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Symmetry and Exact Solutions of (2+1)-Dimensional Generalized Sasa–Satsuma Equation via a Modified Direct Method*

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Abstract *In this paper, first, we employ classic Lie symmetry groups approach to obtain the Lie symmetry groups of the well-known (2+1)-dimensional Generalized Sasa–Satsuma (GSS) equation. Second, based on a modified direct method proposed by Lou [J. Phys. A: Math. Gen.* **38** *(2005) L129], more general symmetry groups are obtained and the relationship between the new solution and known solution is set up. At the same time, the Lie symmetry groups obtained are only special cases of the more general symmetry groups. At last, some exact solutions of GSS equations are constructed by the relationship obtained in the paper between the new solution and known solution.*

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Key words: classic Lie symmetry groups approach, modified CK's direct method, generalized Sasa–Satsuma equation

1 Introduction

Lie symmetry is one of the most powerful methods in every branch of natural science especially in integrable systems.[1,2,3] To find the Lie point symmetry group of a nonlinear system, there is a standard method to find symmetry algebras and groups for almost all the known integrable systems. In previous paper,[4] Clarkson and Kruskal (CK) introduced a simple direct method to derive symmetry reductions of a nonlinear system without using any group theory. For many types of nonlinear systems, those symmetry gained by CK's direct method can also be generated via the classical and nonclassic Lie group approach.[5] In previous paper,[6] the conditional similarity reducation solution can also be obtained by the modified CK's direct method. These facts hint that there is a simple method to find generalized symmetry groups for many types of nonlinear equations.

In this paper, we consider the Generalized Sasa–Satsuma (GSS) equation,[7] which has this following form:

$$\Delta_1 \equiv q_t + q_{xxx} + 6q_xU + 3qU_x = 0\,, \tag{1}$$

$$\Delta_2 \equiv r_t + r_{xxx} + 6r_xU + 3rU_x = 0\,, \tag{2}$$

$$\Delta_3 \equiv U_y - (qr)_x = 0\,, \tag{3}$$

and is a generalization of the Sasa–Satsuma equation introduced recently:[8]

$$q_t + q_{xxx} + 6q_xU + 3qU_x = 0\,, \tag{4}$$

$$U_y - (|q|^2)_x = 0\,, \tag{5}$$

where q is the complex physical field and U is the real potential for Eqs. (1)–(3). When r is taken as the complex conjugate of q, Eqs. (1)–(3) become the Eqs. (4) and (5).

This paper is arranged as follows: In Sec. 2, first, with the method of classic Lie group, the Lie point symmetry of GSS equation can be obtained, then by using the modified direct method, we get a relationship between the new solution and known solution of the GSS equation. Taking some special case, the Lie symmetry of the GSS equation also can be gained, by solving the corresponding characteristic equation, then the reduced equations can be obtained. In Sec. 3, some explict solutions of GSS equation are presented via choosing different parameters. Finally, some conclusions and discussions are given in Sec. 4.

2 Symmetry and Reduction of GSS Equation

In this section, we look for the symmetry of Eqs. (1)–(3) by using the classical Lie method. Let us consider a one-parameter Lie group of infinitesimal transformation

$$\begin{aligned}
x &\to x + \epsilon\xi(x,u,t,q,r,U)\,,\\
y &\to y + \epsilon\eta(x,u,t,q,r,U)\,,\\
t &\to t + \epsilon\tau(x,u,t,q,r,U)\,,\\
q &\to q + \epsilon\phi_1(x,u,t,q,r,U)\,,\\
r &\to r + \epsilon\phi_2(x,u,t,q,r,U)\,,\\
U &\to U + \epsilon\phi_3(x,u,t,q,r,U)\,,
\end{aligned}$$

with a small parameter ϵ, the corresponding vector field associated with the above group of transformations to GSS equation can be written as

$$\mathcal{V} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \tau\frac{\partial}{\partial t} + \phi_1\frac{\partial}{\partial q}$$

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$$+\phi_2\frac{\partial}{\partial r}+\phi_3\frac{\partial}{\partial U}\,,$$

where ξ, η, τ, ϕ_1, ϕ_2, and ϕ_3 are the funcations with respect to $\{x, y, t, q, r, U\}$, the prolongation of the vector field is

$$\begin{aligned}\mathrm{pr}^{(3)}\mathcal{V}=\mathcal{V}&+\phi_1^x\frac{\partial}{\partial q_x}+\phi_1^t\frac{\partial}{\partial q_t}+\phi_1^{xxx}\frac{\partial}{\partial q_{xxx}}+\phi_2^x\frac{\partial}{\partial r_x}\\&+\phi_2^t\frac{\partial}{\partial q_t}+\phi_2^{xxx}\frac{\partial}{\partial r_{xxx}}+\phi_3^x\frac{\partial}{\partial U_x}+\phi_3^y\frac{\partial}{\partial U_y}\,,\end{aligned}$$

where

$$\begin{aligned}\phi_1^x&=D_x(\phi_1-\xi q_x-\eta q_y-\tau q_t)\\&\quad+\xi q_{xx}+\eta q_{xy}+\tau q_{xt}\,,\\\phi_1^t&=D_t(\phi_1-\xi q_x-\eta q_y-\tau q_t)\\&\quad+\xi q_{xt}+\eta q_{yt}+\tau q_{tt}\,,\\\phi_1^{xxx}&=D_{xxx}(\phi_1-\xi q_x-\eta q_y-\tau q_t)\\&\quad+\xi q_{xxxx}+\eta q_{xxxy}+\tau q_{xxxt}\,,\\\phi_2^x&=D_x(\phi_2-\xi r_x-\eta r_y-\tau r_t)\\&\quad+\xi r_{xx}+\eta r_{xy}+\tau r_{xt}\,,\\\phi_2^t&=D_t(\phi_2-\xi r_x-\eta r_y-\tau r_t)\\&\quad+\xi r_{xt}+\eta r_{yt}+\tau r_{tt}\,,\\\phi_2^{xxx}&=D_{xxx}(\phi_2-\xi r_x-\eta r_y-\tau r_t)\\&\quad+\xi_{xxxx}+\eta r_{xxxy}+\tau r_{xxxt}\,,\\\phi_3^x&=D_x(\phi_3-\xi U_x-\eta U_y-\tau U_t)\\&\quad+\xi U_{xx}+\eta U_{xy}+\tau U_{xt}\,,\\\phi_3^y&=D_y(\phi_3-\xi U_x-\eta U_y-\tau U_t)\\&\quad+\xi U_{xy}+\eta U_{yy}+\tau U_{yt}\,.\end{aligned}$$

By using these extensions and applying the prolongation to the GSS equation, the infinitesimal criteria for the invariance of Eqs. (1)–(3) under the group is given by

$$\mathrm{pr}^{(3)}\mathcal{V}\Delta_i|_{\Delta_j=0}=0\quad(i,j=1,2,3)\,.\tag{6}$$

After substituting Eqs. (1)–(3) into Eq. (6) for $i, j = 1, 2, 3$, comparing the coefficients of linearly independent expressions of $\{q, r, U\}$ and let their derivatives be zero (q_t, r_t, U_y and their derivatives have been cancelled by Δ_i ($i = 1, 2, 3$)), we obtain various constraint equations. And solving these equations, one can obtain

$$\begin{aligned}\xi&=\frac{\dot F_1(t)x}{3}+F_4(t),\quad \eta=F_2(y),\quad \tau=F_1(t)\,,\\\phi_1&=-\frac{1}{6}q(6\dot F_2(y)+\dot F_1(t)+6F_3(y))\,,\\\phi_2&=\Big(-\frac{\dot F_1(t)}{6}+F_3(y)\Big)r\,,\\\phi_3&=\frac{\ddot F_1(t)}{18}x+\frac{\dot F_4(t)}{6}-\frac{2\dot F_1(t)}{3}U\,.\end{aligned}$$

If we want to obtain the relation between the new solution and known solution of GSS equation, one has to solve the following ODEs

$$\begin{aligned}\frac{d\bar x}{d\epsilon}&=\xi(\bar x,\bar y,\bar t,\bar q,\bar r,\bar U)\,,\\\frac{d\bar y}{d\epsilon}&=\eta(\bar x,\bar y,\bar t,\bar q,\bar r,\bar U)\,,\\\frac{d\bar t}{d\epsilon}&=\tau(\bar x,\bar y,\bar t,\bar q,\bar r,\bar U)\,,\end{aligned}\tag{7}$$

$$\begin{aligned}\frac{d\bar q}{d\epsilon}&=\phi_1(\bar x,\bar y,\bar t,\bar q,\bar r,\bar U)\,,\\\frac{d\bar r}{d\epsilon}&=\phi_2(\bar x,\bar y,\bar t,\bar q,\bar r,\bar U)\,,\\\frac{d\bar U}{d\epsilon}&=\phi(\bar x,\bar y,\bar t,\bar q,\bar r,\bar U)\,,\end{aligned}\tag{8}$$

with initial solution conditions

$$\begin{aligned}&\bar x|_{\epsilon=0}=x,\quad \bar y|_{\epsilon=0}=y,\quad \bar t|_{\epsilon=0}=t\,,\\&\bar q|_{\epsilon=0}=q,\quad \bar r|_{\epsilon=0}=r,\quad \bar U|_{\epsilon=0}=U\,.\end{aligned}\tag{9}$$

Solving the Eqs. (7)–(9) is difficult,[9] because the equations inclued some arbitrary functions, such as $F_1(t)$, $F_4(t)$, $F_2(y)$, $F_3(y)$. If we want to get the transformation between the new solution and known solution of GSS equation, one may use the modified direct method.

For the GSS equation, in order to search the transformation between the new solution and known solution, it is enough to seek the symmetry reductions in a simple form[10]

$$q=\alpha+\beta Q(\xi,\eta,\tau)\,,\tag{10}$$

$$r=\gamma+\theta R(\xi,\eta,\tau)\,,\tag{11}$$

$$U=\zeta+\phi W(\xi,\eta,\tau)\,,\tag{12}$$

where $\alpha = \alpha(x, y, t)$, $\beta = \beta(x, y, t)$, $\gamma = \gamma(x, y, t)$, $\theta = \theta(x, y, t)$, $\zeta = \zeta(x, y, t)$, $\phi = \phi(x, y, t)$ and ξ, η, τ are functions of $\{x, y, t\}$ to be determined later, under the transformations

$$\begin{aligned}&\{x,y,t,q(x,y,t),r(x,y,t),U(x,y,t)\}\\&\quad\to\{\xi,\eta,\tau,Q(\xi,\eta,\tau),R(\xi,\eta,\tau),W(\xi,\eta,\tau)\}\,,\end{aligned}$$

and satisfy the same PDEs

$$Q_\tau+Q_{\xi\xi\xi}+6Q_\xi W+3QW_\xi=0\,,\tag{13}$$

$$R_\tau+R_{\xi\xi\xi}+6R_\xi W+3RW_\xi=0\,,\tag{14}$$

$$W_\eta-(QR)_\xi=0\,.\tag{15}$$

Substituting Eqs. (10)–(12) along with Eqs. (13)–(15) into Eqs. (1)–(3) and eliminating Q_τ, R_τ, W_η and their higher order derivatives by means of the Eqs. (13)–(15). Let the coefficients of U, V, W and their derivatives be zero, we arrive at some PDEs which are omitted in this paper. By solving the above PDEs, we have the following results:

$$\begin{aligned}\xi&=f_1(t)^2x-6\int f_2(t)f_1(t)dt\,,\\\eta&=\int f_3(y)f_4(y)dy\,,\\\tau&=\int f_1(t)^6dt,\quad \alpha=0,\quad \gamma=0\,,\\\beta&=f_3(y)f_1(t),\quad \theta=f_4(y)f_1(t)\,,\\\zeta&=-\frac{\dot f_1(t)x}{3f_1(t)}+f_2(t),\quad \phi=f_1(t)^4\,.\end{aligned}$$

In summary, by using Eqs. (10)–(12), the following theorem holds:

Theorem 1 If $Q(\xi,\eta,\tau)$, $R(\xi,\eta,\tau)$, $W(\xi,\eta,\tau)$ are the solutions of the GSS equation, then the GSS equation also has solutions as follow:

$$q(x,y,t) = f_3(y)f_1(t)Q(\xi,\eta,\tau)\,,$$
$$r(x,y,t) = f_4(y)f_1(t)R(\xi,\eta,\tau)\,,$$
$$U(x,y,t) = -\frac{\dot{f}_1(t)}{3f_1(t)}x + f_2(t) + f_1(t)^4 W(\xi,\eta,\tau)\,.$$

To obtain the Lie point symmetry group obtained from Theorem 1, we restrict:

$$f_1(t) = 1+\epsilon f(t),\quad f_2(t) = \epsilon F_2(t)\,,$$
$$f_3(y) = 1+\epsilon F_3(y),\quad f_4(y) = 1+\epsilon F_4(y)$$

with an infinitesimal parameter ϵ, then equations in Theorem 1 can be written as:

$$q = Q+\epsilon\delta(Q),\quad r = R+\epsilon\delta(R),\quad U = W+\epsilon\delta(W)\,.$$

Furthermore if setting

$$f(t) = \frac{\dot{F}_1(t)}{6},\quad f_2(t) = -\frac{\dot{H}_4(t)}{6}\,,$$
$$f_3(y)+f_4(y) = \dot{G}_4(y)\,,$$

we also obtain the symmetry

$$\delta(q) = \Big(\frac{\dot{F}_1(t)x}{3}+H_4(t)\Big)Q_x + G_4(y)Q_y + F_1(t)Q_t + \Big(\frac{\dot{F}_1(t)}{6}+F_3(y)\Big)Q\,, \tag{16}$$

$$\delta(r) = \Big(\frac{\dot{F}_1(t)x}{3}+H_4(t)\Big)R_x + G_4(y)R_y + F_1(t)R_t + \Big(\frac{\dot{F}_1(t)}{6}+F_4(y)\Big)R\,, \tag{17}$$

$$\delta(U) = \Big(\frac{\dot{F}_1(t)x}{3}+H_4(t)\Big)W_x + G_4(y)W_y + F_1(t)W_t - \frac{\ddot{F}_1(t)x}{18} - \frac{\dot{H}_4(t)}{6} + \frac{2\dot{F}_1(t)}{3}W\,. \tag{18}$$

If one set some constraint conditions, the Eqs. (16)–(18) are equal to the vector field obtained by classic Lie point group.

The solutions of Eqs. (16)–(18) are obtained by solving the characteristic equation,[5] then one obtain the similarity form of the solution as

$$q = F_1(x,y,t,\Phi_1(X,Y))\,,$$
$$r = F_2(x,y,t,\Phi_2(X,Y))\,,$$
$$U = F_3(x,y,t,\Phi_3(X,Y))\,, \tag{19}$$

where X and Y are similarity variables, the others $\Phi_1(X,Y)$, $\Phi_2(X,Y)$, $\Phi_3(X,Y)$ are dependent variables. Substituting Eqs. (19) into Eqs. (1)–(3), then we can obtain the reduction equations of GSS equation.

By setting $F_1(t)=t$, $H_4(t)=t$, $G_4(y)=y$, $F_3(y)=0$, $F_4(y)=1$ as an example, the invariant surface conditions become

$$\Big(\frac{x}{3}+t\Big)Q_x + yQ_x + tQ_x + \frac{1}{6}Q = 0\,, \tag{20}$$

$$\Big(\frac{x}{3}+t\Big)R_x + yR_x + tR_x + \frac{7}{6}Q = 0\,, \tag{21}$$

$$\Big(\frac{x}{3}+t\Big)W_x + yW_x + tW_x - \frac{1}{6} + \frac{2}{3}W = 0\,. \tag{22}$$

The corresponding characteristic equation is

$$\frac{dx}{x/3+t} = \frac{dy}{y} = \frac{dt}{t} = \frac{dq}{-1/6} = \frac{dr}{-7/6} = \frac{dW}{1/6-(2/3)W}\,. \tag{23}$$

Solving the characteristic Eq. (23), we obtain

$$q = t^{7/6}\Phi_1(X,Y),\quad r = t^{1/6}\Phi_2(X,Y)\,,$$
$$U = \frac{1}{5}t + t^{-2/3}\Phi_3(X,Y)\,, \tag{24}$$

$$X = \frac{5x-3t^2}{5t^{1/3}},\quad Y = \frac{y}{t}\,. \tag{25}$$

Substituting Eqs. (24) and (25) into Eqs. (1)–(3), the reduction of GSS equation read

$$7\Phi_1 + 2X\Phi_{1X} + 6Y\Phi_{1Y} - 6\Phi_{1XXX} - 36\Phi_{1X}\Phi_3 - 18\Phi_1\Phi_{3X} = 0\,, \tag{26}$$

$$\Phi_2 + 2X\Phi_{2X} + 6Y\Phi_{2Y} - 6\Phi_{2XXX} - 36\Phi_{2X}\Phi_3 - 18\Phi_2\Phi_{3X} = 0\,, \tag{27}$$

$$\Phi_{3Y} - \Phi_{1X}\Phi_2 - \Phi_{2X}\Phi_1 = 0\,. \tag{28}$$

As we see, the reduced GSS equation are still complex and to solve the Eqs. (26)–(28) is difficult. However, the above results and discussion set up the relations between new and the known solution, which can construct new solutions. In addition, the more general symmetry group are obtained, which include the classical point symmetry group.

3 Solution of (2+1)-Dimensional GSS Equation

In the paper,[7] the author solve the GSS equation by the Painlevé approach and the final solutions can be given in terms of arbitrary functions as follow

$$q = -\frac{\sqrt{\phi_{2x}}\phi_{1y}}{2F(\phi_1+\phi_2)}\,, \tag{29}$$

$$r = \frac{F\sqrt{\phi_{2x}}}{\phi_1+\phi_2}\,, \tag{30}$$

$$U = -\frac{\phi_{2x}^2}{2(\phi_1+\phi_2)^2} + \frac{\phi_{2xx}}{2(\phi_1+\phi_2)} - \frac{\phi_{2t}+\phi_{2xxx}}{6\phi_{2x}} + \frac{\phi_{2xx}^2}{8\phi_{2x}^2}\,, \tag{31}$$

where ϕ_2 is an arbitrary function of $\{x,t\}$ and ϕ_1 and F are arbitrary functions of y. Also the Eqs. (13)–(15) have the same form solutions:

$$Q = -\frac{\sqrt{\phi_{2\xi}}\phi_{1\eta}}{2F(\phi_1+\phi_2)}\,, \tag{32}$$

$$R = \frac{F\sqrt{\phi_{2\xi}}}{\phi_1+\phi_2}\,, \tag{33}$$

$$W = -\frac{\phi_{2\xi}^2}{2(\phi_1+\phi_2)^2} + \frac{\phi_{2\xi\xi}}{2(\phi_1+\phi_2)} - \frac{\phi_{2\tau}+\phi_{2\xi\xi\xi}}{6\phi_{2\xi}} + \frac{\phi_{2\xi\xi}^2}{8\phi_{2\xi}^2}\,, \tag{34}$$

where ϕ_2 is an arbitrary function of $\{\xi,\tau\}$ and ϕ_1 and F are arbitrary functions of η.

Here, applying the symmetry group Theorem 1 on Eqs. (32)–(34), one can obtain some solutions when the seed solutions of GSS equation are given,

$$q = -f_3(y)f_1(t)\frac{\sqrt{\phi_{2\xi}}\phi_{1\eta}}{2F(\phi_1+\phi_2)}\,, \tag{35}$$

$$r = f_4(y)f_1(t)\frac{F\sqrt{\phi_{2\xi}}}{\phi_1+\phi_2}\,, \tag{36}$$

$$U = -\frac{\dot f_1(t)}{3f_1(t)}x + f_2(t) - f_1(t)^4\Big(\frac{\phi_{2\xi}^2}{2(\phi_1+\phi_2)^2} - \frac{\phi_{2\xi\xi}}{2(\phi_1+\phi_2)} + \frac{\phi_{2\tau}-\phi_{2\xi\xi\xi}}{6\phi_{2\xi}} - \frac{\phi_{2\xi\xi}^2}{8\phi_{2\xi}^2}\Big)\,, \tag{37}$$

where $\xi = f_1(t)^2x - 6\int f_2(t)f_1(t)dt$, $\eta = \int f_3(y)f_4(y)dy$, and $\tau = \int f_1(t)^6 dt$ are given in Sec. 2.

In order to get some explicit results, we choose the arbitrary functions ϕ_1 and ϕ_2 as simple as

$$\phi_1 = a_3e^{\eta}, \qquad \phi_2 = a_0 + a_1e^{k_1\xi-\omega_1\tau}\,,$$

where a_0, a_1, a_3, k_1, ω_1 are arbitrary constants.

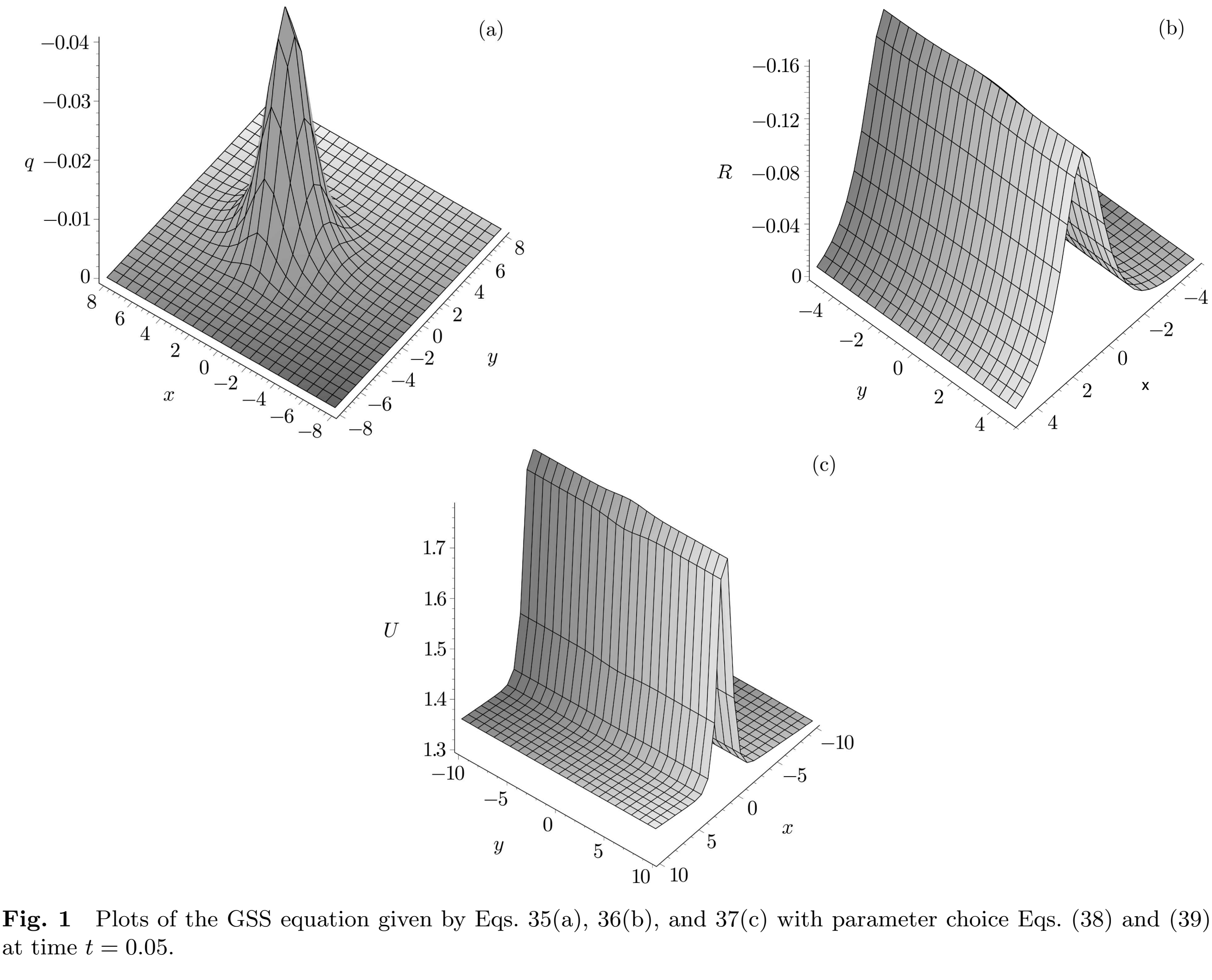

Fig. 1 Plots of the GSS equation given by Eqs. 35(a), 36(b), and 37(c) with parameter choice Eqs. (38) and (39) all at time $t = 0.05$.

Figure 1 displays the solutions structure of Eqs. (36) and (37), with the parameter selections

$$a_0 = 20, \quad a_1 = 20, \quad a_3 = -1, \quad k_1 = 3, \quad \omega_1 = -2, \quad k_3 = -\frac{1}{2}\,, \tag{38}$$

$$f_1(t)=\cos(t)^{1/6},\quad f_2(t)=0,\quad f_3(y)=1,\quad f_4(y)=y,\quad F(\eta)=1\,. \tag{39}$$

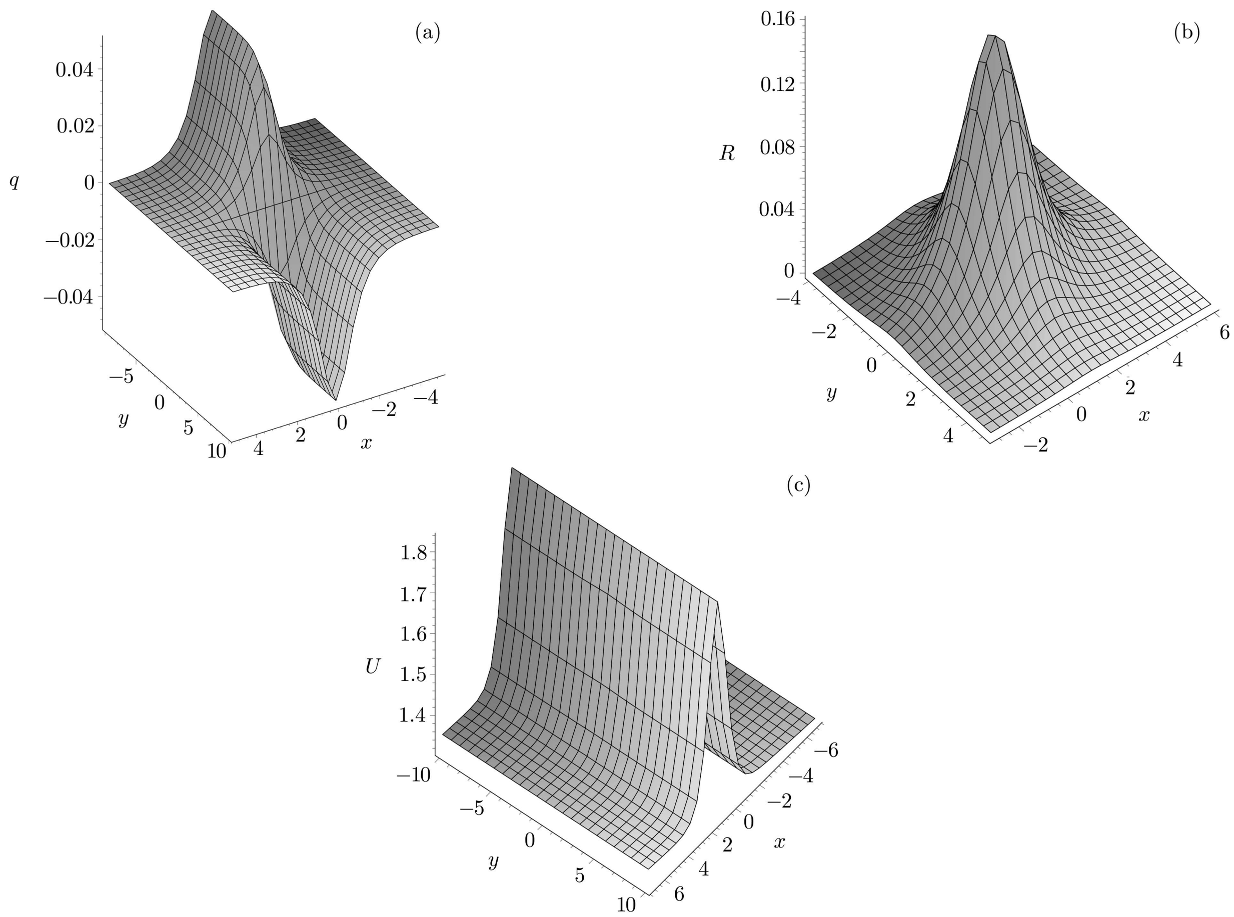

Fig. 2 Plots of the GSS equation given by Eqs. 35(a), 36(b), and 37(c) with the parameter choice Eqs. (40) and (41) all at time $t = 0.05$.

Figure 2 shows the solution structure of Eqs. (35)–(37), for the parameter choice

$$a_0=20,\quad a_1=2,\quad a_3=-1,\quad k_1=\omega_1=2,\quad k_3=\frac{1}{2}\,, \tag{40}$$

$$f_1(t)=\cos(t)^{1/6},\quad f_2(t)=0,\quad f_3(y)=\tanh(y),\quad f_4(y)=-\mathrm{sech(y)},\quad F(\eta)=1\,. \tag{41}$$

For generating the types of periodic wave interaction solution of the GSS equation, the arbitrary functions also can be set as:

$$\phi_1=\sum_{i=1}^{N}[a_{1i}\mathrm{sn}^{\alpha_{1i}}(k_{1i}\eta_{1i},m_{1i})+a_{2i}\mathrm{cn}^{\alpha_{2i}}(k_{1i}\eta_{1i},m_{2i})+a_{3i}\mathrm{dn}^{\alpha_{3i}}(k_{1i}\eta_{1i},m_{3i})]\,,$$

$$\phi_2=b_0+\sum_{i=1}^{N}[b_{1i}\mathrm{sn}^{\beta_{1i}}(\zeta_{1i},m_{1i})+b_{2i}\mathrm{cn}^{\beta_{2i}}(\zeta_{2i},m_{2i})+b_{3i}\mathrm{dn}^{\beta_{3i}}(\zeta_{3i},m_{3i})]\,,$$

where $\zeta_{ij}=l_{ij}\xi+\omega_{ij}\tau$ and l_{ij}, ω_{ij}, a_{ij}, b_{ij}, α_{ij}, β_{ij}, b_0 are arbitrary constants.

4 Summary and Conclusion

In summary, based on the classic Lie groups approach and a modified direct method, we investigate the well-known (2+1)-dimensional Generalized Sasa–Satsuma (GSS) equation. As a result, the more general symmetry groups obtained by a modified direct method include the classical Lie point group by classical method; a transformation of GSS equations solution between the new solution and known solution is set up. By means of the transformation and the known solution, some exact solutions of GSS equations are constructed. It is necessary to point out that the reduced equation by the direct method maybe or not simpler than the original equation. If the reduced equation is relative simply, we can directly use the classic Lie groups approach to solve it and obtain the solution of target equation by the transformation between the new solution and the solution of reduced equation. The method reported here can be applied in principle to all nonlinear systems including both integrable and non-integrable ones and more details will be reported in future.

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