the general symmetry reduction even by compute algebra. The one-parameter optimal system of group invariant solutions is also very difficult to construct. This two questions need our further study in the future.

For case (iii), its seven-dimensional symmetry algebra obtained by us in use of the classic Lie symmetry method is a little different from the six-dimensional symmetry algebra of case (ii), however, for the general reductions of this equation, it is very complicated to operate even by the computer algebra, such as *Maple*. For that we firstly construct a one-parameter optimal system of the seven generators, then give out mostly the corresponding symmetry reductions after the complex and tedious computation. In this way, the original (2+1)-dimensional Klein–Gordon equation is reduced to some (1+1)-dimensional differential equations. Lastly, The general solution of case (iii) from a given solution by the symmetry group transformation is obtianed.

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