

Symbolic computation of exact solutions expressible in rational formal hyperbolic and elliptic functions for nonlinear partial differential equations

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Abstract

With the aid of symbolic computation, some algorithms are presented for the rational expansion methods, which lead to closed-form solutions of nonlinear partial differential equations (PDEs). The new algorithms are given to find exact rational formal polynomial solutions of PDEs in terms of Jacobi elliptic functions, solutions of the Riccati equation and solutions of the generalized Riccati equation. They can be implemented in symbolic computation system *Maple*. As applications of the methods, we choose some nonlinear PDEs to illustrate the methods. As a result, we not only can successfully obtain the solutions found by most existing Jacobi elliptic function methods and Tanh-methods, but also find other new and more general solutions at the same time.

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1. Introduction

The recent development of the nonlinear wave theory clarified the role of soliton in various system. The appearance of solitary wave solutions in nature is quite common. Bell-shaped sech-solutions and kink-shaped tanh-solutions model wave phenomena in fluids, plasmas, elastic media, electrical circuits, optical fibers, chemical reactions, bio-genetics, etc., [1–4].

As we known, travelling wave solutions of many nonlinear PDEs from soliton theory (and beyond) can often be expressed as polynomials of the hyperbolic tangent and secant functions. The Tanh-method provides a straightforward algorithm to compute such particular solutions for a large class of nonlinear PDEs [5–8]. The Tanh-method for, say, a single PDE in $u(x, t)$ works as follows: in a travelling frame of reference, $\xi = k(x + \lambda t)$, one transforms the PDE into an ODE in the new independent variable $T = \tanh \xi$. Since the derivative of \tanh is polynomial in \tanh , i.e., $T' = 1 - T^2$,

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all derivatives of T are polynomials of T . Via a chain rule, the polynomial PDE in $u(x, t)$ is transformed into an ODE in $U(T)$, which has polynomial coefficients in T . One then seeks polynomial solutions of the ODE, thus generating a subset of the set of all solutions. The appeal and success of the Tanh-method lies in the fact that one circumvents integration to get explicit solutions. Recently, the Tanh-methods have been applied to many nonlinear PDEs with various extension [5–14].

Making use of idea of Tanh-method, we use some variables whose derivatives are also polynomial in themselves to replace the hyperbolic functions and present some more general solution form so that it can be used to bring out more types and general formal solutions which contain not only the results obtained by using the various Tanh-methods [11–14], but also other types of solutions. We summary our methods and call them rational expansion methods. For illustration, we apply the generalized method to some nonlinear PDEs and successfully construct new and more general solutions including rational form Jacobi elliptic function solutions, solitary wave solutions and triangular periodic solutions.

The paper is organized as follows: in Section 2, we give the main steps of our algorithms for computing exact solutions of nonlinear polynomial PDEs; In Section 3, we consider the Jacobi elliptic function rational expansion method; In Section 4, we show the Riccati equation rational expansion method; In Section 5, we further extend the Riccati equation rational expansion method to a generalized form. Finally, conclusions are presented.

2. Summary of the rational expansion method

In the following we would like to outline the main steps of our method:

Step 1. Given a system of nonlinear polynomial PDEs with constant coefficients, with some physical fields $u_i(x, y, t)$ in three variables x, y, t ,

$$\Delta(u_i, u_{it}, u_{ix}, u_{iy}, u_{itt}, u_{ixt}, u_{iyt}, u_{ixx}, u_{iyy}, u_{ixy}, \dots) = 0 \quad (2.1)$$

use the wave transformation

$$u_i(x, y, t) = U_i(\xi), \quad \xi = k(x + ly + \lambda t), \quad (2.2)$$

where k, l and λ are constants to be determined later. Then the nonlinear partial differential system (2.1) is reduced to an ordinary differential system

$$\Theta(U_i, U_i', U_i'', \dots) = 0. \quad (2.3)$$

Step 2. We introduce a new ansatz in terms of finite rational formal expansion in the following forms:

$$U_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum_{r_1+r_2=j} a_{r_1 r_2}^{ij} F^{r_1}(\xi) G^{r_2}(\xi)}{(\mu_1 F(\xi) + \mu_2 G(\xi) + 1)^j}, \quad (2.4)$$

where a_{i0}, a_{ij} and b_{ij} ($i = 1, 2, \dots, j = 1, 2, \dots, m_i$) are constants to be determined later and the new variables $F = F(\xi)$ and $G = G(\xi)$ satisfy

$$\frac{dF}{d\xi} = K_1(F, G), \quad \frac{dG}{d\xi} = K_2(F, G), \quad (2.5)$$

where K_1 and K_2 are polynomials of F and G .

Step 3. Determine the m_i of the rational formal polynomial solutions (2.4) by respectively balancing the highest nonlinear terms and the highest-order partial derivative terms in the given system equations (see Refs. [11–14] for details), and then give the formal solutions.

Step 4. Substitute (2.4) into (2.3) along with (2.5) and then set all coefficients of $F^p(\xi)G^q(\xi)$, ($p = 1, 2, \dots, q = 0, 1$) of the resulting system's numerator to be zero to get an over-determined system of nonlinear algebraic equations with respect to $k, \mu_1, \mu_2, a_{i0}, a_{ij}$ and b_{ij} ($i = 1, 2, \dots, j = 1, 2, \dots, m_i$).

Step 5. By solving the over-determined system of nonlinear algebraic equations by use of symbolic computation system *Maple*, we end up with the explicit expressions for $k, \mu_1, \mu_2, a_{i0}, a_{ij}$ and b_{ij} ($i = 1, 2, \dots, j = 1, 2, \dots, m_i$).

Step 6. According to system (2.2) and (2.4), the general solutions of system (2.5) and the conclusions in *Step 5*, we can obtain rational formal exact solutions of system (2.1).

Remark 1. The method proposed here is more general than various exacting method for finding exact solutions of nonlinear PDEs. We can easily draw this conclusion when F and G take particular variables in the following sections. The appeal and success of the method lies in the fact that writing the exact solutions of a nonlinear equation as polynomials of F and G whose derivatives are in closed form, the equation can be changed into a nonlinear system of algebraic equations. The system can be solved with help of symbolic computation.

Remark 2. Recently, there are a lot of papers of subequation method focus their attentions on generalize auxiliary equations or get more solutions of existing auxiliary equations to get more generalized solutions of target equations. But, to our knowledge, most of these methods can be summarized as above rational expansion form (2.4). And according to (2.4), we think we can get more generalized solutions of target equations by using simple auxiliary equation or simple solutions of auxiliary equation.

3. Jacobi elliptic function rational expansion method

In this section we would like to apply our method to obtain rational formal Jacobi elliptic function solutions of nonlinear PDEs, i.e. restricting F and G in Jacobi elliptic functions.

Here $\text{sn}\zeta$, $\text{cn}\zeta$, $\text{dn}\zeta$, $\text{ns}\zeta$, $\text{cs}\zeta$, $\text{sc}\zeta$, $\text{nc}\zeta$, $\text{sd}\zeta$ and $\text{nd}\zeta$ are the Jacobian elliptic sine function, the Jacobian elliptic cosine function and the Jacobian elliptic function of the third kind and other Jacobian functions which is denoted by Glaisher symbols and are generated by these three kinds of functions, namely [15],

$$\text{ns}\zeta = \frac{1}{\text{sn}\zeta}, \quad \text{nc}\zeta = \frac{1}{\text{cn}\zeta}, \quad \text{nd}\zeta = \frac{1}{\text{dn}\zeta}, \quad \text{sd}\zeta = \frac{\text{sn}\zeta}{\text{dn}\zeta}, \quad (3.1.1)$$

$$\text{sc}\zeta = \frac{\text{sn}\zeta}{\text{cn}\zeta}, \quad \text{cs}\zeta = \frac{\text{cn}\zeta}{\text{sn}\zeta}, \quad \text{ds}\zeta = \frac{\text{dn}\zeta}{\text{sn}\zeta}, \quad (3.1.2)$$

which are double periodic and posses the following properties:

1. Properties of triangular function

$$\text{cn}^2\zeta + \text{sn}^2\zeta = \text{dn}^2\zeta + m^2\text{sn}^2\zeta = 1, \quad (3.2.1)$$

$$\text{ns}^2\zeta = 1 + \text{cs}^2\zeta, \quad \text{ns}^2\zeta = m^2 + \text{ds}^2\zeta, \quad (3.2.2)$$

$$\text{sc}^2\zeta + 1 = \text{nc}^2\zeta, \quad m^2\text{sd}^2\zeta + 1 = \text{nd}^2\zeta. \quad (3.2.3)$$

2. Derivatives of the Jacobi elliptic functions

$$\text{sn}'\zeta = \text{cn}\zeta\text{dn}\zeta, \quad \text{cn}'\zeta = -\text{sn}\zeta\text{dn}\zeta, \quad \text{dn}'\zeta = -m^2\text{sn}\zeta\text{cn}\zeta, \quad (3.3.1)$$

$$\text{ns}'\zeta = -\text{ds}\zeta\text{cs}\zeta, \quad \text{ds}'\zeta = -\text{cs}\zeta\text{ns}\zeta, \quad \text{cs}'\zeta = -\text{ns}\zeta\text{ds}\zeta, \quad (3.3.2)$$

$$\text{sc}'\zeta = \text{nc}\zeta\text{dc}\zeta, \quad \text{nc}'\zeta = \text{sc}\zeta\text{dc}\zeta, \quad \text{cd}'\zeta = \text{cd}\zeta\text{nd}\zeta, \quad \text{nd}'\zeta = m^2\text{sd}\zeta\text{cd}\zeta, \quad (3.3.3)$$

where m is a modulus.

3. Properties of limit

$$\lim_{m \rightarrow 1} \text{sn}\zeta = \tanh \zeta, \quad \lim_{m \rightarrow 1} \text{cn}\zeta = \text{sech}\zeta, \quad \lim_{m \rightarrow 1} \text{dn}\zeta = \text{sech}\zeta, \quad (3.4.1)$$

$$\lim_{m \rightarrow 1} \text{ns}\zeta = \coth \zeta, \quad \lim_{m \rightarrow 1} \text{cs}\zeta = \text{csch}\zeta, \quad \lim_{m \rightarrow 1} \text{ds}\zeta = \text{csch}\zeta, \quad (3.4.2)$$

$$\lim_{m \rightarrow 0} \text{sn}\zeta = \sin \zeta, \quad \lim_{m \rightarrow 0} \text{cn}\zeta = \cos \zeta, \quad \lim_{m \rightarrow 0} \text{dn}\zeta = 1, \quad (3.4.3)$$

$$\lim_{m \rightarrow 0} \text{ns}\zeta = \csc \zeta, \quad \lim_{m \rightarrow 0} \text{cs}\zeta = \cot \zeta, \quad \lim_{m \rightarrow 0} \text{ds}\zeta = \csc \zeta. \quad (3.4.4)$$

The Jacobi–Glaisher functions for elliptic function can be found in Ref. [15]. It can easily be seen that the Jacobi elliptic functions satisfy system (2.5).

The six main steps of the Jacobi elliptic function (here just consider the condition $F = \text{sn}\zeta$ and $G = \text{cn}\zeta$) rational expansion method are illustrated with the classical Boussinesq equation, i.e.,

$$\begin{aligned} v_t + vv_x + w_x &= 0, \\ w_t + (wv)_x + \frac{1}{3}v_{xxx} &= 0, \end{aligned} \quad (3.5)$$

where $w - 1$ is the elevation of the water wave, v is the surface velocity of water along x -direction. The equation system (3.5) can be traced back to the works of Broer [16], Kaup [17], Jaulent–Miodek [18], Martinez [19], Kupershmidt [20], etc. A good understanding of all solutions of (3.5) is very helpful for coastal and civil engineers to apply the nonlinear water wave model in a harbor and coastal design. Therefore, finding more types of exact solutions of Eq. (3.5) is of fundamental interest in fluid dynamics. There are several papers devoted to this system of equations [21–24].

According to the *Step 1* in Section 2, we make the following travelling wave transformation:

$$v(x, t) = V(\xi), \quad w(x, t) = W(\xi), \quad \xi = k(x + \lambda t), \tag{3.6}$$

where k and λ are constants to be determined later, and thus system (3.5) becomes

$$\begin{aligned} \lambda V' + VW' + W' &= 0, \\ \lambda W' + (WV)' + \frac{k^2}{3} V''' &= 0. \end{aligned} \tag{3.7}$$

According to *Step 2* in Section 2, we expand the solution of system (3.7) in the form

$$\begin{aligned} V(\xi) &= a_0 + \sum_{j=1}^{m_v} \frac{a_j \operatorname{sn}^j(\xi) + b_j \operatorname{sn}^{j-1}(\xi) \operatorname{cn}(\xi)}{(\mu_1 \operatorname{sn}(\xi) + \mu_2 \operatorname{cn}(\xi) + 1)^j}, \\ W(\xi) &= A_0 + \sum_{j=1}^{m_w} \frac{A_j \operatorname{sn}^j(\xi) + B_j \operatorname{sn}^{j-1}(\xi) \operatorname{cn}(\xi)}{(\mu_1 \operatorname{sn}(\xi) + \mu_2 \operatorname{cn}(\xi) + 1)^j}. \end{aligned} \tag{3.8}$$

According to *Step 3* in Section 2, by balancing $V'''(\xi)$ and $(W(\xi)V(\xi))'$ in system (3.7) and by balancing $W'(\xi)$ and $V(\xi)V'(\xi)$ in system (3.7), we can obtain that $m_v = 1$ and $m_w = 2$. So we have

$$\begin{aligned} V(\xi) &= a_0 + \frac{a_1 \operatorname{sn}(\xi) + b_1 \operatorname{cn}(\xi)}{\mu_1 \operatorname{sn}(\xi) + \mu_2 \operatorname{cn}(\xi) + 1}, \\ W(\xi) &= A_0 + \frac{A_1 \operatorname{sn}(\xi) + B_1 \operatorname{cn}(\xi)}{\mu_1 \operatorname{sn}(\xi) + \mu_2 \operatorname{cn}(\xi) + 1} + \frac{A_2 \operatorname{sn}^2(\xi) + B_2 \operatorname{sn}(\xi) \operatorname{cn}(\xi)}{(\mu_1 \operatorname{sn}(\xi) + \mu_2 \operatorname{cn}(\xi) + 1)^2}, \end{aligned} \tag{3.9}$$

where $a_0, a_1, b_1, A_0, A_1, A_2, B_1, B_2, \mu_1$ and μ_2 are constants to be determined later.

According to *Step 4* in Section 2, with the aid of *Maple*, substituting (3.9) into (3.7), yields a set of algebraic equations for $\operatorname{sn}^i(\xi) \operatorname{cn}^j(\xi)$, ($i = 0, 1, \dots, j = 0, 1$). Setting the coefficients of these terms $\operatorname{sn}^i(\xi) \operatorname{cn}^j(\xi)$ of the resulting system's numerator to be zero yields a set of over-determined algebraic equations with respect to $a_0, a_1, b_1, A_0, A_1, B_1, A_2, B_2, k$ and λ .

According to *Step 5* in Section 2, by use of the *Maple* soft package ‘‘Charsets’’ by Dongming Wang, which is based on the Wu-elimination method [25], solving the over-determined algebraic equations, we get explicit expressions for $a_0, a_1, b_1, A_0, A_1, B_1, A_2, B_2, \mu_1, \mu_2, k$ and λ .

According to *Step 6* in Section 2, we get following solutions of the classical Boussinesq equation:

Family 1

$$\begin{aligned} v_1 &= \frac{\pm(k\mu_2^2 m^2 - k\mu_2^3 - k\mu_2 m^2 + k\mu_2) + \lambda \sqrt{-3\mu_2^4 m^2 + 3\mu_2^4 + 6\mu_2^2 m^2 - 3\mu_2^2 - 3m^2}}{\sqrt{-3\mu_2^4 m^2 + 3\mu_2^4 + 6\mu_2^2 m^2 - 3\mu_2^2 - 3m^2}} \\ &\pm \frac{\sqrt{3\mu_2^2 - 3\mu_2^2 m^2 + 3m^2 k} \operatorname{sn}(\xi)}{3(1 + \mu_2 \operatorname{cn}(\xi))} \pm \frac{\sqrt{-3\mu_2^4 m^2 + 3\mu_2^4 + 6\mu_2^2 m^2 - 3\mu_2^2 - 3m^2 k} \operatorname{cn}(\xi)}{3(1 + \mu_2 \operatorname{cn}(\xi))}, \end{aligned} \tag{3.10.1}$$

$$\begin{aligned} w_1 &= \frac{1}{3} \frac{k^2 m^2}{\mu_2^2 - \mu_2^2 m^2 + m^2} + \frac{(-k^2 \mu_2^2 + k^2 \mu_2^2 m^2 - k^2 m^2) \operatorname{sn}^2(\xi)}{3(1 + \mu_2 \operatorname{cn}(\xi))^2} \\ &- \frac{1}{3} \frac{k^2 \mu_2 \operatorname{cn}(\xi)}{1 + \mu_2 \operatorname{cn}(\xi)} + \frac{1}{3} \frac{\sqrt{\mu_2^2 - \mu_2^2 m^2 + m^2 k^2} \mu_2 (\mu_2^2 m^2 - \mu_2^2 - m^2 + 1) \operatorname{sn}(\xi)}{\sqrt{-\mu_2^4 m^2 + \mu_2^4 + 2\mu_2^2 m^2 - \mu_2^2 - m^2} (1 + \mu_2 \operatorname{cn}(\xi))} \\ &\pm \frac{\sqrt{\mu_2^2 - \mu_2^2 m^2 + m^2 k^2} \sqrt{-\mu_2^4 m^2 + \mu_2^4 + 2\mu_2^2 m^2 - \mu_2^2 - m^2} \operatorname{sn}(\xi) \operatorname{cn}(\xi)}{3(1 + \mu_2 \operatorname{cn}(\xi))^2}, \end{aligned} \tag{3.10.2}$$

where $\xi = k(x + \lambda t)$, k, μ_2 and λ are arbitrary constants.

Family 2

$$v_2 = -\frac{\lambda\sqrt{3\mu_2^4 - 3\mu_2^4m^2 + 6\mu_2^2m^2 - 3\mu_2^2 - 3m^2} \pm (2k\mu_2^3m^2 - 2k\mu_2^3 - 2k\mu_2m^2 + k\mu_2)}{\sqrt{3\mu_2^4 - 3\mu_2^4m^2 + 6\mu_2^2m^2 - 3\mu_2^2 - 3m^2}} \tag{3.11.1}$$

$$\pm 2\frac{\sqrt{3\mu_2^4 - 3\mu_2^4m^2 + 6\mu_2^2m^2 - 3\mu_2^2 - 3m^2}k \operatorname{cn}(\xi)}{3(1 + \mu_2 \operatorname{cn}(\xi))},$$

$$w_2 = \frac{k(m^2 - 2\mu_2^2m^2 - \mu_2^4 + \mu_2^4m^2)}{\mu_2^4m^2 - \mu_2^4 - 2\mu_2^2m^2 + \mu_2^2 + m^2} - \frac{2\mu_2k^2 \operatorname{cn}(\xi)}{3(1 + \mu_2 \operatorname{cn}(\xi))} + 2\frac{(k^2\mu_2^2m^2 - k^2\mu_2^2 - k^2m^2)\operatorname{sn}^2(\xi)}{3(1 + \mu_2 \operatorname{cn}(\xi))^2}, \tag{3.11.2}$$

where $\xi = k(x + \lambda t)$, k , μ_2 and λ are arbitrary constants.

Family 3

$$v_3 = \frac{\pm(k\mu_1m^2 - 2k\mu_1^3 + k\mu_1) - \lambda\sqrt{3m^2 + 3\mu_1^4 - 3m^2\mu_1^2 - 3\mu_1^2}}{\sqrt{3m^2 + 3\mu_1^4 - 3m^2\mu_1^2 - 3\mu_1^2}} \pm 2\frac{\sqrt{3m^2 + 3\mu_1^4 - 3m^2\mu_1^2 - 3\mu_1^2}k \operatorname{sn}(\xi)}{3(\mu_1 \operatorname{sn}(\xi) + 1)}, \tag{3.12.1}$$

$$w_3 = -\frac{k^2(-6m^2\mu_1^2 + 3\mu_1^4m^2 + 3\mu_1^4 - 2\mu_1^6 + m^4 + m^2)}{3(-m^2 - \mu_1^4 + m^2\mu_1^2 + \mu_1^2)}$$

$$+ \frac{(-2k^2\mu_1m^2 - 2k^2\mu_1 + 4k^2\mu_1^3)\operatorname{sn}(\xi)}{3(\mu_1 \operatorname{sn}(\xi) + 1)} - \frac{(2k^2m^2 + 2k^2\mu_1^4 - 2k^2\mu_1^2m^2 - 2k^2\mu_1^2)\operatorname{sn}^2(\xi)}{3(\mu_1 \operatorname{sn}(\xi) + 1)^2}, \tag{3.12.2}$$

where $\xi = k(x + \lambda t)$, k , μ_1 and λ are arbitrary constants.

Family 4

$$v_4 = \frac{\pm(k\mu_1 - k\mu_1^3) - \lambda\sqrt{3\mu_1^4 - 3\mu_1^2 - 3\mu_1^2m^2 + 3m^2}}{\sqrt{3\mu_1^4 - 3\mu_1^2 - 3\mu_1^2m^2 + 3m^2}} \tag{3.13.1}$$

$$\pm \frac{\sqrt{3\mu_1^4 - 3\mu_1^2 - 3\mu_1^2m^2 + 3m^2}k \operatorname{sn}(\xi)}{3(\mu_1 \operatorname{sn}(\xi) + 1)} \pm \frac{\sqrt{3\mu_1^2 - 3m^2}k \operatorname{cn}(\xi)}{3(\mu_1 \operatorname{sn}(\xi) + 1)},$$

$$w_4 = -\frac{k^2(\mu_1^4 - 2\mu_1^2m^2 + m^2)}{3(\mu_1^2 - m^2)} - \frac{(k^2\mu_1 - 2k^2\mu_1^3 + k^2\mu_1m^2)\operatorname{sn}(\xi)}{3(\mu_1 \operatorname{sn}(\xi) + 1)} \pm \frac{1}{3}\frac{\sqrt{\mu_1^2 - m^2}k^2\mu_1(-1 + \mu_1^2)\operatorname{cn}(\xi)}{\sqrt{\mu_1^4 - \mu_1^2 - \mu_1^2m^2 + m^2}(\mu_1 \operatorname{sn}(\xi) + 1)}$$

$$- \frac{(k^2\mu_1^4 - k^2\mu_1^2 - k^2\mu_1^2m^2 + k^2m^2)\operatorname{sn}^2(\xi)}{3(\mu_1 \operatorname{sn}(\xi) + 1)^2} \pm \frac{\sqrt{\mu_1^4 - \mu_1^2 - \mu_1^2m^2 + m^2}k^2\sqrt{\mu_1^2 - m^2}\operatorname{sn}(\xi)\operatorname{cn}(\xi)}{3(\mu_1 \operatorname{sn}(\xi) + 1)^2}, \tag{3.13.2}$$

where $\xi = k(x + \lambda t)$, k , μ_1 and λ are arbitrary constants.

Family 5

$$v_5 = a_0 \pm \frac{(2\lambda + 2a_0)\operatorname{sn}\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right)}{\pm \operatorname{sn}\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right) \pm \operatorname{cn}\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right) + 1}, \tag{3.14.1}$$

$$w_5 = A_0 \pm \frac{2(A_0m^2 - A_0 + \lambda^2m^2 - 2a_0\lambda m^2 - a_0^2m^2)\operatorname{sn}\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right)}{(m^2 - 1)\left(\pm \operatorname{sn}\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right) \pm \operatorname{cn}\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right) + 1\right)}$$

$$\pm \frac{2(2a_0\lambda a_0^2 - A_0m^2 + A_0 + \lambda^2)\operatorname{cn}\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right)}{(m^2 - 1)\left(\pm \operatorname{sn}\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right) \pm \operatorname{cn}\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right) + 1\right)}$$

$$- \frac{(2\lambda^2 + 4a_0\lambda + 2a_0^2)\operatorname{sn}^2\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right)}{\left(\pm \operatorname{sn}\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right) \pm \operatorname{cn}\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right) + 1\right)^2}$$

$$\pm 2\frac{(2a_0\lambda + a_0^2 - A_0m^2 + A_0 + \lambda^2)\operatorname{sn}\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right)\operatorname{cn}\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right)}{(m^2 - 1)\left(\pm \operatorname{sn}\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right) \pm \operatorname{cn}\left(\pm 3\frac{(\lambda+a_0)(x+\lambda t)}{\sqrt{3-3m^2}}\right) + 1\right)^2}, \tag{3.14.2}$$

where a_0 , A_0 and λ are arbitrary constants.

Remark 3. We can easily conclude that when $\mu_2 = 0$, all solutions in [24], can be recovered by our method. The other solutions obtained here, to our knowledge, are all new families of rational formal periodic solution of the classical Boussinesq equation.

Remark 4. If we replace the Jacobi elliptic functions $\text{sn}(\xi)$, $\text{cn}(\xi)$ in the ansätze (3.8) with other pairs of Jacobi elliptic functions such as $\text{sn}(\xi)$ and $\text{dn}(\xi)$; $\text{ns}(\xi)$ and $\text{cs}(\xi)$; $\text{ns}(\xi)$ and $\text{ds}(\xi)$; $\text{sc}(\xi)$ and $\text{nc}(\xi)$; $\text{dc}(\xi)$ and $\text{nc}(\xi)$; $\text{sd}(\xi)$ and $\text{nd}(\xi)$; $\text{cd}(\xi)$ and $\text{nd}(\xi)$ (It is necessary to point out that they are satisfying system (2.5).), therefore other new periodic wave solutions can be obtained for some system. For simplicity, we omit them here.

4. Riccati equation rational expansion method

In this section we would like apply our method to obtain rational form soliton solutions of nonlinear PDEs, i.e., restricting F and G in the solutions of the Riccati equation.

The six main steps of the Riccati equation rational expansion method are illustrated with the $(2 + 1)$ -dimensional dispersive long wave equation (DLWE), i.e.,

$$\begin{aligned} u_{yt} + v_{xx} + (uu_x)_y &= 0, \\ v_t + u_x + (uv)_x + u_{xy} &= 0. \end{aligned} \tag{4.1}$$

The $(2 + 1)$ -dimensional DLWE (4.1) was first derived by Boiti et al. [26] as a compatibility for a “weak” Lax pair. Recently considerable effort has been devoted to the study of this system. In [27], Paquin and Winternitz showed that the symmetry algebra of $(2 + 1)$ -dimensional DLWE (4.1) is infinite-dimensional and possesses a Kac–Moody–Virasoro structure. Some special similarity solutions are also given in [27] by using symmetry algebra and the classical theoretical analysis. The more general symmetry algebra, w_∞ symmetry algebra, is given in [28]. Lou [29] has given nine types of the two-dimensional partial differential equation reductions and thirteen types of the ordinary differential equation reductions by means of the direct and nonclassical method. The system (4.1) have no Painleve property though they are Lax or IST integrable [30]. More recently, Tang et al. [31], by means of the variable separation approach, the abundant localized coherent structures of the system (4.1) are given out. In [32], the possible chaotic and fractal localized structures are revealed for the system (4.1).

According to the *Step 1* in Section 2, we make the following travelling wave transformation

$$u(x, t) = U(\xi), \quad v(x, t) = V(\xi), \quad \xi = k(x + ly + \lambda t), \tag{4.2}$$

where k , l and λ are constants to be determined later, and thus system (4.1) becomes

$$\begin{aligned} \lambda l U'' + V'' + l U'^2 + l U U'' &= 0, \\ \lambda V' + U' + (UV)' + k^2 l U''' &= 0. \end{aligned} \tag{4.3}$$

According to *Step 2* in Section 2, We suppose that DLWE has the following formal travelling wave solution:

$$\begin{aligned} U(\xi) &= a_0 + \sum_{i=1}^{m_u} \frac{a_i \phi^i(\xi) + b_i \phi^{i-1}(\xi) \sqrt{R + \phi^2(\xi)}}{(\mu_1 \phi(\xi) + \mu_2 \sqrt{R + \phi^2(\xi)} + 1)^i}, \\ V(\xi) &= A_0 + \sum_{i=1}^{m_v} \frac{A_i \phi^i(\xi) + B_i \phi^{i-1}(\xi) \sqrt{R + \phi^2(\xi)}}{(\mu_1 \phi(\xi) + \mu_2 \sqrt{R + \phi^2(\xi)} + 1)^i}, \end{aligned} \tag{4.4}$$

and the new variable $\phi = \phi(\xi)$ satisfying the Riccati equation, i.e.,

$$\phi' - (R + \phi^2) = \frac{d\phi}{d\xi} - (R + \phi^2) = 0, \tag{4.5}$$

where R , a_0 , a_i , b_i , A_0 , A_i and B_i ($i = 1, 2, \dots, m_i$) are constants to be determined later.

For the DLWE, According to *Step 3* in Section 2, by balancing the highest nonlinear terms and the highest order partial derivative terms in (4.3), gives $m_u = 1$ and $m_v = 2$. So we have

$$U(\xi) = a_0 + \frac{a_1\phi(\xi) + b_1\sqrt{R + \phi^2(\xi)}}{\mu_1\phi(\xi) + \mu_2\sqrt{R + \phi^2(\xi) + 1}}, \tag{4.6}$$

$$V(\xi) = A_0 + \frac{A_1\phi(\xi) + B_1\sqrt{R + \phi^2(\xi)}}{\mu_1\phi(\xi) + \mu_2\sqrt{R + \phi^2(\xi) + 1}} + \frac{A_2\phi^2(\xi) + B_2\phi(\xi)\sqrt{R + \phi^2(\xi)}}{(\mu_1\phi(\xi) + \mu_2\sqrt{R + \phi^2(\xi) + 1})^2}.$$

According to Step 4 in Section 2, with the aid of Maple, substituting (4.6) along with (4.5) into (4.3), yields a set of algebraic equations for $\phi^i(\xi)(\sqrt{R + \phi^2(\xi)})^j$, ($i = 0, 1, \dots, j = 0, 1$). Setting the coefficients of these terms $\phi^i(\xi)(\sqrt{R + \phi^2(\xi)})^j$ of the resulting system’s numerator to zero yields a set of over-determined algebraic equations.

According to Step 5 in Section 2, by use of the Maple soft package “Charsets” by Dongming Wang, we get the explicit expressions for $a_0, a_1, b_1, A_0, A_1, B_1, A_2, B_2, \mu_1, \mu_2, k$ and λ .

It is well known that the general solutions of Eq. (4.5) are

(1) when $R < 0$

$$\phi(\xi) = -\sqrt{-R} \tanh(\sqrt{-R}\xi), \quad \phi(\xi) = -\sqrt{-R} \coth(\sqrt{-R}\xi) \tag{4.7}$$

(2) when $R = 0$

$$\phi(\xi) = -\frac{1}{\xi}, \tag{4.8}$$

(3) when $R > 0$,

$$\phi(\xi) = \sqrt{R} \tan(\sqrt{R}\xi), \quad \phi(\xi) = -\sqrt{R} \cot(\sqrt{R}\xi). \tag{4.9}$$

Thus according to system (4.2), (4.6)–(4.9) and the conclusions in Step 5, we can obtain following rational formal travelling-wave solutions of the (2 + 1)-dimension DLWE.

Note: Since tan- and cot-type solution appear in pairs with tanh- and coth-type solutions, respectively, we omit them in this paper, i.e., just considering the condition $R < 0$. In addition, some rational solutions are also omitted.

Family 1

$$u_{11} = \frac{-\lambda\sqrt{1 - \mu_2^2 R} \pm k\mu_2 R}{\sqrt{1 - \mu_2^2 R}} \pm 2 \frac{\sqrt{1 - \mu_2^2 R} k \sqrt{-R} \operatorname{isech}(\sqrt{-R}k(x + ly + \lambda t))}{1 \pm \mu_2 \sqrt{-R} \operatorname{isech}(\sqrt{-R}k(x + ly + \lambda t))}, \tag{4.10.1}$$

$$v_{11} = \frac{2lk^2\mu_2^2 R^2 - k^2 lR - 1 + \mu_2^2 R}{1 - \mu_2^2 R} \pm 2 \frac{lk^2\mu_2 R \sqrt{-R} \operatorname{isech}(\sqrt{-R}k(x + ly + \lambda t))}{1 \pm \mu_2 \sqrt{-R} \operatorname{isech}(\sqrt{-R}k(x + ly + \lambda t))} + 2 \frac{k^2 lR \tanh^2(\sqrt{-R}k(x + ly + \lambda t))}{(1 \pm \mu_2 \sqrt{-R} \operatorname{isech}(\sqrt{-R}k(x + ly + \lambda t)))^2}, \tag{4.10.2}$$

$$u_{12} = \frac{-\lambda\sqrt{1 - \mu_2^2 R} \pm k\mu_2 R}{\sqrt{1 - \mu_2^2 R}} \pm 2 \frac{\sqrt{1 - \mu_2^2 R} k \sqrt{-R} \operatorname{csch}(\sqrt{-R}k(x + ly + \lambda t))}{1 \pm \mu_2 R \operatorname{csch}(\sqrt{-R}k(x + ly + \lambda t))}, \tag{4.11.1}$$

$$v_{12} = \frac{2lk^2\mu_2^2 R^2 - k^2 lR - 1 + \mu_2^2 R}{1 - \mu_2^2 R} \pm 2 \frac{lk^2\mu_2 R \sqrt{-R} \operatorname{csch}(\sqrt{-R}k(x + ly + \lambda t))}{1 \pm \mu_2 \sqrt{-R} \operatorname{csch}(\sqrt{-R}k(x + ly + \lambda t))} + 2 \frac{k^2 lR \coth^2(\sqrt{-R}k(x + ly + \lambda t))}{(1 \pm \mu_2 \sqrt{-R} \operatorname{csch}(\sqrt{-R}k(x + ly + \lambda t)))^2}, \tag{4.11.2}$$

where $R < 0, \mu_2, k, l$ and λ are arbitrary constants.

Family 2

$$u_{21} = -\lambda + \frac{-k\sqrt{-R} \tanh(\sqrt{-R}\xi) \pm \sqrt{1 - \mu_2^2 R} k \sqrt{-R} \operatorname{isech}(\sqrt{-R}\xi)}{1 \pm \mu_2 \sqrt{-R} \operatorname{isech}(\sqrt{-R}\xi)}, \tag{4.12.1}$$

$$v_{21} = -k^2 lR - 1 \pm \frac{lk^2 \mu_2 R \sqrt{-R} \operatorname{isech}(\sqrt{-R}\xi)}{1 \pm \mu_2 \sqrt{-R} \operatorname{isech}(\sqrt{-R}\xi)} + \frac{lk^2 R \tanh^2(\sqrt{-R}\xi) \pm lk^2 \sqrt{1 - \mu_2^2 R} \tanh(\sqrt{-R}\xi) R \operatorname{isech}(\sqrt{-R}\xi)}{(1 \pm \mu_2 \sqrt{-R} \operatorname{isech}(\sqrt{-R}\xi))^2}, \tag{4.12.2}$$

$$u_{22} = -\lambda + \frac{-k\sqrt{-R} \coth(\sqrt{-R}\xi) \pm \sqrt{1 - \mu_2^2 R} k \sqrt{-R} R \operatorname{csch}(\sqrt{-R}\xi)}{1 \pm \mu_2 \sqrt{-R} R \operatorname{csch}(\sqrt{-R}\xi)}, \tag{4.13.1}$$

$$v_{22} = -k^2 lR - 1 \pm \frac{lk^2 \mu_2 R \sqrt{-R} R \operatorname{csch}(\sqrt{-R}\xi)}{1 \pm \mu_2 \sqrt{-R} R \operatorname{csch}(\sqrt{-R}\xi)} + \frac{lk^2 R \coth^2(\sqrt{-R}\xi) \pm lk^2 \sqrt{1 - \mu_2^2 R} \tanh(\sqrt{-R}\xi) R \operatorname{csch}(\sqrt{-R}\xi)}{(1 \pm \mu_2 \sqrt{-R} R \operatorname{csch}(\sqrt{-R}\xi))^2}, \tag{4.13.2}$$

where $\xi = k(x + ly + \lambda t)$, $R < 0$, μ_2 , k , l and λ are arbitrary constants.

Family 3

$$u_{31} = \mp 2\mu_1 Rk - \lambda \pm \frac{(2k + 2)Rk\mu_1^2 \sqrt{-R} \tanh(\sqrt{-R}\xi)}{-\mu_1 \sqrt{-R} \tanh(\sqrt{-R}\xi) + 1}, \tag{4.14.1}$$

$$v_{31} = -2k^2 l\mu_1^2 R^2 - 2k^2 lR - 1 - 4 \frac{l(\mu_1^2 R + 1)k^2 R \mu_1 \sqrt{-R} \tanh(\sqrt{-R}\xi)}{-\mu_1 \sqrt{-R} \tanh(\sqrt{-R}\xi) + 1} + 2 \frac{l(\mu_1^2 R + 1)^2 k^2 R \tanh^2(\sqrt{-R}\xi)}{(-\mu_1 \sqrt{-R} \tanh(\sqrt{-R}\xi) + 1)^2}, \tag{4.14.2}$$

$$u_{32} = \mp 2\mu_1 Rk - \lambda \pm \frac{(2k + 2)Rk\mu_1^2 \sqrt{-R} \coth(\sqrt{-R}\xi)}{-\mu_1 \sqrt{-R} \coth(\sqrt{-R}\xi) + 1}, \tag{4.15.1}$$

$$v_{32} = -2k^2 l\mu_1^2 R^2 - 2k^2 lR - 1 - 4 \frac{l(\mu_1^2 R + 1)k^2 R \mu_1 \sqrt{-R} \coth(\sqrt{-R}\xi)}{-\mu_1 \sqrt{-R} \coth(\sqrt{-R}\xi) + 1} + 2 \frac{l(\mu_1^2 R + 1)^2 k^2 R \coth^2(\sqrt{-R}\xi)}{(-\mu_1 \sqrt{-R} \coth(\sqrt{-R}\xi) + 1)^2}, \tag{4.15.2}$$

where $\xi = k(x + ly + \lambda t)$, $R < 0$, μ_1 , k , l and λ are arbitrary constants.

Family 4

$$u_{41} = -\lambda \pm 2 \frac{\sqrt{\mu_1^2 R + 1} k \sqrt{-R} \operatorname{isech}(\sqrt{-R}\xi)}{-\mu_1 \sqrt{-R} \tanh(\sqrt{-R}\xi) + 1}, \tag{4.16.1}$$

$$v_{41} = -2k^2 l\mu_1^2 R^2 - k^2 lR - 1 - 4 \frac{l(\mu_1^2 R + 1)k^2 R \mu_1 \sqrt{-R} \tanh(\sqrt{-R}\xi)}{-\mu_1 \sqrt{-R} \tanh(\sqrt{-R}\xi) + 1} + 2 \frac{l(\mu_1^2 R + 1)^2 k^2 R \tanh^2(\sqrt{-R}\xi)}{(-\mu_1 \sqrt{-R} \tanh(\sqrt{-R}\xi) + 1)^2}, \tag{4.16.2}$$

$$u_{42} = -\lambda \pm 2 \frac{\sqrt{\mu_1^2 R + 1} k \sqrt{-R} R \operatorname{csch}(\sqrt{-R}\xi)}{-\mu_1 \sqrt{-R} \coth(\sqrt{-R}\xi) + 1}, \tag{4.17.1}$$

$$v_{42} = -2k^2 l\mu_1^2 R^2 - k^2 lR - 1 - 4 \frac{l(\mu_1^2 R + 1)k^2 R \mu_1 \sqrt{-R} \coth(\sqrt{-R}\xi)}{-\mu_1 \sqrt{-R} \coth(\sqrt{-R}\xi) + 1} + 2 \frac{l(\mu_1^2 R + 1)^2 k^2 R \coth^2(\sqrt{-R}\xi)}{(-\mu_1 \sqrt{-R} \coth(\sqrt{-R}\xi) + 1)^2}, \tag{4.17.2}$$

where $\xi = k(x + ly + \lambda t)$, $R < 0$, μ_1 , k , l and λ are arbitrary constants.

Family 5

$$u_{51} = -\mu_1 Rk - \lambda - \frac{(k + Rk\mu_1^2) \sqrt{-R} \tanh(\sqrt{-R}\xi) \pm \sqrt{\mu_1^2 R + 1} k \sqrt{-R} \operatorname{isech}(\sqrt{-R}\xi)}{-\mu_1 \sqrt{-R} \tanh(\sqrt{-R}\xi) + 1}, \tag{4.18.1}$$

$$v_{51} = \frac{-(2lk^2R^2\mu_1^3 + 2lk^2R\mu_1)\sqrt{-R} \tanh(\sqrt{-R}\xi) \pm \mu_1 R l \sqrt{\mu_1^2 R + 1} k^2 \sqrt{-R} \operatorname{isech}(\sqrt{-R}\xi)}{-\mu_1 \sqrt{-R} \tanh(\sqrt{-R}\xi) + 1} - \frac{k^2 l R - k^2 l \mu_1^2 R^2 - 1 - \frac{(-k^2 l - 2k^2 l R \mu_1^2 - k^2 l R^2 \mu_1^4) R \tanh^2(\sqrt{-R}\xi)}{(-\mu_1 \sqrt{-R} \tanh(\sqrt{-R}\xi) + 1)^2}}{(-\mu_1 \sqrt{-R} \tanh(\sqrt{-R}\xi) + 1)^2} \pm \frac{(l \sqrt{\mu_1^2 R + 1} k^2 + l \sqrt{\mu_1^2 R + 1} k^2 R \mu_1^2) \tanh(\sqrt{-R}\xi) R \operatorname{isech}(\sqrt{-R}\xi)}{(-\mu_1 \sqrt{-R} \tanh(\sqrt{-R}\xi) + 1)^2}, \tag{4.18.2}$$

$$u_{52} = -\mu_1 R k - \lambda - \frac{(k + R k \mu_1^2) \sqrt{-R} \coth(\sqrt{-R}\xi) \pm \sqrt{\mu_1^2 R + 1} k \sqrt{-R} \operatorname{csch}(\sqrt{-R}\xi)}{-\mu_1 \sqrt{-R} \coth(\sqrt{-R}\xi) + 1}, \tag{4.19.1}$$

$$v_{52} = \frac{-(2lk^2R^2\mu_1^3 + 2lk^2R\mu_1)\sqrt{-R} \coth(\sqrt{-R}\xi) \pm \mu_1 R l \sqrt{\mu_1^2 R + 1} k^2 \sqrt{-R} \operatorname{csch}(\sqrt{-R}\xi)}{-\mu_1 \sqrt{-R} \coth(\sqrt{-R}\xi) + 1} - \frac{k^2 l R - k^2 l \mu_1^2 R^2 - 1 - \frac{(-k^2 l - 2k^2 l R \mu_1^2 - k^2 l R^2 \mu_1^4) R \coth^2(\sqrt{-R}\xi)}{(-\mu_1 \sqrt{-R} \coth(\sqrt{-R}\xi) + 1)^2}}{(-\mu_1 \sqrt{-R} \coth(\sqrt{-R}\xi) + 1)^2} \pm \frac{(l \sqrt{\mu_1^2 R + 1} k^2 + l \sqrt{\mu_1^2 R + 1} k^2 R \mu_1^2) \coth(\sqrt{-R}\xi) R \operatorname{csch}(\sqrt{-R}\xi)}{(-\mu_1 \sqrt{-R} \coth(\sqrt{-R}\xi) + 1)^2}, \tag{4.19.2}$$

where $\xi = k(x + ly + \lambda t)$, $R < 0$, μ_2 , k , l and λ are arbitrary constants.

Family 6

$$u_{61} = -\frac{a_1 R + \lambda + R \lambda}{R + 1} + \frac{-a_1 \sqrt{-R} \tanh(\sqrt{-R}\xi) \pm a_1 \sqrt{-R} \operatorname{isech}(\sqrt{-R}\xi)}{\pm \sqrt{-R} \tanh(\sqrt{-R}\xi) \pm \sqrt{-R} \operatorname{isech}(\sqrt{-R}\xi) + 1}, \tag{4.20.1}$$

$$v_{61} = \frac{-R^2 + la_1^2 R + 1}{(-1 + R)(R + 1)} + \frac{2la_1^2 R \sqrt{-R} \tanh(\sqrt{-R}\xi) \pm 2la_1^2 R \sqrt{-R} \operatorname{isech}(\sqrt{-R}\xi)}{(R^2 - 1)(\pm \sqrt{-R} \tanh(\sqrt{-R}\xi) \pm \sqrt{-R} \operatorname{isech}(\sqrt{-R}\xi) + 1)} - \frac{la_1^2 R \tanh^2(\sqrt{-R}\xi) \pm la_1^2 R i \tanh(\sqrt{-R}\xi) \operatorname{sech}(\sqrt{-R}\xi)}{(R - 1)(\pm \sqrt{-R} \tanh(\sqrt{-R}\xi) \pm \sqrt{-R} \operatorname{isech}(\sqrt{-R}\xi) + 1)^2}, \tag{4.20.2}$$

$$u_{62} = -\frac{a_1 R + \lambda + R \lambda}{R + 1} + \frac{-a_1 \sqrt{-R} \coth(\sqrt{-R}\xi) \pm a_1 \sqrt{-R} \operatorname{csch}(\sqrt{-R}\xi)}{\pm \sqrt{-R} \coth(\sqrt{-R}\xi) \pm \sqrt{-R} \operatorname{csch}(\sqrt{-R}\xi) + 1}, \tag{4.21.1}$$

$$v_{62} = \frac{-R^2 + la_1^2 R + 1}{(-1 + R)(R + 1)} + \frac{2la_1^2 R \sqrt{-R} \coth(\sqrt{-R}\xi) \pm 2la_1^2 R \sqrt{-R} \operatorname{csch}(\sqrt{-R}\xi)}{(R^2 - 1)(\pm \sqrt{-R} \coth(\sqrt{-R}\xi) \pm \sqrt{-R} \operatorname{csch}(\sqrt{-R}\xi) + 1)} - \frac{la_1^2 R \coth^2(\sqrt{-R}\xi) \pm la_1^2 R \coth(\sqrt{-R}\xi) \operatorname{csch}(\sqrt{-R}\xi)}{(R - 1)(\pm \sqrt{-R} \coth(\sqrt{-R}\xi) \pm \sqrt{-R} \operatorname{csch}(\sqrt{-R}\xi) + 1)^2}, \tag{4.21.2}$$

where $\xi = k(x + ly + \lambda t)$, $k = \pm \frac{a_1}{R+1}$, $R < 0$, a_1 , l and λ are arbitrary constants.

Family 7

$$u_{71} = -\frac{a_1 R + \lambda + R \lambda}{R + 1} - \frac{a_1 (R + 1) \sqrt{-R} \tanh(\sqrt{-R}\xi) \pm a_1 (R - 1) \sqrt{-R} \operatorname{isech}(\sqrt{-R}\xi)}{(R + 1)(\pm \sqrt{-R} \tanh(\sqrt{-R}\xi) \pm \sqrt{-R} \operatorname{isech}(\sqrt{-R}\xi) + 1)}, \tag{4.22.1}$$

$$v_{71} = -\frac{R^2 + 2R + la_1^2 R + 1}{(R + 1)^2} - 2 \frac{la_1^2 R \sqrt{-R} \tanh(\sqrt{-R}\xi)}{(R + 1)^2 (\pm \sqrt{-R} \tanh(\sqrt{-R}\xi) \pm \sqrt{-R} \operatorname{isech}(\sqrt{-R}\xi) + 1)} + \frac{(R + 1) la_1^2 R \tanh^2(\sqrt{-R}\xi) \pm la_1^2 (R - 1) \tanh(\sqrt{-R}\xi) R \operatorname{isech}(\sqrt{-R}\xi)}{(R + 1)^2 (\pm \sqrt{-R} \tanh(\sqrt{-R}\xi) \pm \sqrt{-R} \operatorname{isech}(\sqrt{-R}\xi) + 1)^2}, \tag{4.22.2}$$

$$u_{72} = -\frac{a_1 R + \lambda + R \lambda}{R + 1} - \frac{a_1 (R + 1) \sqrt{-R} \coth(\sqrt{-R}\xi) \pm a_1 (R - 1) \sqrt{-R} \operatorname{csch}(\sqrt{-R}\xi)}{(R + 1)(\pm \sqrt{-R} \coth(\sqrt{-R}\xi) \pm \sqrt{-R} \operatorname{csch}(\sqrt{-R}\xi) + 1)}, \tag{4.23.1}$$

$$v_{72} = -\frac{R^2 + 2R + la_1^2R + 1}{(R + 1)^2} - 2\frac{la_1^2R\sqrt{-R}\coth(\sqrt{-R}\xi)}{(R + 1)^2(\pm\sqrt{-R}\coth(\sqrt{-R}\xi) \pm \sqrt{-R}\operatorname{Rcsch}(\sqrt{-R}\xi) + 1)} + \frac{(R + 1)la_1^2R\coth^2(\sqrt{-R}\xi) \pm la_1^2(R - 1)\coth(\sqrt{-R}\xi)\operatorname{Rcsch}(\sqrt{-R}\xi)}{(R + 1)^2(\pm\sqrt{-R}\coth(\sqrt{-R}\xi) \pm \sqrt{-R}\operatorname{Rcsch}(\sqrt{-R}\xi) + 1)^2}, \tag{4.23.2}$$

where $\xi = k(x + ly + \lambda t)$, $k = \pm \frac{a_1}{R+1}$, $R < 0$, a_1 , l and λ are arbitrary constants.

Family 8

$$u_{81} = \frac{1}{2}b_1R - \lambda \pm \frac{b_1\sqrt{-R}\operatorname{isech}(\sqrt{-R}\xi)}{\pm\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm \sqrt{-R}\operatorname{isech}(\sqrt{-R}\xi) + 1}, \tag{4.24.1}$$

$$v_{81} = \frac{4 + lb_1^2R^3 + lb_1^2R - 4R}{4(R - 1)} + \frac{2lb_1^2R\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm lb_1^2R\sqrt{-R}i(R + 1)\operatorname{sech}(\sqrt{-R}\xi)}{2(R - 1)(\pm\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm \sqrt{-R}\operatorname{isech}(\sqrt{-R}\xi) + 1)} - \frac{lb_1^2(R + 1)R\tanh^2(\sqrt{-R}\xi) \pm 2lb_1^2R^2i\tanh(\sqrt{-R}\xi)\operatorname{sech}(\sqrt{-R}\xi)}{2(R - 1)(\pm\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm \sqrt{-R}\operatorname{isech}(\sqrt{-R}\xi) + 1)^2}, \tag{4.24.2}$$

$$u_{82} = \frac{1}{2}b_1R - \lambda \pm \frac{b_1\sqrt{-R}\operatorname{csch}(\sqrt{-R}\xi)}{\pm\sqrt{-R}\tanh(\sqrt{-R}\xi) \pm \sqrt{-R}\operatorname{csch}(\sqrt{-R}\xi) + 1}, \tag{4.25.1}$$

$$v_{82} = \frac{4 + lb_1^2R^3 + lb_1^2R - 4R}{4(R - 1)} + \frac{2lb_1^2R\sqrt{-R}\coth(\sqrt{-R}\xi) + lb_1^2R\sqrt{-R}(R + 1)\operatorname{csch}(\sqrt{-R}\xi)}{2(R - 1)(\pm\sqrt{-R}\coth(\sqrt{-R}\xi) \pm \sqrt{-R}\operatorname{csch}(\sqrt{-R}\xi) + 1)} - \frac{lb_1^2(R + 1)R\coth^2(\sqrt{-R}\xi) \pm 2lb_1^2R^2\coth(\sqrt{-R}\xi)\operatorname{csch}(\sqrt{-R}\xi)}{2(R - 1)(\pm\sqrt{-R}\coth(\sqrt{-R}\xi) \pm \sqrt{-R}\operatorname{csch}(\sqrt{-R}\xi) + 1)^2}, \tag{4.25.2}$$

where $\xi = k(x + ly + \lambda t)$, $k = \pm \frac{1}{2}b_1$, $R < 0$, b_1 , l and λ are arbitrary constants.

Remark 5. The solution (4.10) can be reduced to the solution 3 in [12], the solution (4.11) can be reduced to the solution 4 in [12], when $k = 1$ and $\mu_2 = 0$. The solution (4.12) can be reduced to the solution 9 in [12], the solution (4.13) can be reduced to the solution 10 in [12], when $k = 1$ and $\mu_2 = 0$. The solution (4.16) can be reduced to the solution 1 in [12], the solution (4.17) can be reduced to the solution 2 in [12], when $k = 1$ and $\mu_1 = 0$. So we recover all the hyperbolic function solutions obtain in [12], and of course, we can recover all the triangular function solutions when $R > 0$. The other solutions obtained here, to our knowledge, are all new families of exact solutions of the (2 + 1)-dimension DLWE.

5. Generalized Riccati equation rational expansion method

In this section we would like extend the Riccati equation rational expansion method, i.e., using generalized ansatz to replace (2.4) and solutions of generalized Riccati equation to replace the variables in (2.4).

Consider the (2 + 1)-dimension Burgers system, i.e.,

$$\begin{aligned} -u_t + uu_y + \alpha v u_x + \beta u_{yy} + \alpha \beta u_{xx} &= 0, \\ u_x - v_y &= 0. \end{aligned} \tag{5.1}$$

According to the Step 1 in Section 2, we make the following travelling wave transformation

$$u(x, y, t) = U(\xi), \quad v(x, y, t) = V(\xi), \quad \xi = k(x + ly + \lambda t), \tag{5.2}$$

where k and λ are constants to be determined later, and thus system (5.1) becomes

$$\begin{aligned} -\lambda U' + lUU' + \alpha VU' + \beta k l^2 U'' + \alpha \beta k U'' &= 0, \\ U' - lV' &= 0. \end{aligned} \tag{5.3}$$

According to *Step 2* in Section 2, We suppose that $(2 + 1)$ -dimension Burgers system has the following formal travelling wave solution:

$$U(\xi) = a_0 + \sum_{i=1}^{m_u} \frac{a_i \phi^i(\xi) + b_i \phi^{i-1}(\xi) \phi'(\xi) + c_i \frac{1}{\phi(\xi)}}{(\mu_1 \phi(\xi) + \mu_2 \phi'(\xi) + \mu_3 \frac{1}{\phi} + 1)^i},$$

$$V(\xi) = A_0 + \sum_{i=1}^{m_v} \frac{A_i \phi^i(\xi) + B_i \phi^{i-1}(\xi) \phi'(\xi) + C_i \frac{1}{\phi(\xi)}}{(\mu_1 \phi(\xi) + \mu_2 \phi'(\xi) + \mu_3 \frac{1}{\phi} + 1)^i}$$
(5.4)

and the new variable $\phi = \phi(\xi)$ satisfying

$$\phi' - (h_1 + h_2 \phi^2) = \frac{d\phi}{d\xi} - (h_1 + h_2 \phi^2) = 0,$$
(5.5)

where $R, a_0, a_i, b_i, c_i, A_0, A_i, B_i$ and C_i ($i = 1, 2, \dots, m_i$) are constants to be determined later.

For the $(2 + 1)$ -dimension Burgers system, According to *Step 3* in Section 2, by balancing the highest nonlinear terms and the highest order partial derivative terms in (5.3), gives $m_u = 1$ and $m_v = 1$. So we have

$$U(\xi) = a_0 + \frac{a_1 \phi^i(\xi) + b_1 \phi'(\xi) + c_1 \frac{1}{\phi(\xi)}}{\mu_1 \phi(\xi) + \mu_2 \phi'(\xi) + \mu_3 \frac{1}{\phi(\xi)} + 1},$$

$$V(\xi) = A_0 + \frac{A_1 \phi^i(\xi) + B_1 \phi'(\xi) + C_1 \frac{1}{\phi(\xi)}}{\mu_1 \phi(\xi) + \mu_2 \phi'(\xi) + \mu_3 \frac{1}{\phi(\xi)} + 1}.$$
(5.6)

According to *Step 4* in Section 2, with the aid of *Maple*, substituting (5.6) along with (5.5) into (5.3), yields a set of algebraic equations for $\phi^i(\xi)$, ($i = 0, 1, \dots$). Setting the coefficients of these terms $\phi^i(\xi)$ of the resulting system's numerator to zero yields a set of over-determined algebraic equations.

According to *Step 5* in Section 2, by use of the *Maple* soft package “Charsets” by Dongming Wang, we get the explicit expressions for $a_0, a_1, b_1, c_1, A_0, A_1, B_1, C_1, \mu_1, \mu_2, \mu_3, k, l$ and λ . we know that the general solutions of Eq. (5.5) are

(1) When $h_1 = \frac{1}{2}$ and $h_2 = -\frac{1}{2}$

$$\phi(\xi) = \tanh(\xi) \pm \operatorname{isech}(\xi), \quad \phi(\xi) = \coth(\xi) \pm \operatorname{csch}(\xi).$$
(5.7)

(2) When $h_1 = h_2 = \pm \frac{1}{2}$

$$\phi(\xi) = \sec(\xi) \pm \tan(\xi), \quad \phi(\xi) = \csc(\xi) \pm \cot(\xi).$$
(5.8)

(3) When $h_1 = 1$ and $h_2 = -1$

$$\phi(\xi) = \tanh(\xi), \quad \phi(\xi) = \coth(\xi).$$
(5.9)

(4) When $h_1 = h_2 = 1$

$$\phi(\xi) = \tan(\xi).$$
(5.10)

(5) When $h_1 = h_2 = -1$

$$\phi(\xi) = \cot(\xi).$$
(5.11)

(6) When $h_1 = 0$ and $h_2 \neq 0$

$$\phi(\xi) = -\frac{1}{h_2 \xi + c_0}.$$
(5.12)

Thus according to system (5.2), (5.7)–(5.12) and the conclusions in *Step 5*, we can obtain following rational formal travelling-wave solutions of $(2 + 1)$ -dimension Burgers system.

Note: Here like Section 4, we omit tan-, cot-type and rational solutions and just list new solutions compared with the result in Section 4.

Family 1

$$u_{11} = a_0 + \frac{\frac{l\beta k}{2\mu_2^2}(4\mu_1^2 - \mu_2^2)(4\mu_1^2 + \mu_2)(\tanh(\xi) \pm \operatorname{isech}(\xi))^2}{\mu_1(\tanh(\xi) \pm \operatorname{isech}(\xi))^2 + \mu_2(\tanh(\xi) \pm \operatorname{isech}(\xi))(\operatorname{sech}^2(\xi) \mp \operatorname{isech}(\xi)\tanh(\xi)) + \tanh(\xi) \pm \operatorname{isech}(\xi) + \mu_3} + \frac{l\beta k\mu_1(8\mu_1^2 - 2\mu_2^2)(\tanh(\xi) \pm \operatorname{isech}(\xi))(\operatorname{sech}^2(\xi) \mp \operatorname{isech}(\xi)\tanh(\xi)) - \mu_2 c_1}{\mu_2(\mu_1(\tanh(\xi) \pm \operatorname{isech}(\xi))^2 + \mu_2(\tanh(\xi) \pm \operatorname{isech}(\xi))(\operatorname{sech}^2(\xi) \mp \operatorname{isech}(\xi)\tanh(\xi)) + \tanh(\xi) \pm \operatorname{isech}(\xi) + \mu_3)}, \tag{5.13.1}$$

$$v_{11} = A_0 + \frac{\frac{\beta k}{2\mu_2^2}(4\mu_1^2 - \mu_2^2)(4\mu_1^2 + \mu_2)(\tanh(\xi) \pm \operatorname{isech}(\xi))^2}{\mu_1(\tanh(\xi) \pm \operatorname{isech}(\xi))^2 + \mu_2(\tanh(\xi) \pm \operatorname{isech}(\xi))(\operatorname{sech}^2(\xi) \mp \operatorname{isech}(\xi)\tanh(\xi)) + \tanh(\xi) \pm \operatorname{isech}(\xi) + \mu_3} + \frac{\beta k\mu_1(8\mu_1^2 - 2\mu_2^2)(\tanh(\xi) \pm \operatorname{isech}(\xi))(\operatorname{sech}^2(\xi) \mp \operatorname{isech}(\xi)\tanh(\xi)) - \frac{\mu_2}{l}c_1}{\mu_2(\mu_1(\tanh(\xi) \pm \operatorname{isech}(\xi))^2 + \mu_2(\tanh(\xi) \pm \operatorname{isech}(\xi))(\operatorname{sech}^2(\xi) \mp \operatorname{isech}(\xi)\tanh(\xi)) + \tanh(\xi) \pm \operatorname{isech}(\xi) + \mu_3)}, \tag{5.13.2}$$

$$u_{12} = a_0 + \frac{\frac{l\beta k}{2\mu_2^2}(4\mu_1^2 - \mu_2^2)(4\mu_1^2 + \mu_2)(\coth(\xi) \pm \operatorname{csch}(\xi))^2}{\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi))^2 + \mu_2(\coth(\xi) \pm \operatorname{csch}(\xi))(\operatorname{csch}^2(\xi) \pm \operatorname{csch}(\xi)\coth(\xi)) + \coth(\xi) \pm \operatorname{csch}(\xi) + \mu_3} + \frac{l\beta k\mu_1(8\mu_1^2 - 2\mu_2^2)(\coth(\xi) \pm \operatorname{csch}(\xi))(\operatorname{csch}^2(\xi) \mp \operatorname{csch}(\xi)\coth(\xi)) - \mu_2 c_1}{\mu_2(\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi))^2 + \mu_2(\coth(\xi) \pm \operatorname{csch}(\xi))(\operatorname{csch}^2(\xi) \pm \operatorname{csch}(\xi)\coth(\xi)) + \coth(\xi) \pm \operatorname{csch}(\xi) + \mu_3)}, \tag{5.14.1}$$

$$v_{12} = A_0 + \frac{\frac{\beta k}{2\mu_2^2}(4\mu_1^2 - \mu_2^2)(4\mu_1^2 + \mu_2)(\coth(\xi) \pm \operatorname{csch}(\xi))^2}{\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi))^2 + \mu_2(\coth(\xi) \pm \operatorname{csch}(\xi))(\operatorname{csch}^2(\xi) \pm \operatorname{csch}(\xi)\coth(\xi)) + \coth(\xi) \pm \operatorname{csch}(\xi) + \mu_3} + \frac{\beta k\mu_1(8\mu_1^2 - 2\mu_2^2)(\coth(\xi) \pm \operatorname{csch}(\xi))(\operatorname{csch}^2(\xi) \mp \operatorname{csch}(\xi)\coth(\xi)) - \frac{\mu_2}{l}c_1}{\mu_2(\mu_1(\coth(\xi) \pm \operatorname{csch}(\xi))^2 + \mu_2(\coth(\xi) \pm \operatorname{csch}(\xi))(\operatorname{csch}^2(\xi) \pm \operatorname{csch}(\xi)\coth(\xi)) + \coth(\xi) \pm \operatorname{csch}(\xi) + \mu_3)}, \tag{5.14.2}$$

where $\xi = k(x + ly + \lambda t)$, $a_0 = \frac{2\beta k l^2 \mu_1 \mu_2^2 + 2\beta k l^2 \mu_2 \mu_1 - 8\mu_1^3 \alpha \beta k - 8\mu_1^3 \beta k l^2 + 2\alpha \beta k \mu_2 \mu_1 + 2\alpha \beta k \mu_1 \mu_2^2 - \alpha A_0 \mu_2^2 + \lambda \mu_2^2}{\mu_2^2 l}$, $\mu_3 = -\frac{\mu_1(8\mu_1^2 - \mu_2^2 - 2\mu_2)}{\mu_2^2}$, $c_1 = -\frac{l\beta k(4\mu_1^2 - \mu_2)(8\mu_1^2 - \mu_2^2 - 2\mu_2)(8\mu_1^2 - \mu_2^2)}{2\mu_2^4}$, μ_1, μ_2, l and λ are arbitrary constants.

Family 2

$$u_{21} = a_0 + \frac{2l\beta k(-\mu_2^2 + \mu_1^2)(2\mu_1^2 + \mu_2)\tanh^2(\xi)}{\mu_2^2(\mu_1\tanh^2(\xi) + \mu_2\tanh(\xi)\operatorname{sech}^2(\xi) + \mu_3 + \tanh(\xi))} + \frac{4l\beta k\mu_1\mu_2^3(-\mu_2^2 + \mu_1^2)\tanh(\xi)\operatorname{sech}^2(\xi) - 2l\beta k(2\mu_1^2 - \mu_2)(-\mu_2^2 + 2\mu_1^2 - \mu_2)(-\mu_2^2 + \mu_1^2)}{\mu_2^4(\mu_1\tanh^2(\xi) + \mu_2\tanh(\xi)\operatorname{sech}^2(\xi) + \mu_3 + \tanh(\xi))}, \tag{5.15.1}$$

$$v_{21} = A_0 + \frac{2\beta k(-\mu_2^2 + \mu_1^2)(2\mu_1^2 + \mu_2)\tanh^2(\xi)}{\mu_2^2(\mu_1\tanh^2(\xi) + \mu_2\tanh(\xi)\operatorname{sech}^2(\xi) + \mu_3 + \tanh(\xi))} + \frac{4\beta k\mu_1\mu_2^3(-\mu_2^2 + \mu_1^2)\tanh(\xi)\operatorname{sech}^2(\xi) - 2\beta k(2\mu_1^2 - \mu_2)(-\mu_2^2 + 2\mu_1^2 - \mu_2)(-\mu_2^2 + \mu_1^2)}{\mu_2^4(\mu_1\tanh^2(\xi) + \mu_2\tanh(\xi)\operatorname{sech}^2(\xi) + \mu_3 + \tanh(\xi))}, \tag{5.15.2}$$

$$u_{22} = a_0 + \frac{2l\beta k(-\mu_2^2 + \mu_1^2)(2\mu_1^2 + \mu_2)\coth^2(\xi)}{\mu_2^2(\mu_1\coth^2(\xi) + \mu_2\coth(\xi)\operatorname{csch}^2(\xi) + \mu_3 + \coth(\xi))} + \frac{4l\beta k\mu_1\mu_2^3(-\mu_2^2 + \mu_1^2)\coth(\xi)\operatorname{csch}^2(\xi) - 2l\beta k(2\mu_1^2 - \mu_2)(-\mu_2^2 + 2\mu_1^2 - \mu_2)(-\mu_2^2 + \mu_1^2)}{\mu_2^4(\mu_1\coth^2(\xi) + \mu_2\coth(\xi)\operatorname{csch}^2(\xi) + \mu_3 + \coth(\xi))}, \tag{5.16.1}$$

$$v_{22} = A_0 + \frac{2\beta k(-\mu_2^2 + \mu_1^2)(2\mu_1^2 + \mu_2)\coth^2(\xi)}{\mu_2^2(\mu_1\coth^2(\xi) + \mu_2\coth(\xi)\operatorname{csch}^2(\xi) + \mu_3 + \coth(\xi))} + \frac{4\beta k\mu_1\mu_2^3(-\mu_2^2 + \mu_1^2)\coth(\xi)\operatorname{csch}^2(\xi) - 2\beta k(2\mu_1^2 - \mu_2)(-\mu_2^2 + 2\mu_1^2 - \mu_2)(-\mu_2^2 + \mu_1^2)}{\mu_2^4(\mu_1\coth^2(\xi) + \mu_2\coth(\xi)\operatorname{csch}^2(\xi) + \mu_3 + \coth(\xi))}, \tag{5.16.2}$$

where $\zeta = k(x + ly + \lambda t)$, $a_0 = \frac{4\beta k l^2 \mu_1 \mu_2^2 + 2\beta k l^2 \mu_2 \mu_1 - 4\mu_1^3 \alpha \beta k - 4\mu_1^3 \beta k l^2 + 2\alpha \beta k \mu_2 \mu_1 + 4\alpha \beta k \mu_1 \mu_2^2 - \alpha A_0 \mu_2^2 + \lambda \mu_2^2}{\mu_2^2 l}$, $\mu_3 = -\frac{\mu_1(-\mu_2^2 + 2\mu_1^2 - \mu_2)}{\mu_2^2}$, μ_1, μ_2, l and λ are arbitrary constants.

Remark 6. From this example, we can easily see that the ansatz (2.4) can be directly extended by adding some F_i in its numerator and denominator, and the only restriction of the added F_i is their derivatives are polynomial in $F, G, F_1, F_2, \dots, F_i, \dots$

6. Summary and conclusions

In summary, we have proposed a unified algebraic method: rational expansion method with symbolic computation, which greatly exceeds the applicability of the existing methods in obtaining travelling wave solutions of the nonlinear PDEs.

In Section 3, when $\mu_1 = \mu_2 = 0$, our method just reduces to the Jacobi elliptic function expansion method [33].

In Section 4, when $\mu_1 = \mu_2 = 0$, our method just reduces to the Tanh-method [5–8,11,12]; when $\mu_1 = 0$, our method just reduce to the projective Riccati equation expansion method [34–36].

As we can see in Sections 3–5, we can naturally extend our method to generalized form, such as: we can introduce a new more generalized solution form of (2.1) as

$$U_i(\zeta) = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum_{r_1+\dots+r_n=j} a_{r_1\dots r_n}^{ij} F_1^{r_1}(\zeta_1) \cdots F_n^{r_n}(\zeta_n)}{\sum_{l_1+\dots+l_n=j} b_{l_1\dots l_n}^{ij} F_1^{l_1}(\zeta_1) \cdots F_n^{l_n}(\zeta_n) + b_{ij}}, \tag{6.1}$$

where $a_{i0}, a_{r_1\dots r_n}^{ij}, b_{l_1\dots l_n}^{ij}$ and ζ_i are differentiable function to be determined and $\frac{dF_i}{d\zeta_i} = K_i(F_i, \dots, F_n)$, where K_i are polynomial of F_i and ζ_i are not need to be equal. It is clear to see that (6.1) is also satisfying solving the recurrent relation or derivative relation for the terms of polynomial for computation closed. And we think this algorithm cover all algorithms of Function expansion method. The argument will be given in the following paper.

The feature of our proposed method is that, we firstly propose a more general ansatz, in which the closed-form solutions of nonlinear PDEs that can be expressed as a finite series of some Jacobi elliptic function or solutions of subequation (like the Riccati equation) in rational form. The method provides us with new and more general travelling wave solutions that cannot be obtained by the Tanh-method, Jacobi elliptic function expansion methods, the homogeneous balance method and projective Riccati equation expansion method. This method is also computerizable. The key idea is to decompose PDE system into a system of algebraic equations or an ordinary differential equation, which allows us to perform complicated and tedious algebraic computation by using symbolic computation. Lastly, we point out that our method also can be applied to a large class of either integrable or nonintegrable nonlinear coupled systems. The details for these cases will be investigated in our future works.

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