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The function cascade synchronization scheme for discrete-time hyperchaotic systems

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ABSTRACT

In this paper, the function cascade synchronization scheme is proposed to investigate the discrete-time hyperchaotic systems. By choosing some different error functions and with the aid of symbolic–numeric computation, the proposed scheme is applied to achieve the function cascade synchronization for two discrete-time hyperchaotic systems: the generalized Hénon map and the discrete-time Rössler system, respectively. Numerical simulations are used to verify the effectiveness and feasibility of the proposed technique.

1. Introduction

Since the pioneering works of Pecora and Carroll [1], Ott et al. [2], chaos control and synchronization have attracted extensive attention from various areas [1–6]. Recently, great efforts have been devoted to realize chaos synchronization for two identical and different chaotic systems because of their great potential applications in secure communication, chemical and biological systems, telecommunications, system identification, etc. At the same time, many powerful methods have been proposed, such as OGY method [2], backstepping design technique [7–9], linear and nonlinear feedback approaches [10–13], adaptive control approach [4,14–16], active control approach [17,18], cascade synchronization method [19,20] as well as the function cascade synchronization method [21,22], etc., which are continuous-time synchronization schemes. In fact, many mathematical models of neural networks, biological process, physical process and chemical process, etc., were defined using discrete-time dynamical systems [23–25].Thus more and more attention has been paid to the chaos (hyperchaos) synchronization in discrete-time dynamical systems [23–32].

In recent time, hyperchaotic systems have become the hot topic and many works have been done for hyperchaotic synchronization [21,22,29–34]. A hyperchaotic system is defined as a chaotic system with at least two positive exponents, implying that its dynamics are expanded in several different directions simultaneously. It means that the hyperchaotic system has more complex dynamical behaviors, which can be used to improve the security of chaotic communication systems. Therefore, they are more suitable for some special engineering applications such as chaos-based encryption and secure communication.

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In this paper, combined of function cascade synchronization scheme [21,22] and Q–S synchronization method [30,31], the function cascade synchronization scheme for the continuous-time chaotic systems is successfully extended to the discrete-time systems. By constructing function scalar controllers and choosing some different error functions, we can easily realize the function cascade synchronization for the discrete-time hyperchaotic systems with the aid of symbolic–numeric computation. In order to verify its effectiveness of the proposed method, we apply it to two discrete-time hyperchaotic systems: the generalized Hénon map [23] and the discrete-time Rössler system [25], respectively.

It is organized as follows: In Section 2, the function cascade synchronization method for the discrete-time hyperchaotic systems is introduced. In Section 3, the proposed scheme is applied to achieve the function cascade synchronization for two discrete-time hyperchaotic systems and numerical simulations are used to verify its effectiveness. In Section 4, conclusions are followed.

2. The function cascade synchronization scheme for discrete-time hyperchaotic systems

The function cascade synchronization for the continuous-time chaotic system can be seen in Refs. [21,22]. Combined of the Q–S synchronization method [30,31], we extend the function cascade synchronization method to the 3D discrete-time dynamical systems. It is defined as the following.

Consider the hyperchaotic system:

$$\begin{cases} x_1(k+1) = f(x_1(k), x_2(k), x_3(k)), \\ x_2(k+1) = g(x_1(k), x_2(k), x_3(k)), \\ x_3(k+1) = h(x_1(k), x_2(k), x_3(k)). \end{cases} \tag{1}$$

Firstly, copy the first two equations of (1) and we get a sub-response system:

$$\begin{cases} X_1(k+1) = f(X_1(k), X_2'(k), x_3(k)) + u_1(k), \\ X_2'(k+1) = g(X_1(k), X_2'(k), x_3(k)) + u_2(k), \end{cases} \tag{2}$$

where variable x_3 is a corresponding signal afforded by the original system, $u_1(k)$ and $u_2(k)$ are the desired scalar controllers that make the drive system (1) synchronize with the sub-response system (2) by choosing suitable scaling function factors.

Now, we define the error functions $e_1(k) = X_1(k) - Q_1(x_1(k))x_1(k)$, $e_2(k) = X_2'(k) - Q_2(x_2(k))x_2(k)$. The error dynamical system between the drive system (1) and response system (2) is

$$\begin{cases} e_1(k+1) = X_1(k+1) - Q_1(x_1(k+1))x_1(k+1) \\ \qquad\qquad\quad = f(X_1(k), X_2'(k), x_3(k)) - Q_1(f(x_1(k), x_2(k), x_3(k)))f(x_1(k), x_2(k), x_3(k)) + u_1(k), \\ e_2(k+1) = X_2'(k+1) - Q_2(x_1(k+1))x_2(k+1) \\ \qquad\qquad\quad = g(X_1(k), X_2'(k), x_3(k)) - Q_2(g(x_1(k), x_2(k), x_3(k)))g(x_1(k), x_2(k), x_3(k)) + u_2(k). \end{cases} \tag{3}$$

We choose the scalar controllers $u_1(k)$ and $u_2(k)$ as

$$\begin{cases} u_1(k) = c_{11}e_1(k) + c_{12}e_2(k) - f(X_1(k), X_2'(k), x_3(k)) + Q_1(f(x_1(k), x_2(k), x_3(k)))f(x_1(k), x_2(k), x_3(k)), \\ u_2(k) = c_{21}e_1(k) + c_{22}e_2(k) - g(X_1(k), X_2'(k), x_3(k)) + Q_2(g(x_1(k), x_2(k), x_3(k)))g(x_1(k), x_2(k), x_3(k)). \end{cases} \tag{4}$$

Substituting (4) into (3), we get the error system

$$\begin{pmatrix} e_1(k+1) \\ e_2(k+1) \end{pmatrix} = C \begin{pmatrix} e_1(k) \\ e_2(k) \end{pmatrix}, \quad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}. \tag{5}$$

In fact, it is easy to prove that for any given parameters c_{ij} $(i,j = 1,2)$ in (5), there always exists at least one family of parameters c_{ij} such that all eigenvalues of matrix C are in the unit disc for the origin, which indicates that the error system (5) is globally asymptotically stable. For example, the simplest case is

$$0 < c_{ii} < 1, \quad c_{ij} = 0 \quad (i \neq j \text{ and } i,j = 1,2). \tag{6}$$

Take the Lyapunov function as

$$L_1(k) = |e_1(k)| + c_1|e_2(k)|, \quad c_1 > 0. \tag{7}$$

So we have the derivative of $L_1(k)$

$$\begin{aligned} \Delta L_1(k) &= L_1(k+1) - L_1(k) = |e_1(k+1)| + c_1|e_2(k+1)| - |e_1(k)| - c_1|e_2(k)| \\ &\leqslant [|c_{11}| + c_1|c_{21}| - 1]|e_1(k)| + [|c_{12}| + c_1|c_{22}| - c_1]|e_2(k)|. \end{aligned} \tag{8}$$

From (8), we know if the parameters c_1, c_{ij} $(i,j = 1,2)$ satisfy

$$|c_{11}| + c_1|c_{21}| < 1, \quad |c_{12}| + c_1|c_{22}| < c_1, \tag{9}$$

then $\Delta L_1(k)$ is negative definite, denoting that the system is globally asymptotically stable and

$$\begin{cases} \lim\limits_{k\to+\infty} \|e_1(k)\| = \lim\limits_{k\to+\infty} \|X_1(k) - Q_1(x_1(k))x_1(k)\| = 0, \\ \lim\limits_{k\to+\infty} \|e_2(k)\| = \lim\limits_{k\to+\infty} \|X_2'(k) - Q_2(x_2(k))x_2(k)\| = 0. \end{cases} \tag{10}$$

That is to say the synchronization is achieved between the discrete-time hyperchaotic systems (1) and (2) with the scalar controller $(u_1(k), u_2(k))$ given in (4).

Secondly, copy another response system:

$$\begin{cases} X_2(k+1) = g(X_1(k), X_2(k), X_3(k)) + u_3(k), \\ X_3(k+1) = h(X_1(k), X_2(k), X_3(k)) + u_4(k), \end{cases} \tag{11}$$

where $X_1(k)$ is the drive variable corresponding to system (2) and $u_3(k),\ u_4(k)$ are the scalar controllers determined later.

In order to achieve the discrete-time synchronization between systems (1) and (11), we make the same analysis above. Defining the necessary error functions $e_3(k) = X_2(k) - Q_3(x_2(k))x_2(k),\ e_4(k) = X_3(k) - Q_4(k)x_3(k)$. The corresponding Lyapunov function is chosen as

$$L_2(k) = |e_3(k)| + c_2|e_4(k)|, \quad c_2 > 0. \tag{12}$$

Here we set

$$\begin{cases} u_3(k) = c_{31}e_3(k) + c_{32}e_4(k) - g(X_1(k), X_2(k), X_3(k)) + Q_3(g(x_1(k), x_2(k), x_3(k)))g(x_1(k), x_2(k), x_3(k)), \\ u_4(k) = c_{41}e_3(k) + c_{42}e_4(k) - h(X_1(k), X_2(k), X_3(k)) + Q_4(h(x_1(k), x_2(k), x_3(k)))h(x_1(k), x_2(k), x_3(k)), \end{cases} \tag{13}$$

then the derivative of Lyapunov function is

$$\begin{aligned} \Delta L_2(k) &= L_2(k+1) - L_2(k) = |e_3(k+1)| + c_2|e_4(k+1)| - |e_3(k)| - c_2|e_4(k)| \\ &\leqslant [|c_{31}| + c_2|c_{42}| - 1]|e_3(k)| + [|c_{32}| + c_2|c_{42}| - c_2]|e_4(k)|. \end{aligned} \tag{14}$$

When the parameters $c_2, c_{31}, c_{32}, c_{41}, c_{42}$ satisfy

$$|c_{31}| + c_2|c_{42}| < 1, \quad |c_{32}| + c_2|c_{42}| < c_2, \tag{15}$$

$\Delta L_2(k)$ is negative and the error functions $e_3(k), e_4(k)$ approach to zero. That is to say the synchronization is achieved between systems (1) and (11).

From the above, we know that the discrete-time hyperchaotic system (1) synchronize with the systems (2) and (11) with the controllers chosen in (4) and (13), i.e.

$$\begin{aligned} &\lim_{k\to+\infty} \|X_1(k) - Q_1(x_1(k))x_1(k)\| = 0, \quad \lim_{k\to+\infty} \|X_2'(k) - Q_2(x_2(k))x_2(k)\| = 0, \\ &\lim_{k\to+\infty} \|X_2(k) - Q_3(x_2(k))x_2(k)\| = 0, \quad \lim_{k\to+\infty} \|X_3(k) - Q_4(x_3(k))x_3(k)\| = 0. \end{aligned}$$

So we can get

$$\begin{cases} \lim\limits_{k\to+\infty} \|X_1(k) - Q_1(x_1(k))x_1(k)\| = 0, \\ \lim\limits_{k\to+\infty} \|X_2(k) - Q_3(x_2(k))x_2(k)\| = 0, \\ \lim\limits_{k\to+\infty} \|X_3(k) - Q_4(x_3(k))x_3(k)\| = 0. \end{cases} \tag{16}$$

Therefore, we can predict that the function cascade synchronization is achieved for the discrete-time hyperchaotic system.

Remark 1. In our scheme, we choose the error function $e_i(k) = X_i(k) - Q_i(x_i(k))x_i(k)$ as a special function form, so we name it the function cascade synchronization in discrete-time systems. When we set $Q_i(x_i(k)) = 1$ or $Q_i(x_i(k)) = \alpha$, the cascade synchronization methods known [19,20] will appear, respectively. In addition, more parameters $\{c_{ij} | 1 \leqslant i, j \leqslant 4\}$ are needed in our scheme, which enlarge the scope of the scalar controllers $u_i(k)$ in the sense of synchronization. Therefore, our scheme is different from the known techniques [7–9,23–34].

Remark 2. Although the ansätz of the error function here is general, how to choose an error function is the key problem naturally. Even if the availability of computer symbolic systems like *Maple* or *Mathematica* allows us to perform the complicated and tedious algebraic calculation and differential calculation on a computer, in general, it is very difficult, sometime impossible, to find all class of specified error function $Q_i(x_i(k))$. As the calculation goes on, in order to drastically simplify the work or make the work feasible to obtain controllers $u_i(k)$. Experience tells us that the error function $Q_i(x_i(k))$ is chosen as some special function forms, such as in the forms of $\tanh x, \operatorname{sech} x$, x^n and $\sum a_n x^n$, e^x, a trial-and-error basis.

Remark 3. As is known, a continuous system is a chaotic system that must be a three-dimensional dynamical system at least. However, a discrete-time system is a chaotic system that must be a two-dimensional dynamical system at least. Generally speaking, for the chaotic systems owning the same dimension and nonlinear property, the discrete-time system is always more complex than the continuous one, therefore, the method proposed in this paper is not a trivial extension to discrete systems. In fact, the scheme can be applied to investigate the synchronization more than 3D dynamical systems and the tracking problems in the discrete-time systems.

3. Two examples of the function cascade synchronization for the discrete-time hyperchaotic systems

In the following, we will apply the function cascade synchronization method to the discrete-time hyperchaotic systems: the generalized Hénon map [23] and the Rössler system [25], respectively. We choose the error functions in the forms of $\tanh x, \operatorname{sech} x, x^n$ and $\sum a_n x^n$. Numerical simulations are followed to illustrate the effectiveness of the proposed method.

3.1. The 3D generalized Hénon map

The 3D generalized Hénon map [23] discovered by Hitzl and Zele is described as

$$\begin{cases} x_1(k+1) = -bx_2(k), \\ x_2(k+1) = 1 + x_3(k) - ax_2(k)^2, \\ x_3(k+1) = x_1(k) + bx_2(k). \end{cases} \tag{17}$$

Here a, b are the control parameters of the discrete-time system. This system has a hyperchaotic attractor when $a = 1.07$ and $b = 0.3$.

According to the above main idea, we copy (17) and obtain the first response system:

$$\begin{cases} X_1(k+1) = -bX_2'(k) + u_1(k), \\ X_2'(k+1) = 1 + x_3(k) - aX_2'(k)^2 + u_2(k), \end{cases} \tag{18}$$

where $x_3(k)$ is the state variable of the original system (17) and $u_1(k), u_2(k)$ are the control functions determined later.

Let us define the Lyapunov function as

$$L_1(k) = |e_1(k)| + c_1|e_2(k)|, \quad c_1 > 0, \tag{19}$$

where $e_1(k) = X_1(k) - x_1(k)\operatorname{sech}^2 x_1(k),\ e_2(k) = X_2'(k) + (1 + x_2(k))x_2(k)$.Our aim is to find the controllers $u_1(k), u_2(k)$ that make the drive system (17) globally asymptotically synchronize with the response system (18).

The derivative of the corresponding Lyapunov function $L_1(k)$ is

$$\begin{aligned} \Delta L_1(k) &= L_1(k+1) - L_1(k) \\ &= |e_1(k+1)| + c_1|e_2(k+1)| - |e_1(k)| - c_1|e_2(k)| \\ &= |X_1(k+1) - x_1(k+1)\operatorname{sech}^2 x_1(k+1)| + c_1|X_2'(k+1) + (1 + x_2(k+1))x_2(k+1)| \\ &\quad - |X_1(k) - x_1(k)\operatorname{sech}^2 x_1(k)| - c_1|X_2'(k) + (1 + x_2(k))x_2(k)|. \end{aligned} \tag{20}$$

We choose the controllers $u_1(k), u_2(k)$ as

$$\begin{aligned} u_1(k) &= bX_2'(k) - bx_2(k)\operatorname{sech}(-bx_2(k)) + c_{11}[X_1(k) - x_1(k)\operatorname{sech}^2 x_1(k)] + c_{12}[X_2'(k) + x_2(k) + x_2(k)^2], \\ u_2(k) &= aX_2'(k) - 3 - 4x_3(k) - x_3(k)^2 + ax_2(k)^2[3 + 2x_3(k) - ax_2(k)^2] \\ &\quad + c_{21}[X_1(k) - x_1(k)\operatorname{sech}^2 x_1(k)] + c_{22}[X_2'(k) + x_2(k) + x_2(k)^2]. \end{aligned} \tag{21}$$

Substituting (17), (18) and (21) as well as the error functions $e_1(k)$ and $e_2(k)$ into (20), we can get

$$\Delta L_1(k) = L_1(k+1) - L_1(k) \leqslant [|c_{11}| + c_1|c_{21}| - 1]|e_1(k)| + [|c_{12}| + c_1|c_{22}| - c_1]|e_2(k)|. \tag{22}$$

When setting the parameters c_{ij} $(i, j = 1, 2)$ satisfy

$$|c_{11}| + c_1|c_{21}| < 1, |c_{12}| + c_1|c_{22}| < c_1, \tag{23}$$

$\Delta L_1(k)$ is negative definite and the error functions $e_1(k),\ e_2(k)$ asymptotically tend to zero, which says that the synchronization is achieved between the system (17) and (18).

Next we take (17) as the drive system, another copied response system is

$$\begin{cases} X_2(k+1) = 1 + X_3(k) - aX_2(k)^2 + u_3(k), \\ X_3(k+1) = X_1(k) + bX_2(k) + u_4(k), \end{cases} \tag{24}$$

where $u_3(k), u_4(k)$ are the controller functions to make the system (17) synchronize with the system (24).

We define the error functions in this form:

$$\begin{cases} e_3(k) = X_2(k) - \alpha x_2(k), \\ e_4(k) = X_3(k) + x_3(k)^3 \end{cases} \tag{25}$$

and the Lyapunov function is

$$L_2(k) = |e_3(k)| + c_2|e_4(k)|. \tag{26}$$

For simplicity, we omit the calculation procedure and only give the final results of the controllers:

$$\begin{aligned} u_3(k) &= \alpha[1 + x_3(k)] - 1 - X_3(k) + a[X_2(k)^2 - \alpha x_2(k)^2] + c_{31}[X_2(k) - \alpha x_2(k)] + c_{32}[X_3(k) + x_3(k)^3], \\ u_4(k) &= -X_1(k) - x_3(k)^3 - 3bx_1(k)^2x_2(k) - b[X_2(k) + b^2x_2(k)^3] \\ &\quad - 3b^2x_1(k)x_2(k)^2 + c_{41}[X_2(k) - \alpha x_2(k)] + c_{42}[X_3(k) + x_3(k)^3]. \end{aligned} \tag{27}$$

With this choice, we can get the derivative of the Lyapunov function

$$\begin{aligned} \Delta L_2(k) &= L_2(k+1) - L_2(k) = |e_3(k+1)| + c_2|e_4(k+1)| - |e_3(k)| - c_2|e_4(k)| \\ &\leqslant [|c_{31}| + c_2|c_{42}| - 1]|e_3(k)| + [|c_{32}| + c_2|c_{42}| - c_2]|e_4(k)| \end{aligned} \tag{28}$$

with the parameters satisfying

$$|c_{31}| + c_2|c_{42}| < 1, \quad |c_{32}| + c_2|c_{42}| < c_2. \tag{29}$$

So we can get the error functions $\lim_{k\to+\infty}\|e_3(k)\| = 0, \lim_{k\to+\infty}\|e_4(k)\| = 0$, which denotes the synchronization is achieved between the systems (17) and (24).

In this way, we get

$$\begin{cases} \lim\limits_{k\to+\infty}\|e_1(k)\| = \lim\limits_{k\to+\infty}|X_1(k) - x_1(k)\mathrm{sech}^2 x_1(k)| = 0, \\ \lim\limits_{k\to+\infty}\|e_3(k)\| = \lim\limits_{k\to+\infty}|X_2(k) - \alpha x_2(k)| = 0, \\ \lim\limits_{k\to+\infty}\|e_4(k)\| = \lim\limits_{k\to+\infty}|X_3(k) + x_3(k)^3| = 0. \end{cases} \tag{30}$$

So we predict that the function cascade synchronization is achieved for the discrete-time Hénon map.

In the following, in order to verify the effectiveness of the above controllers, we draw the numerical simulation figures. The parameters are taken as $(a,b) = (1.07,3), c_1 = 0.5, c_2 = 0.5, c_{11} = \frac{1}{3}, c_{12} = 0.2, c_{21} = -1, c_{22} = 0.5, c_{31} = 0.5, c_{32} = 0.1, c_{41} = 0.1, c_{42} = -0.1$, which satisfy (23) and (29), and the initial values are chosen as $(x_1(0), x_2(0), x_3(0)) = (0.2, 0.7, 0.06)$ and $(X_1(0), X_2(0), X_3(0)) = (0.5, -0.01, 0.08)$, respectively. Fig. 1 displays the discrete-time hyperchaotic attractors of the drive system and response system. The states of the error functions are displayed in Fig. 2a–c.

3.2. 3D discrete-time Rössler system

Take the 3D discrete-time Rössler system [25] as

$$\begin{cases} x_1(k+1) = 3.8x_1(k)[1 - x_1(k)] - 0.05[x_3(k) + 0.35][1 - 2x_2(k)], \\ x_2(k+1) = 3.78x_2(k)[1 - x_2(k)] + 0.2x_3(k), \\ x_3(k+1) = 0.1[1 - 1.9x_1(k)][(x_3(k) + 0.35)(1 - 2x_2(k)) - 1]. \end{cases} \tag{31}$$

The steps are similar to the generalized Hénon map, we consider the first copied response system as

$$\begin{cases} X_1(k+1) = 3.8X_1(k)[1 - X_1(k)] - 0.05[x_3(k) + 0.35][1 - 2X_2'(k)] + u_1(k), \\ X_2'(k+1) = 3.78X_2'(k)[1 - X_2'(k)] + 0.2x_3(k) + u_2(k). \end{cases} \tag{32}$$

Here $X_1(k), X_2'(k), x_3(k)$ are the state variables and $u_1(k), u_2(k)$ are the external controllers designed later.

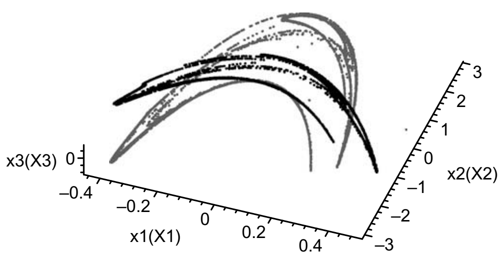

Fig. 1. Hyperchaotic attractors for the discrete-time generalized Hénon map. The black and small figure is for the drive system and the big one is the attractor for the response system. For the special error functions, the attractor for the response system is extended outside a little compared with the drive system. However, it is obvious that the function cascade synchronization is achieved from this figure.

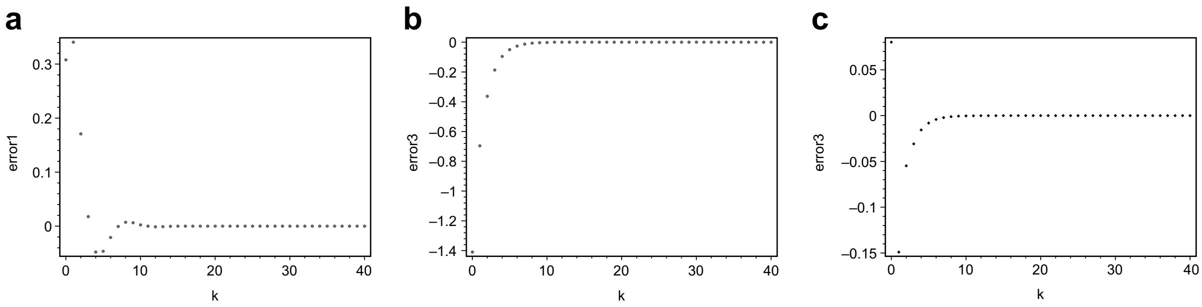

Fig. 2. The states of the hyperchaotic synchronization error functions: (a) $e_1(k) = X_1(k) - x_1(k)\mathrm{sech}^2 x_1(k)$, (b) $e_3(k) = X_2(k) - \alpha x_2(k)$, (c) $e_4(k) = X_3(k) + x_3(k)^3$.

We define the Lyapunov function $L_1(k)$ is the same as (19)

$$L_1(k) = |e_1(k)| + c_1|e_2(k)|, \quad c_1 > 0$$

with $e_1(k) = X_1(k) + x_1(k)^2, e_2(k) = X_2'(k) + x_2(k)\tanh x_2(k)$.

When we choose the controllers as

$$\begin{aligned} u_1(k) &= 0.01719375 + 0.133x_1(k) + 0.001225x_2(k) + 0.4825x_3(k) - 14.573x_1(k)^2 \\ &\quad - 0.001225x_2(k)^2 - 0.0025x_3(k)^2 - 0.035X_2'(k) + 3.8X_1(k)[X_1(k) - 1] \\ &\quad + 0.76x_1(k)x_2(k)x_3(k)[x_1(k) - 1] + 0.01x_2(k)x_3(k)[1 - x_2(k)][x_3(k) + 0.7], \\ u_2(k) &= 3.78X_2'(k)[X_2'(k) - 1] + c_{21}[X_1(k) + x_1(k)^2] \\ &\quad + c_{22}[X_2(k) - x_2(k)\tanh x_2(k)] - 0.2x_3(k) + \tanh[3.78x_2(k)^2 \\ &\quad - 3.78x_2(k) - 0.2x_3(k)][3.78x_2(k)^2 - 3.78x_2(k) - 0.2x_3(k)]. \end{aligned} \tag{33}$$

Substituting the drive system (31), the response system (32) and the controllers (33) into the derivative of the Lyapunov function $\Delta L_1(k)$, we can get the result

$$\Delta L_1(k) = L_1(k+1) - L_1(k) \leqslant [|c_{11}| + c_1|c_{21}| - 1]|e_1(k)| + [|c_{12}| + c_1|c_{22}| - c_1]|e_2(k)|.$$

If we set the parameters satisfying $|c_{11}| + c_1|c_{21}| < 1, |c_{12}| + c_1|c_{22}| < c_1$ we can know that $\Delta L_1(k) \leqslant 0$. So we predict that the synchronization between system (31) and (32) is achieved and the error functions $e_1(k), e_2(k)$ asymptotically tend to zero.

In the following we depict another system as the response system of (31) in this form:

$$\begin{cases} X_2(k+1) = 3.78X_2(k)[1 - X_2(k)] + 0.2X_3(k) + u_3(k), \\ X_3(k+1) = 0.1[1 - 1.9X_1(k)][(X_3(k) + 0.35)(1 - 2X_2(k)) - 1] + u_4(k). \end{cases} \tag{34}$$

According to the Lyapunov stable theory, we choose the Lyapunov function $L_2(k)$ as

$$L_2(k) = |e_3(k)| + c_2|e_4(k)|, \quad c_2 > 0,$$

where the error functions are in the style of $e_3(k) = X_2(k) - x_2(k)\tanh^2 x_2(k), e_4(k) = X_3(k) - \alpha x_3(k)$.

With the aid of symbolic computation system *Maple*, we can easily work out the controllers desired $u_3(k)$ and $u_4(k)$ that make the system (31) globally asymptotically synchronize with the system (34). Here we omit the heavy the calculation and just give the final results:

$$\begin{aligned} u_3(k) &= 3.78X_2(k)[X_2(k) - 1] - 0.2X_3(k) + c_{31}[X_2(k) - x_2(k)\tanh^2 x_2(k)] + c_{32}[X_3(k) - \alpha x_3(k)] \\ &\quad + \tanh^2[3.78x_2(k)^2 - 3.78x_2(k) - 0.2x_3(k)][3.78x_2(k) - 3.78x_2(k)^2 - 0.2x_3(k)], \\ u_4(k) &= 0.065(1 - \alpha) - 0.1235[X_1(k) - \alpha x_1(k)] + 0.07[X_2(k) - \alpha x_2(k)] - 0.1[X_3(k) - \alpha x_3(k)] \\ &\quad - 0.133[X_1(k)X_2(k) - \alpha x_1(k)x_2(k)] + 0.19[X_1(k)X_3(k) - \alpha x_1(k)x_3(k)] \\ &\quad + 0.2[X_2(k)X_3(k) - \alpha x_2(k)x_3(k)] + c_{41}[X_2(k) - x_2(k)\tanh^2 x_2(k)] \\ &\quad + c_{42}[X_3(k) - \alpha x_3(k)] - 0.38[X_1(k)X_2(k)X_3(k) - \alpha x_1(k)x_2(k)x_3(k)]. \end{aligned}$$

When the parameters are chosen as $|c_{31}| + c_2|c_{42}| < 1, |c_{32}| + c_2|c_{42}| < c_2$, the derivative of the Lyapunov function $\Delta L_2(k)$ satisfies

$$\begin{aligned} \Delta L_2(k) &= L_2(k+1) - L_2(k) = |e_3(k+1)| + c_2|e_4(k+1)| - |e_3(k)| - c_2|e_4(k)| \\ &\leqslant [|c_{31}| + c_2|c_{42}| - 1]|e_3(k)| + [|c_{32}| + c_2|c_{42}| - c_2]|e_4(k)| \leqslant 0. \end{aligned} \tag{35}$$

So the system (31) asymptotically synchronizes with the system (34), that is to say $\lim_{k\to+\infty}\|e_3(k)\| = 0$ and $\lim_{k\to+\infty}\|e_4(k)\| = 0$. Therefore, we have

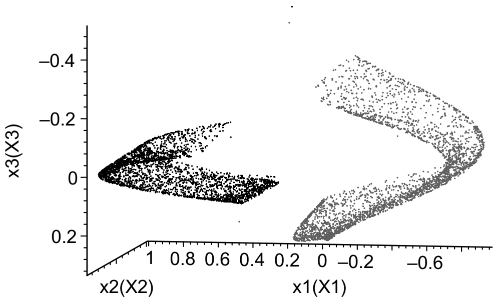

Fig. 3. Hyperchaotic attractors for the discrete-time Rössler system. The more black figure is the attractor for the drive system and the other figure is for the response system.

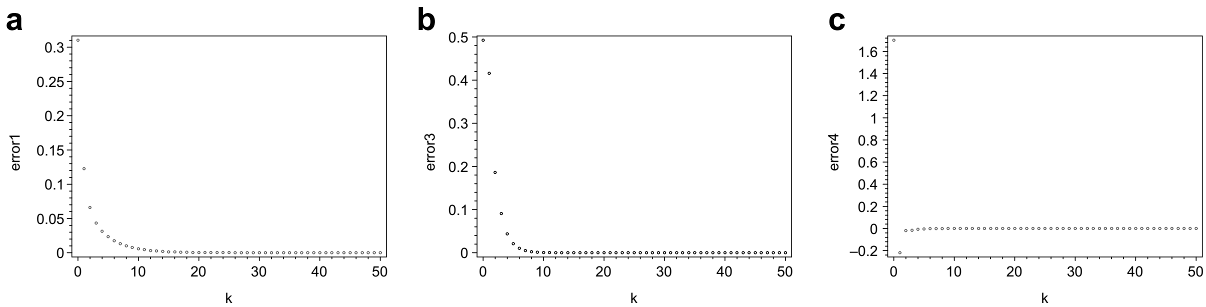

Fig. 4. The states of the hyperchaotic synchronization error functions: (a) $e_1(k) = X_1(k) + x_1(k)^2$, (b) $e_3(k) = X_2(k) - x_2(k)\tanh^2 x_2(k)$, (c) $e_4(k) = X_3(k) - \alpha x_3(k)$.

$$\begin{cases} \lim\limits_{k\to+\infty} \|e_1(k)\| = \lim\limits_{k\to+\infty} |X_1(k) + x_1(k)^2| = 0, \\ \lim\limits_{k\to+\infty} \|e_3(k)\| = \lim\limits_{k\to+\infty} |X_2(k) - x_2(k)\tanh^2 x_2(k)| = 0, \\ \lim\limits_{k\to+\infty} \|e_4(k)\| = \lim\limits_{k\to+\infty} |X_3(k) - \alpha x_3(k)| = 0, \end{cases} \tag{36}$$

which indicate the function cascade synchronization is achieved for the discrete-time Rössler system.

Numerical simulation results are used in order to verify the effectiveness of the proposed scheme for the Rössler system. The initial values are chosen as $(x_1(0), x_2(0), x_3(0)) = (0.1, 0.2, -0.5)$ and $(X_1(0), X_2(0), X_3(0)) = (0.3, 0.5, 0.2)$ and the parameters as $c_1 = 1.2, c_2 = 0.3, c_{11} = 0.5, c_{12} = -0.2, c_{21} = -1/3, c_{22} = 0.5, c_{31} = 0.5, c_{32} = 0.1, c_{41} = 0.1, c_{42} = -0.1$. Fig. 3 displays the discrete-time hyperchaotic attractors for the Rössler system. Fig. 4a–c is the error function figures.

4. Conclusions

In summary, this paper has successfully extended the cascade synchronization from the continuous-time systems to the discrete-time dynamical systems. The proposed method are used to realize the discrete-time hyperchaotic synchronization of the 3D generalized Hénon map and the 3D Rössler system, respectively. It is necessary to point out that here we use the scheme presented to study hyperchaotic system, generally speaking, the method by us is effective to synchronize the discrete-time chaotic system, whether it is hyperchaotic system or not. Numerical simulations are implemented to show the effectiveness of the method. Our scheme contains many parameters c_{ij} which provide a larger choice scope in the controllers. In addition, as for the error functions, we can have many different choices, such as α, $\tanh x$, x^n and some simple polynomial $\sum a_n x^n$, $\sum a_n \tanh^n x$, etc., which allow us to adjust these desired scaling error functions to obtain the reliable results needed. So we predict that our scheme is also a powerful method and different form the former synchronization methods. Whether this scheme can be applied to complex networks and fractional dimension will be further studied.

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