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Generalized Projective Synchronization Between Rössler System and New Unified Chaotic System*

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Abstract *Based on symbolic computation system Maple and Lyapunov stability theory, an active control method is used to projectively synchronize two different chaotic systems — Lorenz–Chen–Lü system (LCL) and Rössler system, which belong to different dynamic systems. In this paper, we achieve generalized projective synchronization between the two different chaotic systems by directing the scaling factor onto the desired value arbitrarily. To illustrate our result, numerical simulations are used to perform the process of the synchronization and successfully put the orbits of drive system (LCL) and orbits of the response system (Rössler system) in the same plot for understanding intuitively.*

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Key words: Rössler system, Lorenz–Chen–Lü system, generalized projective synchronization, scaling factor, active control

1 Introduction

In 1990, Pecora and Carroll[1] presented the chaos synchronization method to synchronize two identical chaotic systems with different initial values. Chaos synchronization has become an active research subject in nonlinear science and attracted much attention in many fields. Different types of synchronization behaviors have been discovered because of potential applications. Recently, projective synchronization has been first reported by Mainieri and Rehacek[2] in partially linear systems, where the drive and response vectors evolve proportionally in a scale vector. The early projective synchronization is usually observable only in a class of systems with partial-linearity,[3] but recently some researchers[4−7] have achieved control of the projective synchronization in a general class of chaotic systems including non-partially-linear systems, and termed this projective synchronization as "generalized projective synchronization" (GPS). Li[8] showed GPS between Lorenz system and Chen's system.

Recently, Lü *et al.*[9] presented a new unified chaotic system (called Lorenz–Chen–Lü (LCL) system), which is a kind of unique and unified classification between the Lorenz system and Chen system, both in theory and in simulation. In fact, the unified system is likely to be the simplest chaotic system that bridges the gap between Lorenz system and Chen system, and contributes to a better understanding of the correlation between Lorenz system and Chen system.

In this paper, we study the generalized projective synchronization between two different chaotic systems: Rössler system and Lorenz–Chen–Lü system by means of *Maple* and the active scheme in Ref. [6]. Numerical simulations are used to verify the effectiveness of the method, i.e. there exist controllers to make Lorenz–Chen–Lü system and other different known chaotic systems synchronize. It is necessary to point out that the known synchronization realized by the active scheme only is between the same class of chaotic systems[6] or two different chaotic system[6] but they belong to the same dynamic system with different parameters.[10] In Sec. 2, using active control, a controlled drive slave is constructed. The GPS of the two different chaotic systems is theoretically analyzed. In Sec. 3, we simulate the two chaotic systems after synchronization. Finally, the conclusion of the paper is given.

2 GPS of Rössler System and LCL System

Consider the following two different chaotic systems: Rössler system and LCL system, as drive system and response system in the generalized projective synchronization studied by us respectively.

The Rössler system is given by

$$\dot{x} = -y - z\,, \quad \dot{y} = x + ay\,,$$
$$\dot{z} = b + xz - cz\,, \tag{1}$$

which arises from work in chemical kinetics[11] and is simpler than the Lorenz model.[12] When $a = b = 0.2$, $c = 5.7$, Rössler system (1) has a chaotic attractor as shown in Fig. 1(a).

Recently, Lü *et al.*[9] presented the new unified chaotic system (called Lorenz–Chen–Lü (LCL) system).

$$\dot{x} = (25\alpha + 10)(y - x)\,,$$

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$$\dot{y} = (28 - 35\alpha)x - xz + (29\alpha - 1)y\,,$$
$$\dot{z} = xy - \frac{8+\alpha}{3}z\,, \qquad (2)$$

where $\alpha \in [-0.016, 1.15]$. When $-0.016 \le \alpha < 0.8$, LCL system (2) belongs to the generalized Lorenz system; when $\alpha = 0.8$, LCL system (2) belongs to the generalized Lü system; when $0.8 < \alpha \le 1.15$, LCL system (2) belongs to the generalized Chen system. Figure 1(b) displays the chaotic attractor of the generalized Lorenz system $\alpha = 0.2$. We only give one attractor of the LCL system, and similarly you can get others.

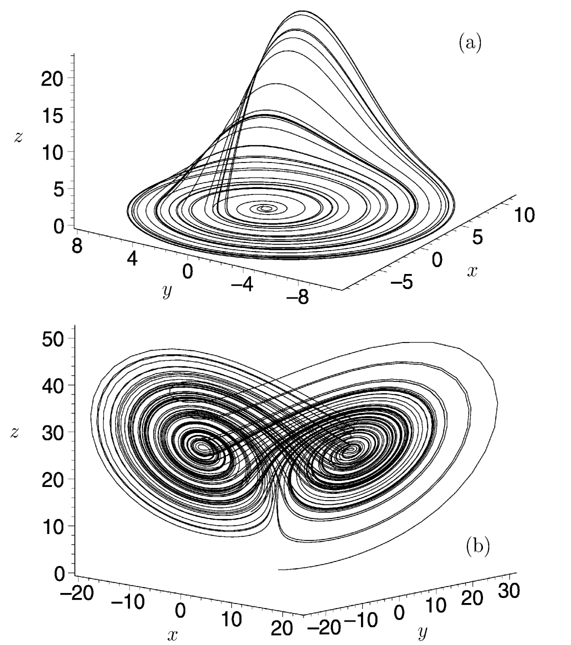

Fig. 1 (a) Rössler chaotic attractor with initial states: $x(0) = 0.1$, $y(0) = 0.2$, $z(0) = 0.3$; (b) LCL chaotic attractor ($\alpha = 0.2$) with initial values: $x(0) = 0.2$, $y(0) = 0.3$, $z(0) = 0.5$.

The projective synchronization means that the drive and response vectors synchronize up to a scaling factor m, that is, the vectors become proportional.[2] Recently, definition of the generalized projective synchronization is given.[6,7] In the following we would like to introduce the generalized projective synchronization. Consider the following chaotic system:

$$\dot{x} = f(x)\,, \quad \dot{y} = g(x, u(x,y))\,, \qquad (3)$$

where $x,\ y \in R^n$, $u : R^{2n} \to R^n$ and $u(0,0) = 0$, $g(x, u(0,0)) = f(x)$. x stands for the drive system, and y stands for the response system. If there exists a nonzero constant m such that

$$\lim_{t\to\infty} \|x - my\| = 0\,,$$

then we call them "generalized projective synchronization", and we call m a "scaling factor".

Now, take Rössler system and LCL system into consideration. The drive system (2)

$$\dot{x_1} = (25\alpha + 10)(y_1 - x_1)\,,$$
$$\dot{y_1} = (28 - 35\alpha)x_1 - x_1 z_1 + (29\alpha - 1)y_1\,,$$
$$\dot{z_1} = x_1 y_1 - \frac{8+\alpha}{3}z_1\,, \qquad (4)$$

and Rössler system is the response system,

$$\dot{x_2} = -y_2 - z_2 + u_1\,,$$
$$\dot{y_2} = x_2 + a y_2 + u_2\,,$$
$$\dot{z_2} = b + x_2 z_2 - c z_2 + u_3\,, \qquad (5)$$

where u_i $(i = 1,2,3)$ are the control functions to be determined later.

In the following and by means of *Maple*, we will use three steps to seek the control functions u_i $(i = 1,2,3)$ by means of techniques from active control theory and feedback-stepping method, so that the synchronization occurs between Eqs. (4) and (5):

Step 1 Introduce the error variables e_i $(i = 1,2,3)$ in the following form:

$$e_1 = x_1 - m x_2\,, \quad e_2 = y_1 - m y_2\,, \quad e_3 = z_1 - m z_2\,, \quad (6)$$

where m is a desired scaling factor. Then one obtains the error dynamical system by substituting Eqs. (4) and (5) into Eq. (6),

$$\dot{e}_1 = -e_2 - e_3 + (25\alpha + 11)y_1 - (25\alpha + 10)x_1 + z_1 - m u_1\,,$$
$$\dot{e}_2 = e_1 + a e_2 + (27 - 35\alpha)x_1 + (29\alpha - 1 - a)y_1 - x_1 z_1 - m u_2\,,$$
$$\dot{e}_3 = -\Big(\frac{8}{3} + \frac{1}{3}\alpha\Big)e_3 + x_1 y_1 - mb - m\Big(\frac{8}{3} + \frac{1}{3}\alpha\Big)(z_1 - e_3)/m - (x_1 - e_1)(z_1 - e_3)/m - m u_3\,. \qquad (7)$$

Step 2 Referring to the original methods of active control, we must define the three control functions u_i $(i = 1,2,3)$ as follows:

$$m u_1 = (25\alpha + 11)y_1 - (25\alpha + 10)x_1 + z_1 - v_1\,,$$
$$m u_2 = (27 - 35\alpha)x_1 + (29\alpha - 1 - a)y_1 - x_1 z_1 - v_2\,,$$
$$m u_3 = x_1 y_1 - mb - \frac{1}{m}\Big(\frac{8}{3} + \frac{1}{3}\alpha\Big)(z_1 - e_3) - \frac{1}{m}(x_1 - e_1)(z_1 - e_3) - v_3\,. \qquad (8)$$

Hence the error system (7) becomes

$$\dot{e_1} = v_1 - e_2 - e_3\,, \quad \dot{e_2} = v_2 + e_1 + a e_2\,,$$
$$\dot{e_3} = v_3 - \Big(\frac{8}{3} + \frac{1}{3}\alpha\Big)e_3\,. \qquad (9)$$

Obviously, the error system (9) is a linear system, if v_i $(i = 1,2,3)$ are in the form of the error e_1, e_2, and e_3. If

$\lim_{t\to\infty}\|e_i\| = 0$, it implies that GPS of two different chaotic systems as mentioned above is achieved with a scaling factor m.

Step 3 In order to make the error system (9) become a linear system and stabilize it, we must determine v_i $(i = 1,2,3)$ properly. There are many possible choices for the controllers v_1, v_2, and v_3. But all those choices must satisfy the condition that the feedback system should have all eigenvalues with negative real parts. In this paper, we choose

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ -1 & -a-1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{10}$$

In this choice, the three eigenvalues of the system (9) are -1, -1, and $-(8/3+\alpha/3)$. Obviously, these three eigenvalue are all negative. That is, Rössler system and LCL system projectively synchronize with the scaling factor m.

Above all, from Eqs. (8) and (10) we can observe u_i $(i = 1,2,3)$,

$$\begin{aligned} u_1 &= \frac{1}{m}(25\alpha+11)y_1 - \frac{1}{m}(25\alpha+10)x_1 + \frac{z_1}{m} + \frac{e_1}{m} - \frac{e_2}{m} - \frac{e_3}{m}\,, \\ u_2 &= \frac{1}{m}(27-35\alpha)x_1 + \frac{1}{m}(29\alpha-1-a)y_1 - \frac{x_1z_1}{m} + \frac{e_1}{m} + \frac{(a+1)e_2}{m}\,, \\ u_3 &= \frac{x_1y_1}{m} - b - \frac{1}{m^2}\Big(\frac{8}{3}+\frac{1}{3}\alpha\Big)(z_1-e_3) - \frac{1}{m^2}(x_1-e_1)(z_1-e_3)\,. \end{aligned} \tag{11}$$

Let us define the Lyapunov function as

$$V = \frac{1}{2}(e_1^2+e_2^2+e_3^2)\,. \tag{12}$$

If the Lyapunov function (11) satisfies the conditions

$$\begin{aligned} &V > 0\,, \quad \text{if } (e_1,e_2,e_3) \neq 0\,, \\ &V = 0\,, \quad \text{if } (e_1,e_2,e_3) = 0 \end{aligned} \tag{13}$$

and

$$\begin{aligned} &\dot V < 0\,, \quad \text{if } (e_1,e_2,e_3) \neq 0\,, \\ &\dot V = 0\,, \quad \text{if } (e_1,e_2,e_3) = 0\,, \end{aligned} \tag{14}$$

then e_1, e_2, and e_3 will asymptotically tend to zero.

It is easy to prove that the Lyapnov function V in Eq. (12) satisfies the conditions of Eqs. (13) and (14).

Proof

$$V = \frac{1}{2}(e_1^2+e_2^2+e_3^2)\,,$$

obviously, it satisfies Eq. (13). From the above discussions, we obtain

$$\dot e_1 = -e_1\,, \quad \dot e_2 = -e_2\,, \quad \dot e_3 = -\frac{8+\alpha}{3}e_3\,, \tag{15}$$

then

$$\begin{aligned} \dot V &= e_1\dot e_1 + e_2\dot e_2 + e_3\dot e_3 \\ &= e_1(-e_1) + e_2(-e_2) + e_3\Big(-\frac{8+\alpha}{3}e_3\Big) \\ &= -e_1^2 - e_2^2 - \frac{8+\alpha}{3}e_3^2\,, \end{aligned}$$

so $\dot V < 0$. That is, V satisfies Eq. (14). From Lyapnov theory, we know that e_i $(i = 1,2,3)$ tends to zero asymptotically. Therefore, the generalized projective synchronization of two different chaotic systems is achieved.

3 Numerical Simulation

In this section, based on the above method and symbolic computation *Maple*, we use the numerical simulation to perform the process of the synchronization and the effectiveness of the above-designed controller successfully. Arbitrarily given the initial states of the two different chaotic systems — LCL system and Rössler system, $(0.2, 0.3, 0.5)$ and $(0.1, 0.2, 0.3)$ respectively, we can choose $m = -2$. Figure 2 displays the time response of the synchronization error e_i $(i = 1,2,3)$. Obviously, e_1, e_2, and e_3 converge to zero finally after the controller is activated. That is, all the state variables tend to be synchronized in proportion. Figure 3 depicts the projection of the synchronized attractors onto the y-z plane.

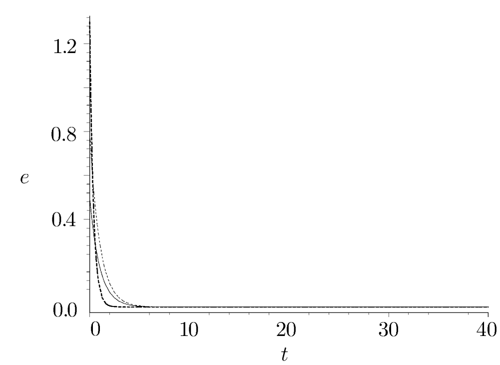

Fig. 2 The error e_i $(i = 1,2,3)$. The solid line is e_1, the dotted line is e_2, and the dash-dotted line is e_3.

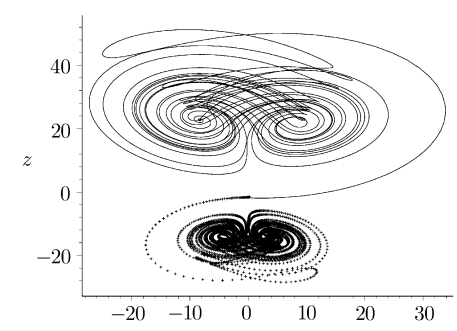

Fig. 3 The projection of the synchronized attractors onto the y-z plane after transient states.

From Fig. 3, we can easily observe the ratio of the amplitudes of the two systems tends to a constant scaling factor. A natural problem is whether the two different chaotic systems completely synchronize if we make the response system multiplied by the scaling factor. So we can get the result in Fig. 4.

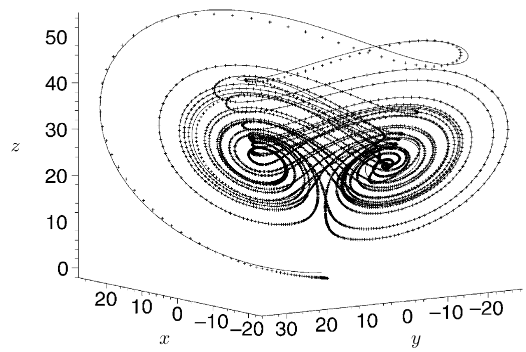

Fig. 4 Complete synchronization of the two different chaotic systems with the scaling factor m. The solid line denotes the trajectory of the drive system — LCL system, and the dotted line denotes the trajectory of Rössler system synchronized with active controllers and scaling factor.

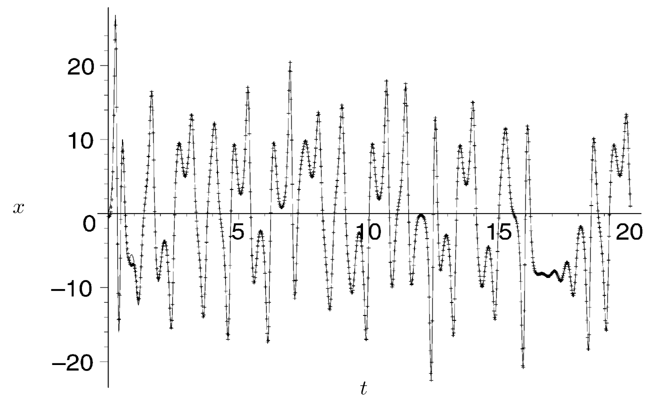

Fig. 5 Complete synchronization of the two different chaotic systems with the scaling factor m on t-x plane. The solid line denotes for LCL system, and the dotted line denotes for Rössler system synchronized with active controllers and scaling factor.

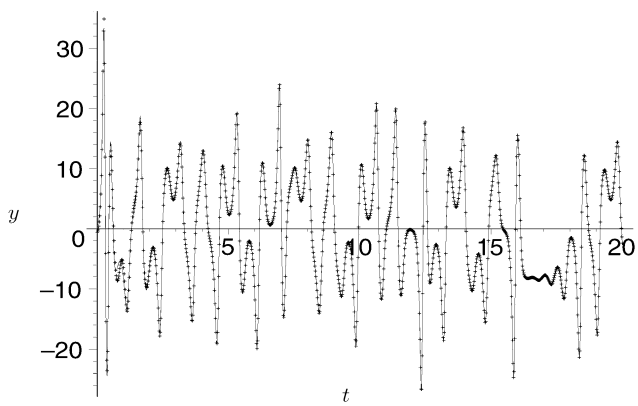

Fig. 6 Complete synchronization of the two different chaotic systems with the scaling factor m on t-y plane. The solid line denotes for LCL system, and dotted line denotes for Rössler system synchronized with active controllers and scaling factor.

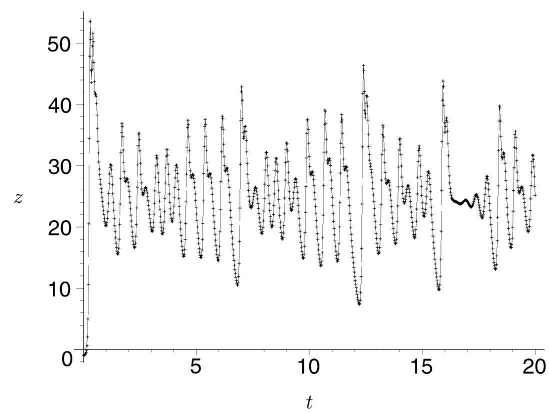

Fig. 7 Complete synchronization of the two different chaotic systems with the scaling factor m on t-z plane. The solid line denotes for LCL system, and dotted line denotes for Rössler system synchronized with active controllers and scaling factor.

In the following, we give Rössler system and LCL system the projective synchronization on t-x plane, t-y plane, and t-z plane in Figs. 5, 6, and 7, respectively.

Remark Recently, based on the backstepping design and Lyapunov stability theory, Yan[13−15] developed a series of systematic, concrete, and the automatic scheme to investigates the Q-S synchronization, for example, between Rössler system and LCL system in Ref. [13]. It is obvious that the method used above can be powerful performed in the computer.

4 Summary and Conclusion

In summary, based on the symbolic computation, we choose Lorenz–Chen–Lü system as drive system and Rössler system as response system to realize the generalized projective synchronization of two different chaotic dynamic systems with the active control method, and make them completely synchronize with a scaling factor. Numerical simulations are used to verify the effectiveness of the method for complete different chaotic systems, and there exist controllers to make the Lorenz–Chen–Lü system and other different known chaotic systems synchronize. And it can be used for other chaotic systems and hyperchaotic systems.

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