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Function Projective Synchronization of Two Identical New Hyperchaotic Systems*

LI Xin[1,3] and CHEN Yong[1,2,3,†]

[1]Nonlinear Science Center and Department of Mathematics, Ningbo University, Ningbo 315211, China

[2]Institute of Theoretical Computing, East China Normal University, Shanghai 200062, China

[3]Key Laboratory of Mathematics Mechanization, the Chinese Academy of Sciences, Beijing 100080, China

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Abstract *A function projective synchronization of two identical hyperchaotic systems is defined and the theorem of sufficient condition is given. Based on the active control method and symbolic computation Maple, the scheme of function projective synchronization is developed to synchronize the two identical new hyperchaotic systems constructed by Yan up to a scaling function matrix with different initial values. Numerical simulations are used to verify the effectiveness of the scheme.*

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1 Introduction

Chaos control and synchronization have been intensively and extensively studied in recent decades, due to their wide potential applications in many areas such as secure communication, information processing, and chemical reactions, etc.[1−5] Since the first chaotic system found by Lorenz in a three-dimensional autonomous system in 1963,[6] many important chaotic systems have been found and constructed.[7−9] In particular, since Rössler[10] first introduced the Rössler hyperchaotic system, which meant that the system possesses more than one positive Lyapunov exponents, many chaotic systems have been reported in nonlinear field, people focus on studying and constructing hyperchaotic systems, defined as a chaotic system with more than one positive Lyapunov exponent. There exist some well-known hyperchaotic systems such as hyperchaotic Rössler system,[10] hyperchaotic Chua's circuit,[11] hyperchaotic Matsumoto–Chua–Kobayashi circuit,[12] hyperchaotic Tamasevicius–Namajunas–Cenys system,[13] hyperchaotic Tamasevicius–Cenys–Namajunas system,[14] hyperchaotic Chen system,[15] Hénon-like map,[16] 4D hyperchaotic oscillator and hyperchaotic oscillator with gyrators,[17] and so on. Since hyperchaotic system has the characteristics of high capacity, high security and high efficiency, it has been broadly applied in nonlinear circuits, secure communications, lasers, neural networks, biological systems, and so on. Recently, Yan[18] obtained a new hyperchaotic system by introducing an additional state, and adding two nonlinear terms of the original states and one linear term of the new state to the Chen chaotic system,[19] finally globally exponential hyperchaos (time-delayed) synchronization and control for this hyperchaotic system are studied.

In this paper, by introducing a new more general synchronization form than known project projective synchronization,[20−25] we develop the function projective synchronization (FPS) to investigate the synchronization of a new hyperchaotic system constructed by Yan.[18] Numerical simulation results are given for illustration and verification.

The rest of the paper is arranged as follows. In Sec. 2, we give the introduction of function projective synchronization. In Sec. 3, using active control, a controlled drive slave is constructed. The FPS of the two identical hyperchaotic systems is theoretically analyzed, and some numerical simulations are displayed. Finally, the conclusion of the paper is given.

2 Function Projective Synchronization Between Two Identical Chaotic (Hyperchaotic) Systems

Projective synchronization[20] is that the drive and response vectors evolve in a proportional scale – the vectors become proportional. The early projective synchronization is usually observable only in a class of systems with partial-linearity,[21] but recently some researchers[22−25] have achieved control of the projective synchronization in a general class of chaotic systems including non-partially-linear systems, and termed this projective synchronization as "generalized projective synchronization" (GPS). Recently, the modification of projective synchronization is proposed[25] to synchronize two identical systems up to

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†E-mail: chenyong@nbu.edu.cn

a scaling constant matrix. Based on the previous projective synchronization, function projective synchronization (FPS) is defined characterized by a scaling function matrix. Here, we give the introduction of FPS as follows.

Let $\dot{x} = F(x,t)$ be the drive chaotic system, and $\dot{y} = F(y,t) + U$ is the response system, where $x = (x_1(t), x_2(t), \ldots, x_m(t))^{\mathrm{T}}$, $y = (y_1(t), y_2(t), \ldots, y_m(t))^{\mathrm{T}}$, $U = (u_1(x,y), u_2(x,y), \ldots, u_m(x,y))$ is a controller to be determined later. Denote $e_i = x_i - f_i(x)y_i$, $(i = 1,2,\ldots,m)$, $f_i(x)$ $(i = 1,2,\ldots,m)$ are functions of x. If $\lim_{t\to\infty} \|e\| = 0$, $e = (e_1, e_2, \ldots, e_m)$, there exists *function projective synchronization (FPS)* between these two identical chaotic (hyperchaotic) systems, and we call f a "scaling function matrix".

Consider the drive system in the form of

$$\dot{x} = Ax + h(x,t)\,, \tag{1}$$

where $x \in R^n$, A is an $m \times m$ constant matrix, and $h : R^n \to R^n$ is a nonlinear function. Assume that the response system coupled with Eq. (1) is as follows:

$$\dot{y} = Ay + h(y,t) + U\,, \tag{2}$$

where $y \in R^n$, and U is a controller to be determined later.

Theorem 1 For an invertible diagonal function matrix f, function projective synchronization between the two systems (1) and (2) will occur, if the following conditions are satisfied:

(i) $U = f^{-1}h(x,t) + (f^{-1}Af - A)y + f^{-1}B(x - fy) - h(y,t) - f^{-1}gy$, where $g = \mathrm{diag}(\dot{f}_1, \dot{f}_2, \ldots, \dot{f}_m)$, and $B \in R^{m\times m}$;

(ii) The real parts of all the eigenvalues of $(A - B)$ are negative.

Proof From $e = x - fy$ in definition of FPS, one can get

$$\begin{aligned}\dot{e} &= \dot{x} - f\dot{y} - gy\\ &= Ax + h(x,t) - f(Ay + h(y,t) + U) - gy\\ &= Ax + h(x,t) - fAy - fh(y,t) - gy - h(x,t) - Afy\\ &\quad + fAy - B(x - fy) + fh(y,t) + gy\\ &= (A - B)e\,.\end{aligned}$$

For a feasible control, the feedback B must be selected such that all the eigenvalues of $(A-B)$, if any, have negative real parts. Thus, if the matrix $(A-B)$ is in full rank, the system $\dot{e}$ is asymptotically stable at the origin, which implies (1) and (2) are in the state of function projective synchronization.

In this case, the active control method[26] is usually adopted to obtain the gain matrix B for any specified eigenvalues of $(A - B)$.

Remark When $f_1 = f_2 = \cdots = f_m = 1$, $f_1 = f_2 = \cdots = f_m = \alpha$ and $f_1 = \alpha_1$, $f_2 = \alpha_2, \cdots, f_m = \alpha_m$, CS, generalized projective synchronization and modified projective synchronization will appear, respectively. And the scaling function matrix f has no effect on the eigenvalues of $(A-B)$, that is to say, one can adjust the scaling function matrix arbitrarily during control without worrying about the control robustness.

3 FPS of Two Identical Hyperchaotic Systems

At the first, we introduce a new hyperchaotic system constructed by Yan[18] from Chen system. The 4-dimensional nonlinear dynamical system is introduced as

$$\begin{aligned}\dot{x} &= a(y - x)\,,\\ \dot{y} &= (c - a)x - xz_c y\,,\\ \dot{z} &= -bz + xy - yz + xz - w\,,\\ \dot{w} &= -dw + yz - xz\,.\end{aligned} \tag{3}$$

As we all know, a "regular" chaotic system has one positive Lyapunov exponent. Hyperchaotic systems with more than one positive Lyapunov exponent, on the other hand, are more complex and also play a significant role in nonlinear science. When $a = 37$, $b = 3$, $c = 26$, $d = 38$, computation shows that system (3) has the following Lyapunov exponents: $\lambda_1 = 1.319$, $\lambda_2 = 0.146$, $\lambda_3 = -20.148$, $\lambda_4 = -56.337$. The two positive Lyapunov exponents indicate that system (3) is hyperchaotic. Figures 1(a) $\sim$ 1(d) displays the projections of the hyperchaotic attrator of system (3) in the x-y-z space, the x-y-w space, the x-z-w space and the y-z-w space, respectively.

In this section, we use active control method to make two identical hyperchaotic Chen systems globally synchronized.

The drive and response systems are defined as follows:

$$\begin{aligned}\dot{x_1} &= a(y_1 - x_1)\,,\\ \dot{y_1} &= (c - a)x_1 - x_1z_1 + cy_1\,,\\ \dot{z_1} &= -bz_1 + x_1y_1 - y_1z_1 + x_1z_1 - w_1\,,\\ \dot{w_1} &= -dw_1 + y_1z_1 - x_1z_1\,;\end{aligned} \tag{4}$$

and

$$\begin{aligned}\dot{x_2} &= a(y_2 - x_2) + u_1(t)\,,\\ \dot{y_2} &= (c - a)x_2 - x_2z_2 + cy_2 + u_2(t)\,,\\ \dot{z_2} &= -bz_2 + x_2y_2 - y_2z_2 + x_2z_2 - w_2 + u_3(t)\,,\\ \dot{w_2} &= -dw_2 + y_2z_2 - x_2z_2 + u_4(t)\,.\end{aligned} \tag{5}$$

Here is written as $\dot{x} = Ax + h(x,t)$, where

$$A = \begin{pmatrix} -a & a & 0 & 0\\ c-a & c & 0 & 0\\ 0 & 0 & b & -1\\ 0 & 0 & 0 & -d \end{pmatrix}$$

and $h(x,t) = (0, -x_1z_1, x_1y_1 - y_1z_1 + x_1z_1, y_1z_1 - x_1z_1)^{\mathrm{T}}$.

Referring to the original methods of active control, we choose $f(x) = \mathrm{diag}(1, 1 + \tanh {z_1}^2, \tanh z_1, \tanh z_1)$, that is

$$
\begin{aligned}
e_1 &= x_1 - x_2\,,\\
e_2 &= y_1 - y_2(1 + \tanh {z_1}^2)\,,\\
e_3 &= x_1 - z_2 \tanh z_1\,,\\
e_4 &= w_1 - w_2 \tanh z_1\,.
\end{aligned}
\tag{6}
$$

By feedback stepping method, we choose

$$
B = \begin{pmatrix} -a + k_1 & a & 0 & 0 \\ c - a & c + k_2 & 0 & 0 \\ 0 & 0 & b + k_3 & -1 \\ 0 & 0 & 0 - d + k_4 & \end{pmatrix}.
$$

It is easy to get

$$
A - B = \begin{pmatrix} -k_1 & 0 & 0 & 0 \\ 0 & -k_2 & 0 & 0 \\ 0 & 0 & -k_3 & 0 \\ 0 & 0 & 0 & -k_4 \end{pmatrix},
$$

where k_1, k_2, k_3, and k_4 are all positive real numbers or complex numbers with positive real parts. Here we choose $k_1 = 2$, $k_2 = 3$, $k_3 = 3$, and $k_4 = 4$. And correspondingly the four control functions u_i, $(i = 1, 2, 3)$ are in the form as follows:

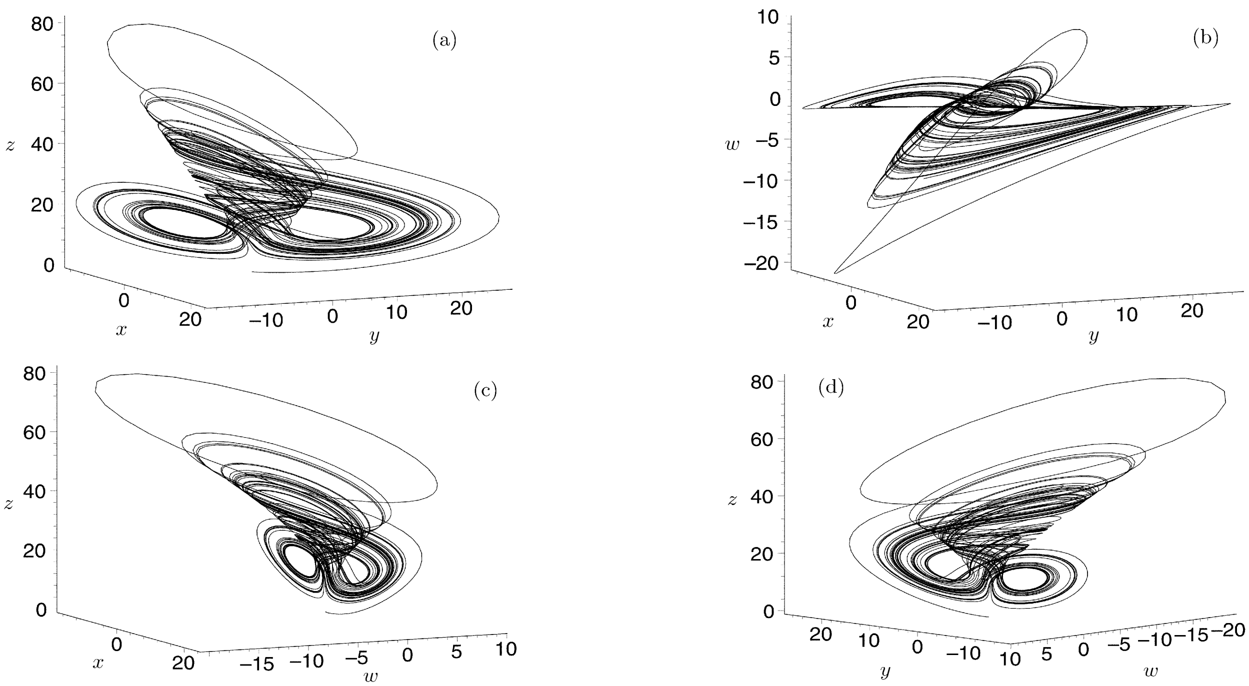

Fig. 1 Simulated phase portraits of the hyperchaotic system (3) with parameter values $a = 37$, $b = 3$, $c = 26$, $d = 38$, projective in (a) the x-y-z space; (b) the x-y-w space; (c) the x-z-w space; and (d) the y-z-w space.

$$
U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix}
-(a - k_1)e_1 + a(y_1 - y_2) \\
-[(c-a)x_2 - x_2 z_2 + c y_2] + \dfrac{(c-a)x_1 - x_1 z_1 + c y_1}{1 + \tanh {z_1}^2} \\
+ \dfrac{-2 y_2 \tanh z_1 (1 - \tanh z_1)^2 (-b z_1 + x_1 y_1 - y_1 z_1 + x_1 z_1 - w_1) + k_2 e_2}{1 + \tanh {z_1}^2} \\
b z_2 - x_2 y_2 + y_2 z_2 - x_2 z_2 + w_2 + (x_1 y_1 z_2 - b z_1 z_2 - y_1 z_1 z_2 + x_1 z_1 z_2 - z_2 w_1) \tanh z_1 \\
+ \dfrac{-b z_1 + x_1 y_1 - y_1 z_1 + x_1 z_1 - w_1 + b z_1 z_2 - x_1 y_1 z_2 + y_1 z_1 z_2 - x_1 z_1 z_2 + z_2 w_1 + k_3 e_3)}{\tanh z_1} \\
d w_2 - y_2 z_2 + x_2 z_2 + (-b z_1 w_2 + x_1 y_1 w_2 - y_1 z_1 w_2 + x_1 z_1 w_2 - w_1 w_2) \tanh z_1 \\
+ \dfrac{-d w_1 + y_1 z_1 - x_1 z_1 + b z_1 w_2 + y_1 z_1 w_2 - x_1 z_1 w_2 + w_1 w_2 + k_4 e_4}{\tanh z_1}
\end{pmatrix}.
\tag{7}
$$

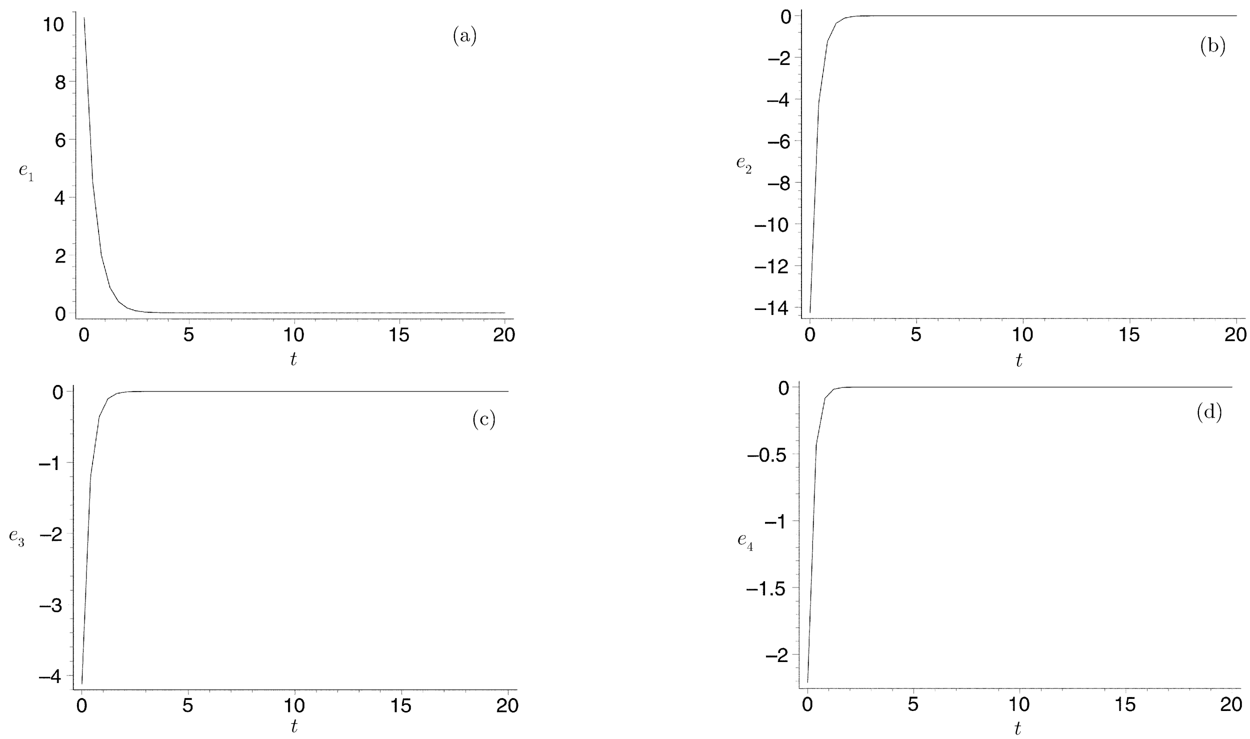

Fig. 2 (a) $\sim$ (d): The error orbits of e_1, e_2, e_3, and e_4.

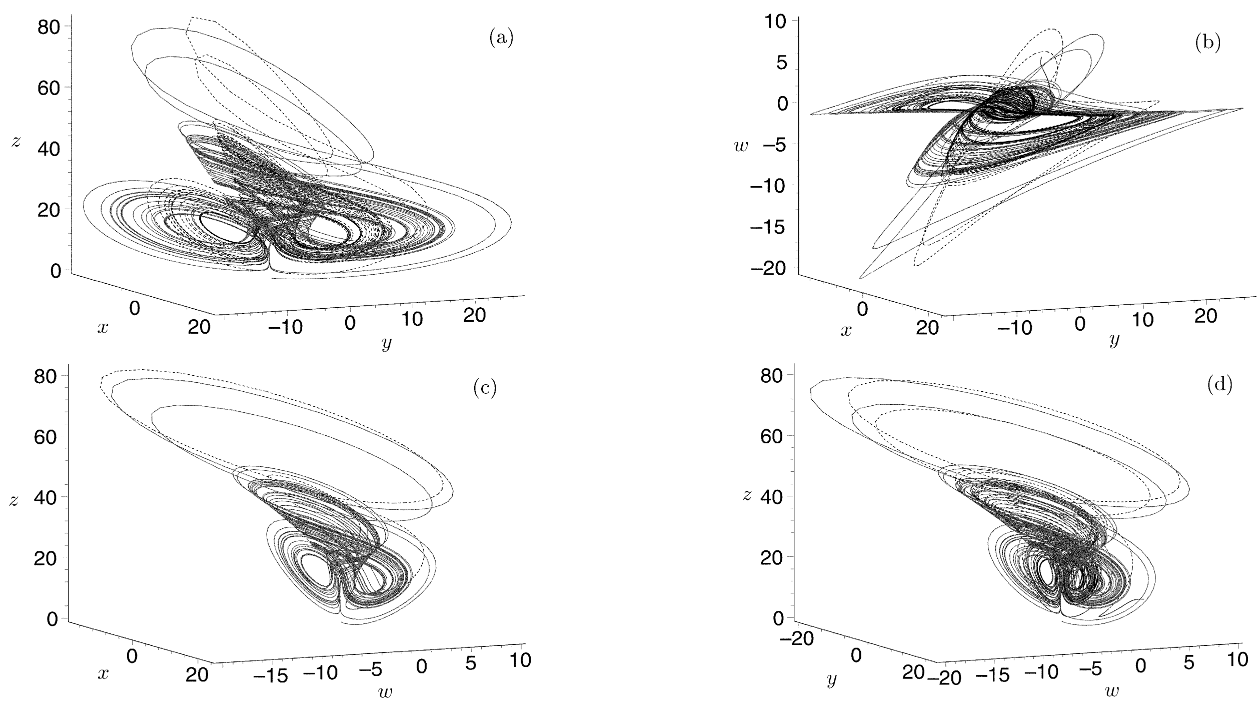

Fig. 3 (a) $\sim$ (d): The solid line denotes for the drive system, the dashed line denotes for the response system.

Based on symbolic computation system *Maple* and Lyapunov stability theory, we give some numerical simulations to illustrate our results. Figures 2(a) $\sim$ 2(d) show the numerical simulation of the error e, and figures 3(a) $\sim$ 3(d) reveal the numerical global synchronization between them with initial values (0.2,0.3,0.5,0.1) and $(-10, 12, 10, 5)$, respectively.

Figures 4(a) $\sim$ 4(f) show the projections of the drive and response hyperchaotic attractors on six coordinate planes.

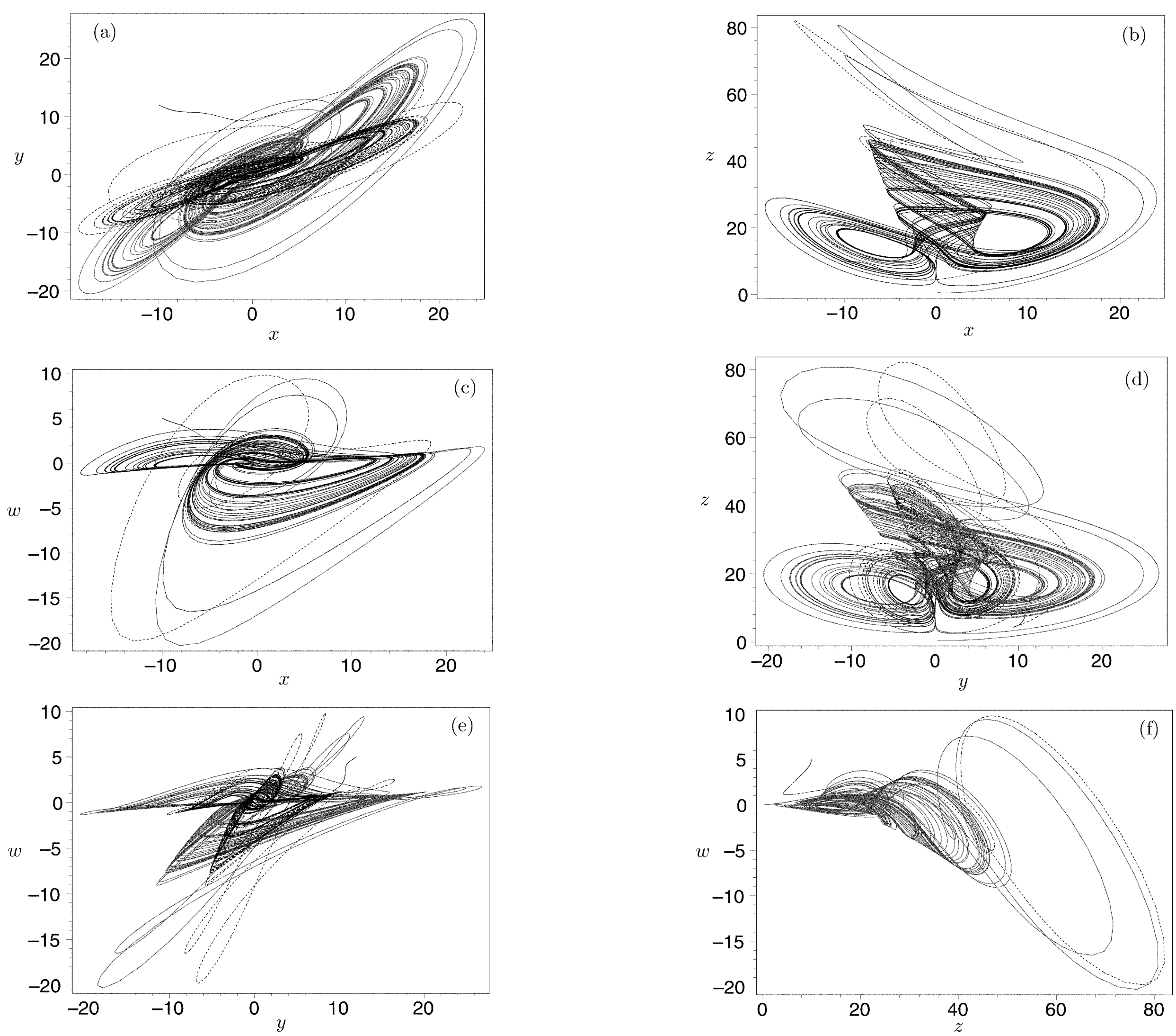

Fig. 4 (a) $\sim$ (f): The solid line denotes for the drive system, the dashed orbits denotes for the response system.

Remark In fact, $f(x)$ of the error functions has many choices, such as constant α, $\tanh x$, x^i, $\sinh x$, etc. Numeric simulations are used to verify the effectiveness of the proposed synchronization techniques. If one wants to get other results, one can choose the $f(x)$ properly. For example, $f(x) = \mathrm{diag}(-1, 2 + z_1(t), \tanh(z_1(t)), 1)$, we can get the numerical simulations, please see Figs. 5 and 6.

4 Conclusions

Based on function projective synchronization method and symbolic computation *Maple*, new two identical hyperchaotic systems constructed by Yan[18] globally are synchronized up to a scaling function matrix f. We choose two group function f to show more generality of function projective synchronization than known projective synchronization. Numerical simulations are carried out to verify the effectiveness of the proposed controller. Under different choices of function matrix f, some interesting pictures are obtained to illustrate function projective synchronization of master system and slave system in the same coordinate after transient states. With the aid of symbolic-numeric computation, the scheme can be used for other chaotic systems and hyperchaotic systems.

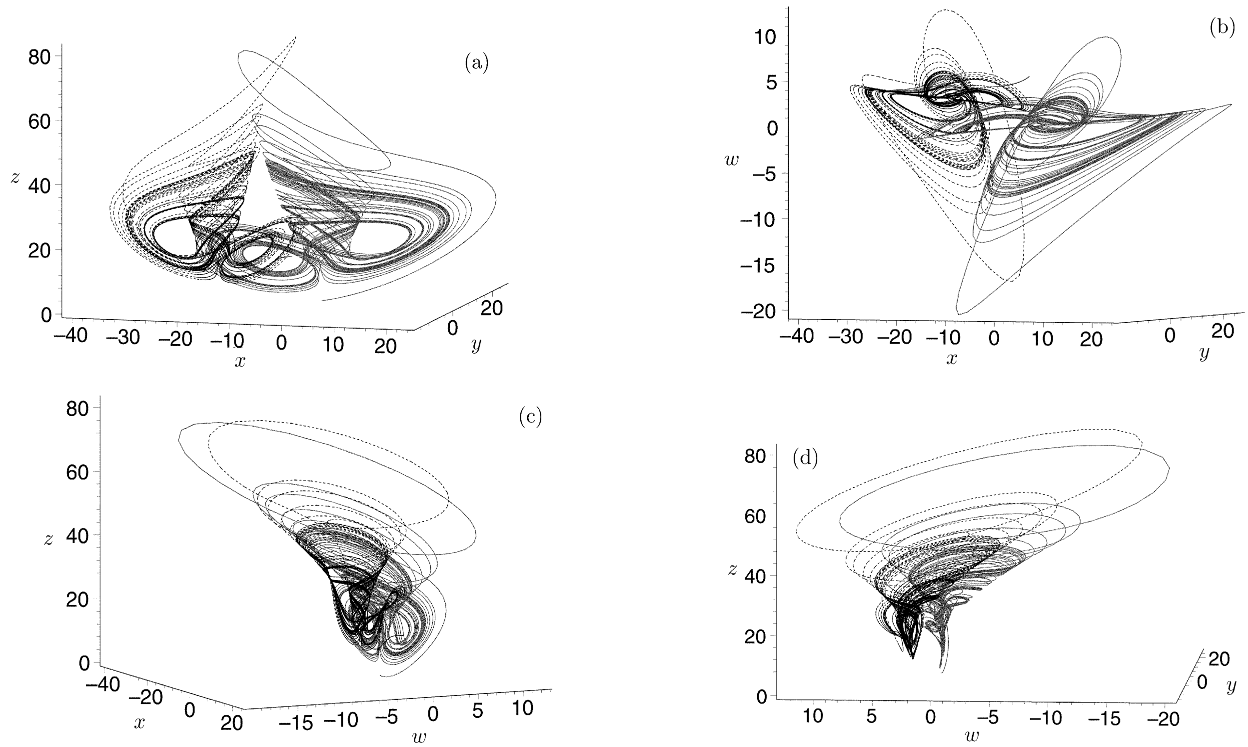

Fig. 5 (a) ∼ (d): The solid line denotes for the drive system, the dashed line denotes for the response system.

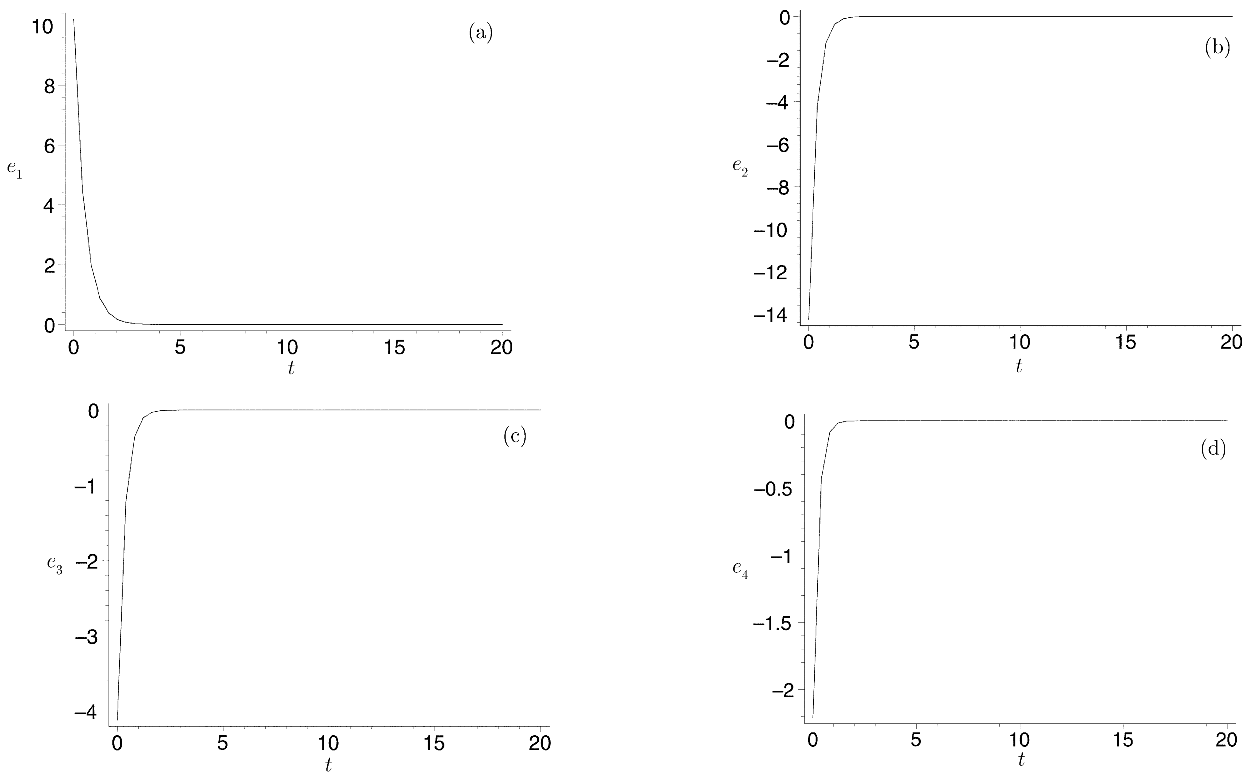

Fig. 6 (a) ∼ (d): The error orbits of e_1, e_2, e_3, and e_4.

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