Function Projective Synchronization and Its Applications

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***Abstract*—In this paper, function projective synchronization of continuous-time chaotic system is introduced. Based on symbolic computation $Maple$ and the chaotic controlling methods: the cascade synchronization approach and active control method, function projective synchronization scheme is developed to realize the function synchronization of two identical unified chaotic system and function projective synchronization between the chua circle and the disk dynamo model. Numerical simulations are used to verify the effectiveness of the scheme.**

I. INTRODUCTION

SINCE the synchronization of the chaotic system by means of nonlinear driving is discovered [1], the synchronization problem in chaotic systems has been intensively and extensively studied in recent decades due to its potential applications and many referenced cited therein. Up to now, there exist many types of chaos synchronization in dynamical systems [2-6]. In particular, amongst all kinds of chaos synchronization, projective synchronization reported by Mainieri and Rehacek [7] is one of the most noticeable ones that the drive and response vectors evolve in a proportional scale–the vectors become proportional.

The early projective synchronization is usually observable only in a class of systems with partial-linearity [8], but recently some researchers [9-10] have achieved control of the projective synchronization in a general class of chaotic systems including non-partially-linear systems, and termed this projective synchronization as "generalized projective synchronization" (GPS). Recently, the modify of projective synchronization [11] is proposed to synchronize two identical systems up to a scaling constant matrix.

In this paper, based on the previous work [7-12], and by introducing a new more general synchronization form than one in the above scheme, we propose a new and more generalized definition: function projective synchronization (FPS), where the response of the synchronized dynamical states synchronize up to a scaling function matrix f. Different controlling methods: the cascade synchronization approach [13-16] and active control method [17] are used to realized the synchronization. Numerical simulation results are given for illustration and verification.

II. FUNCTION PROJECTIVE SYNCHRONIZATION

Recently, based on the previous projective synchronization and the function projective synchronization between two identical chaotic systems proposed by us [18], we extend to synchronize two different chaotic (hyperchaotic) systems up to a scaling function matrix. Similarly, the function projective synchronization is characterized by a scaling function matrix.

Definition. Let $\dot{x} = F(x,t)$ is the drive chaotic system, and $\dot{y} = G(y,t) + U$ is the response system, where $x = (x_1(t), x_2(t), ..., x_m(t))^T$, $y = (y_1(t), y_2(t), ..., y_m(t))^T$, and $U = (u_1(x,y), u_2(x,y), ..., u_m(x,y))$ is a controller to be determined later. Denote $e_i = x_i - f_i(x)y_i, (i = 1,2,..,m)$, $f_i(x)(i = 1,2,..,m)$ are functions of x. If $lim_{t\to\infty}\|e\| = 0, e = (e_1, e_2, ..., e_m)$, we call there exists function projective synchronization (FPS) between these two different chaotic systems", and we call f a "scaling function matrix".

Consider the drive system in the form of

$$\dot{x} = A_1 x + h_1(x,t) \tag{1}$$

Assume that the response system is as follows:

$$\dot{y} = A_2 y + h_2(y,t) + U \tag{2}$$

where $x, y \in R^n$, A_1, A_2 are $m \times m$ constant matrixes, $h_1, h_2 : R^m \to R^m$ are nonlinear function vectors, and U is a controller to be determined later.

Theorem. For an invertible diagonal function matrix f, function projective synchronization between the two systems (1) and (2) will occur, if the following conditions are satisfied:

(i) $U = f^{-1}h_1(x,t) + (f^{-1}A_1 f - A_2)y + f^{-1}B(x - fy) - h_2(y,t) - f^{-1}gy$, where $g = diag(\dot{f_1}, \dot{f_2}, ..., \dot{f_m})$, and $B \in R^{m\times m}$;

(ii) The real parts of all the eigenvalues of $(A_1 - B)$ are negative.

Proof. From $e = x - fy$ in definition of FPS, one can get

$\dot{e} = \dot{x} - f\dot{y} - gy = A_1x + h_1(x,t) - f(A_2y + h_2(y,t) + U) - gy$

$= A_1x + h_1(x,t) - fA_2y - fh_2(y,t) - gy - h_1(x,t) - A_1fy + fA_2y - B(x - fy) + fh_2(y,t) + gy$

$= (A_1 - B)e$

Regards with the Lyapnov stability theory and for a feasible control, the feedback B must be selected such that all the eigenvalues of $(A_1 - B)$, have negative real parts. Thus, if the controllability matrix $(A_1 - B)$ is in full rank, the system $\dot{e}$ is

asymptotically stable at the origin, which implies (1) and (2) are in the state of the function projective synchronization.
It is necessary to point out that the scaling function matrix f also has no effect on the eigenvalues of $(A_1 - B)$ like the modified projective synchronization. Thus one can adjust the scaling matrix arbitrarily during control without worrying about the control robustness. The FPS is more general: when $f_1 = f_2 = ... = f_m = 1$, $f_1 = f_2 = ... = f_m = \alpha$ and $f_1 = \alpha_1, f_2 = \alpha_2, ..., f_m = \alpha_m$, the complete synchronization, the projective synchronization and the modified projective synchronization will appear, respectively.

III. FPS OF THE UNIFIED CHAOTIC SYSTEM WITH THE CASCADE SYNCHRONIZATION METHOD

In the following, we apply the cascade synchronization approach to the unified chaotic system [19].
The unified chaotic system is described by

$$\begin{cases} \dot{x} = (25\beta + 10)(y - x), \\ \dot{y} = (28 - 35\beta)x - xz + (29\beta - 1)y, \\ \dot{z} = xy - \frac{8+\beta}{3} z \end{cases} \tag{3}$$

where β is parameter. The system is chaotic for any$\beta \in [0, 1]$. System (3) is called the general Lorenz system [20], Lü system [21] and Chen system [22], when $\beta \in [0, 0.8)$, when $\beta = 0.8$, and when $\beta \in$ "$(0.8, 1]$, respectively.
Now, the drive system and the response one are defined below, respectively.

$$\begin{cases} \dot{x_1} = (25\beta + 10)(y_1 - x_1), \\ \dot{y_1} = (28 - 35\beta)x_1 - x_1 z_1 + (29\beta - 1)y_1, \\ \dot{z_1} = x_1 y_1 - \frac{8+\beta}{3} z_1 \end{cases} \tag{4}$$

and

$$\begin{cases} \dot{x_3} = (25\beta + 10)(y_2 - x_3) + u_1, \\ \dot{y_2} = (28 - 35\beta)x_3 - x_3 z_1 + (29\beta - 1)y_2 + u_2 \end{cases} \tag{5}$$

where $(u_1, u_2)^T$ is a desired controller designed below.
In the following, we define the error functions as the difference between the system (4) and (5),

$$e_1 = x_1 - x_3, e_2 = y_1 - (1 + \tanh(z_1)^2)y_2,$$

the errors become zero if the projective synchronization appears. Here by the active method, we get

$$\begin{cases} u_1 = -25\beta x_1(t) + 10y_1(t) - 7x_1(t) + 25\beta x_3(t) - 10y_2(t) \\ \quad +7x_3(t), \\ u_2 = \frac{1}{3(1+\tanh(z_1(t))^2)}[84x_1(t) - 105\beta x_1(t) - 3x_1(t)z_1(t) \\ \quad +87\beta y_1(t) + 9y_1(t) - 6\tanh(z_1(t))y_2(t)x_1(t)y_1(t) \\ \quad +16\tanh(z_1(t))y_2(t)z_1(t) + 2\beta\tanh(z_1(t))y_2(t)z_1(t) \\ \quad +6\tanh(z_1(t))^3 y_2(t)x_1(t)y_1(t) - 16\tanh(z_1(t))^3 y_2(t) \\ \quad \times z_1(t) - 2\beta\tanh(z_1(t))^3 y_2(t)z_1(t) - 84x_3(t) + 105\beta \\ \quad \times x_3(t) + 3x_3(t)z_1(t) - 87\beta y_2(t) - 9y_2(t) - 84x_2(t) \\ \quad \times \tanh(z_1(t))^2 + 105\beta\tanh(z_1(t))^2 x_2(t) + 3x_3(t) \\ \quad \times \tanh(z_1(t))^2 z_1(t) - 87\beta\tanh(z_1(t))^2 y_2(t) - 9y_2(t) \\ \quad \times \tanh(z_1(t))^2] \end{cases}$$

Then we regard (5) as the drive system, and the response one as

$$\begin{cases} \dot{x_2} = (25\beta + 10)(y_2 - x_2) + u_3, \\ \dot{z_2} = x_2 y_2 - \frac{8+\beta}{3} z_2 + u_4 \end{cases} \tag{6}$$

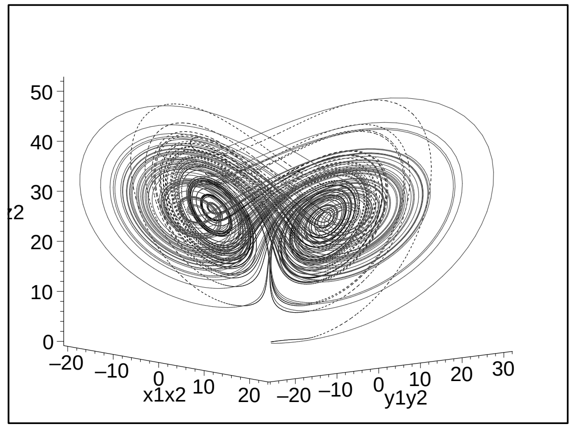

Fig. 1. " − " denotes for the drive system, " − . − " denotes for the response system

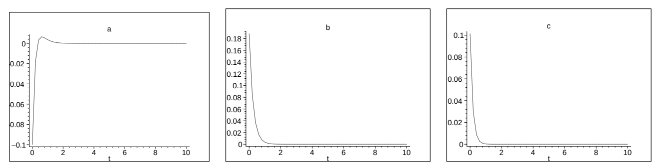

Fig. 2. (a) the error state $e_1(t)$; (b) the error state $e_2(t)$; (c) the error state $e_3(t)$

where $(u_3, u_4)^T$ is a desired controller. The error states: $e_3 = x_3 - x_2, e_4 = z_1 - \tanh(z_1)z_2$.
We set

$$\begin{cases} u_3 = -(25\beta + 10)y_2(t) + 2x_3(t) + (25\beta + 10)y_1(t) - \\ \quad (25\beta + 7)x_1(t) + (25\beta + 5)x_2(t), \\ u_4 = -\frac{\tanh(z_1)}{3}[-3x_1(t)y_1(t) - 10z_1(t) + \beta z_1(t) + 3z_2(t) \\ \quad \times x_1(t)y_1(t) - 8z_1(t)z_2(t) - \beta z_1(t)z_2(t) - 3x_1(t)y_1(t) \\ \quad \times z_2(t)\tanh(z_1(t))^2 + 8z_1(t)z_2(t)\tanh(z_1(t))^2 + \beta z_1(t) \\ \quad \times z_2(t)\tanh(z_1(t))^2 - \beta z_2(t)\tanh(z_1(t)) + 10\tanh(z_1(t)) \\ \quad \times z_2(t) + 3x_2(t)y_2(t)\tanh(z_1(t))] \end{cases}$$

For the initial condition $(x_1, y_1, z_1) = (0.1, 0.5, 0.2)$, and $(x_2, y_2, z_2) = (0.2, 0.3, 0.5)$, we give the numerical simulations for the general Lorenz ($\beta = 0.2$), Lü ($\beta = 0.8$), and Chen system ($\beta = 1$).
Case I: $\beta = 0.2$**, Lorenz chaotic system** The numerical simulations are displayed in Fig.1 and Fig.2. In Fig.1, we give the orbits of the identical unified chaotic systems with different initial values in the same coordinate. And in Fig.2, the orbits of the three error states are given.
Case II: $\beta = 0.8$**, Lü chaotic system** The numerical simulations are displayed in Fig.3 and Fig.4. In Fig.3, we give the orbits of the identical unified chaotic systems with different initial values in the same coordinate. And in Fig.4, the orbits of the three error states are given.
Case III: $\beta = 1$**, Chen chaotic system** The numerical simulations are displayed in Fig.5 and Fig.6. In Fig.5, we give the orbits of the identical unified chaotic systems with different initial values in the same coordinate. And in Fig.6, the orbits of the three error states are given.

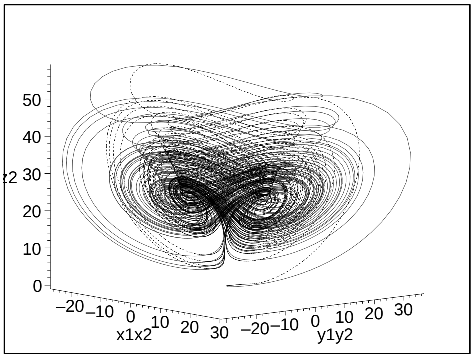

Fig. 3. " − " denotes for the drive system, " − . − " denotes for the response system

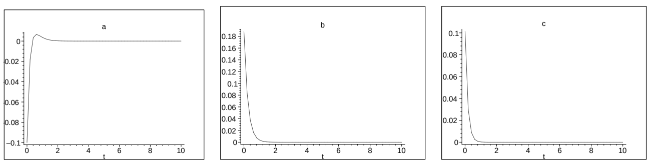

Fig. 4. (a) the error state $e_1(t)$; (b) the error state $e_2(t)$; (c) the error state $e_3(t)$

IV. FPS OF TWO DIFFERENT CHAOTIC SYSTEMS WITH ACTIVE CONTROL METHOD

In the following, we use the active control method to synchronize the chua circle [23] and the disk dynamo model [24] to the fixed scaling function matrix.
The drive system (chua circle) is described as follows:

$$\begin{cases} \dot{x_1} = \alpha(y_1 - x_1 - f(x_1)), \\ \dot{y_1} = x_1 - y_1 + z_1, \\ \dot{z_1} = -\beta y_1 \end{cases} \tag{7}$$

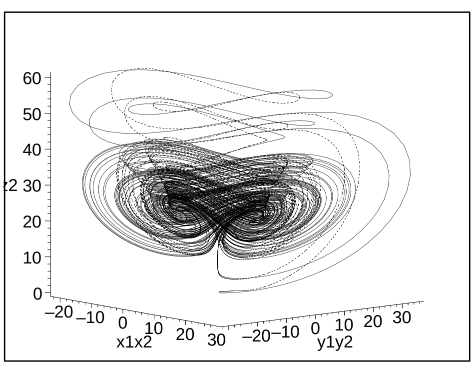

Fig. 5. " − " denotes for the drive system, " − . − " denotes for the response system

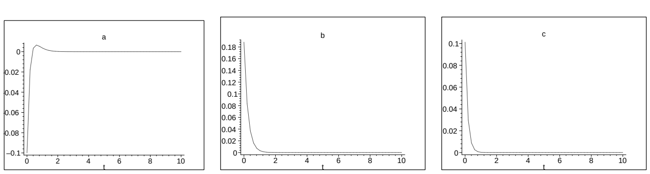

Fig. 6. (a) the error state $e_1(t)$; (b)the error state $e_2(t)$; (c) the error state $e_3(t)$

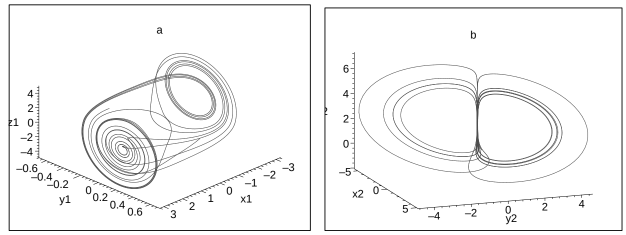

Fig. 7. (a) the attractor of the chua circle; (b) the attractor of disk dynamo model, that is (11) with $u_i = 0$

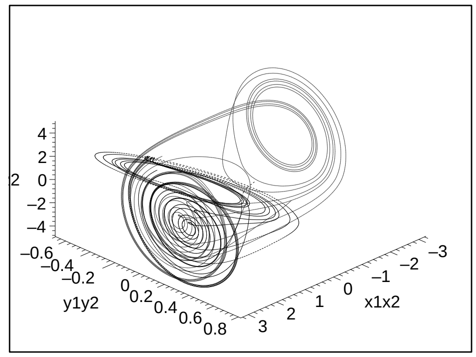

Fig. 8. " − " denotes for the drive system, " − . − " denotes for the response system

where

$$f(x_1) = \begin{cases} bx_1 + a - b, if x_1 \geq 1, \\ bx_1 - a + b, if x_1 \leq -1, \\ ax_1, otherwise \end{cases}$$

And the response system (disk dynamo model) is introduced as below

$$\begin{cases} \dot{x_2} = -2x_2 + y_2 z_2 + u_1, \\ \dot{y_2} = -2y_2 + x_2(z_2 - 1) + u_2, \\ \dot{z_2} = 1 - x_2 y_2 + u_3 \end{cases} \tag{8}$$

The two above attractors are shown in Fig.7

We choose the error states are as followed

$$\begin{cases} e_1 = x_1 - \tanh(x_1)x_2, \\ e_2 = y_1 - \tanh(y_1)y_2, \\ e_3 = z_1 - z_2 \end{cases} \tag{9}$$

By using the active control method, we can get $u_i (i = 1, 2, 3)$

$$\begin{cases} u_1 = \frac{1}{\tanh(x_1(t))}[\alpha y_1(t) - \alpha x_1(t) - \alpha f(x_1(t)) - \alpha x_2(t) y_1(t) \\ \quad +\alpha x_2(t) x_1(t)) + \alpha x_2(t) f(x_1(t)) + \alpha x_2(t) \tanh(x_1(t))^2 \\ \quad \times y_1(t) - \alpha x_2(t) \tanh(x_1(t))^2 x_1(t) - \alpha x_2(t) \tanh(x_1(t))^2 \\ \quad \times f(x_1(t)) + 2\tanh(x_1(t)) x_2(t) - \tanh(x_1(t)) y_2(t) z_2(t) \\ \quad +3e_1(t)], \\ u_2 = x_1(t) - y_1(t) + z_1(t) + 2y_2(t) - x_2(t) z_2(t) + x_2(t) \\ \quad +4e_2(t), \\ u_3 = -\beta y_1(t) - 1 + x_2(t) y_2(t) + 5e_3(t) \end{cases}$$

Set $\alpha = 10, \beta = 15.68, a = -1.2768$, and $b = -0.6888$, we get the FPS of the two chaotic system mentioned above, and show them in Fig.8. The error states are shown in Fig.9.

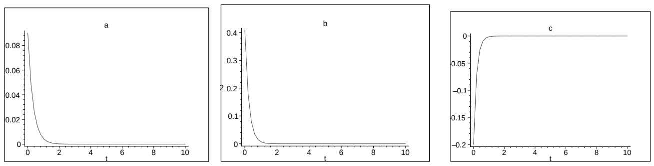

Fig. 9. (a) the error state $e_1(t)$; (b) the error state $e_2(t)$; (c) the error state $e_3(t)$

V. Conclusion

A function projective synchronization of two identical or different chaotic (hyperchaotic) systems is defined, which is more general than the definition known. Based on symbolic computation $Maple$ and chaos controlling methods: the cascade synchronization method and the active control method, we show how to synchronize two chaotic systems, and make them globally synchronize up to a scaling function matrix f. Numerical simulations are carried out to verify the effectiveness of the proposed controller. With the aid of symbolic-numeric computation, the scheme can be used for other chaotic systems and hyperchaotic systems.

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