Commun. Theor. Phys. (Beijing, China) **48** (2007) pp. 987–992
© International Academic Publishers Vol. 48, No. 6, December 15, 2007

An Extended Subequation Rational Expansion Method and Solutions of (2+1)-Dimensional Cubic Nonlinear Schrödinger Equation*

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(Received January 19, 2007)

Abstract *An extended subequation rational expansion method is presented and used to construct some exact analytical solutions of the (2+1)-dimensional cubic nonlinear Schrödinger equation. From our results, many known solutions of the (2+1)-dimensional cubic nonlinear Schrödinger equation can be recovered by means of some suitable selections of the arbitrary functions and arbitrary constants. With computer simulation, the properties of new non-travelling wave and coefficient function's soliton-like solutions, and elliptic solutions are demonstrated by some plots.*

PACS numbers: 02.30.Jr
Key words: (2+1)-d cubic nonlinear Schrödinger equation, soliton solution, elliptic function soltuions

1 Introduction

In the past several decades, both mathematicians and physicists have made many attempts to investigating the travelling wave solutions of nonlinear evolution equations (NLEEs), which are widely used to describe many important phenomena and dynamic processes in physics, mechanics, chemistry, biology, etc. With the development of soliton theory, many effective methods have been presented, such as inverse scattering method,[1] Bäcklund transformation,[2] Darboux transformation,[3,4] variable separation approach,[5] homogeneous balance method,[6] Hirota bilinear method,[7,8] similarity reduction method,[9] etc.

The tanh method[10,11] is considered to be one of the most straightforward and effective algorithm to obtain solitary wave solutions for a large NLEEs. Since Fan presented the extended tanh method,[12] whose key idea is to use the solutions of a Riccati equation as subequation to replace the tanh function in the tanh method, much work[13,14] has been concentrated on various extensions and applications of the tanh method, such as improved tanh method by Yan,[13] generalized extended tanh-function method by Chen *et al.*,[14] and improved extended tanh method by Fan.[15] The basic purpose of the above work was to simplify the routine calculation of the method or obtain more general travelling wave solutions. Recently, we presented a series of rational subequation expansion method, such as the Jacobi elliptic function rational expansion method,[16] the Riccati equation rational expansion method,[17] and elliptic equation rational expansion method,[18] in which the ansatzes are firstly expressed as rational form. In Ref. [19], on the basis of the methods in Refs. [10] $\sim$ [18], a generalized method was established by a more general form and used to construct some exact solutions of the nonlinear Schrödinger equation in inhomogeneous optical fiber media.

In this paper, we would like to further extend the method[19] to construct some exact analytical solutions for the (2+1)-dimensional cubic nonlinear Schrödinger (NLS) equation,[20−22]

$$\mathrm{i}\psi_t + \alpha(\psi_{xx} + \psi_{yy}) + \beta|\psi|^2\psi + \tau\psi = 0\,, \tag{1}$$

where α, β, and τ are constants, $\psi(x, y, z)$ is a complex function, and subscripts x, y, t represent partial derivatives. Equation (1) is an extension of the following two-dimensional cubic nonlinear NLS equation

$$\mathrm{i}\phi_t + \alpha(\psi_{xx} + \psi_{yy}) + \beta|\psi|^2\psi = 0\,. \tag{2}$$

In Eq. (1), the constant τ plays the role of absorption coefficient, acts as a defocusing mechanism, and depends on some physical parameters. The complex function $\psi(x, y, t)$ in the mathematical model of (2+1)-dimensional cubic NLS equation (1) arises in many physical applications and also $\psi(x, y, t)$ has different physical meanings in different branches of physics. It may be an electromagnetic potential and the NLS equation then describes, for instance, the collapse of Langmuir waves with collisional damping. In other applications, $\psi(x, y, t)$ can be a complex order parameter, describing various physical phenomena close to critical stability, in the context of the complex Ginzburg–Landau equation where τ plays the role of the instability parameter. To our knowledge, some exact soliton-like solutions of Eq. (1) have been reported by Saied *et al.*[23] through symmetry reduction method.[24,25] In Ref. [26], six families of exact analytical solutions for (2+1)-dimensional cubic NLS Eq. (1) are reported by the extended projective Riccati equation method. The present work is motivated by the desire to seek for some new exact solutions for Eq. (1).

The rest of this paper is organized as follows. In Sec. 2, an extended subequation rational method with symbolic computation for constructing exact analytical solutions of nonlinear evolution equations is proposed. In Sec. 3, we

*The project supported by Natural Science Foundation of Zhejiang Province of China under Grant Nos. Y604056 and 605408, the Doctoral Foundation of Ningbo City under Grant No. 2005A61030 and Ningbo Natural Science Foundation under Grant No. 2007A610049

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apply the method to the (2+1)-dimensional cubic NLS Eq. (1) and obtain a broad class of exact solution, and further investigate the main features of various analytical solutions obtained by using direct computer simulations. Finally a short summary and discussion is given.

2 Extended Subequation Rational Expansion Method

Now we establish an extended subsequation rational expansion method with symbolic computation as follows.

Given a nonlinear partial differential equations (PDEs) with three variables, $\{x, y, t\}$

$$p(u_t, u_x, u_y, u_{xt}, u_{yt}, u_{xy}, u_{tt}, u_{xx}, u_{yy}, \ldots) = 0\,. \tag{3}$$

Step 1 We assume that the solutions of Eq. (3) are as follows:

$$u(x, y, t) = \frac{A_0 + \sum_{i=1}^{M} \Phi(\xi)^{i-1}[A_i\Phi(\xi) + B_i\Phi'(\xi)]}{1 + \sum_{j=1}^{N} \Phi(\xi)^{j-1}[\mu_j + \nu_j\Phi'(\xi)]}\,, \tag{4}$$

where $A_0 = A_0(x, y, t)$, $A_i = A_i(x, y, t)$, $B_i = B_i(x, y, t)$, $(i = 1, \ldots, m)$, $\xi = \xi(x, y, t)$ are all differentiable functions of (x, y, t), μ_j and ν_j are arbitrary constants and new variable $\Phi(\xi)$ satisfying

$$\Phi'(\xi) = [h_0 + h_1\Phi(\xi) + h_2\Phi^2(\xi) + h_3\Phi^3(\xi) + h_4\Phi^4(\xi)]^{1/2}. \tag{5}$$

The parameter M and N can be determined by balancing the highest-order derivative term and the nonlinear terms in Eq. (3).

Step 2 Substituting Eq. (4) along with Eq. (5) into Eq. (3), extracting the numerator of the obtained system, setting the coefficients of $\Phi^i(\xi)[\Phi'(\xi)]^j$, $(i = 0, 1, \ldots;$ $j = 0, 1)$ to zero, we obtain a set of over-determined PDEs (or ODEs) with regard to the differential functions A_0, A_i, B_i $(i = 1, \ldots, M)$, and ξ.

Step 3 Solving the over-determined PDEs by use of a symbolic computation system-Maple, we would end up the explicit expressions for A_0, A_i, B_i, $(i = 1, \ldots, M)$, μ_j and ν_j $(j = 1, \ldots, N)$, and ξ or the constraints among them.

Step 4 By using the results obtained in the above steps and the various solutions of Eq. (5), we can derive rich solutions for Eq. (3).

By considering the different values of h_0, h_1, h_2, h_3, and h_4, we can derive that equation (5) admits a series of fundamental solutions. For simplification, we only list some solitary wave solutions and Jacobi and Weierstrass doubly periodic solutions as follows:

(i) ***Solitary wave solutions***

(a) *Bell shaped solitary wave solutions*

$$\Phi(\xi) = \sqrt{\frac{-h_2}{h_4}}\mathrm{sech}(\sqrt{h_2}\xi),$$
$$h_0 = h_1 = h_3 = 0, h_2 > 0, h_4 < 0\,; \tag{6}$$

$$\Phi(\xi) = -\frac{h_2}{h_3}\mathrm{sech}^2\Big(\frac{\sqrt{h_2}}{2}\xi\Big),$$
$$h_0 = h_1 = h_4 = 0, h_2 > 0\,. \tag{7}$$

(b) *Kink shaped solitary wave solutions*

$$\Phi(\xi) = \sqrt{\frac{-h_2}{h_4}}\tanh\Big(\sqrt{\frac{-h_2}{2}}\xi\Big),$$

$$h_0 = \frac{h_2^2}{4h_4}, \quad h_1 = h_3 = 0, \quad h_2 < 0, \quad h_4 > 0\,. \tag{8}$$

(c) *Solitary wave solutions*

When $h_0 = h_1 = 0$, equation (5) has a solution as follows:

$$\Phi(\xi) = -\frac{h_2\sec^2(\sqrt{-h_2}\xi/2)}{2\sqrt{-h_2h_4}\tan(\sqrt{-h_2}\xi/2) + h_3}, \quad h_2 < 0\,; \tag{9}$$

$$\Phi(\xi) = -\frac{h_2\mathrm{sech}^2(\sqrt{h_2}\xi/2)}{2\sqrt{h_2h_4}\tanh(\sqrt{h_2}\xi/2) - h_3}, \quad h_2 > 0\,. \tag{10}$$

(ii) ***Jacobi and Weierstrass doubly periodic solutions***

When $h_1 = h_3 = 0$, we have the following solutions for Eq. (5):

$$\Phi(\xi) = \sqrt{\frac{-h_2m^2}{h_4(2m^2-1)}}\mathrm{cn}\Big(\sqrt{\frac{h_2}{2m^2-1}}\xi\Big),$$
$$h_4 < 0, \quad h_2 > 0, \quad h_0 = \frac{h_2^2m^2(1-m^2)}{h_4(2m^2-1)^2}\,, \tag{11}$$

$$\Phi(\xi) = \sqrt{\frac{-h_2}{h_4(2-m^2)}}\mathrm{dn}\Big(\sqrt{\frac{h_2}{2-m^2}}\xi\Big),$$
$$h_4 < 0, \quad h_2 > 0, \quad h_0 = \frac{h_2^2(1-m^2)}{h_4(2-m^2)^2}\,, \tag{12}$$

$$\Phi(\xi) = \sqrt{\frac{-h_2m^2}{h_4(1+m^2)}}\mathrm{sn}\Big(\sqrt{\frac{-h_2}{1+m^2}}\xi\Big),$$
$$h_4 > 0, \quad h_2 < 0, \quad h_0 = \frac{h_2^2m^2}{h_4(1+m^2)^2}\,, \tag{13}$$

where m is a modulus;

$$\Phi(\xi) = \wp\Big(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\Big), \quad h_2 = h_4 = 0, \quad h_3 > 0\,, \tag{14}$$

where $g_2 = -4h_1/h_3$ and $g_3 = -4h_0/h_3$ are called invariants of Weierstrass elliptic function.

The Jacobi elliptic functions are doubly periodic and possess properties of triangular functions:

$$\mathrm{sn}^2(\xi) + \mathrm{cn}^2(\xi) = 1, \quad \mathrm{dn}^2(\xi) = 1 - m^2\mathrm{sn}^2(\xi),$$
$$\mathrm{sn}'(\xi) = \mathrm{cn}(\xi)\mathrm{dn}(\xi)\,, \tag{15}$$
$$\mathrm{cn}'(\xi) = -\mathrm{sn}(\xi)\mathrm{dn}(\xi), \quad \mathrm{dn}'(\xi) = -m^2\mathrm{sn}(\xi)\mathrm{cn}(\xi)\,. \tag{16}$$

When $m \to 1$, the Jacobi functions degenerate to the hyperbolic functions, i.e.

$$\mathrm{sn}(\xi) \to \tanh(\xi), \quad \mathrm{cn}(\xi) \to \mathrm{sech}(\xi)\,. \tag{17}$$

When $m \to 0$, the Jacobi functions degenerate to the hyperbolic functions, i.e.

$$\mathrm{sn}(\xi) \to \sin(\xi), \qquad \mathrm{cn}(\xi) \to \cos(\xi)\,. \tag{18}$$

Remark 1 The new ansatz (4) presented in the paper is more general than the ansatz in tanh method, various extended tanh-function methods and projective Riccati equation method.[10−18]

Remark 2 In order that the PDEs derived in Step 2 can be solved, we usually choose special forms of A_i, B_i, and ξ (as we do in Sec. 3).

3 Exact Analytical Solutions and Computer Simulations

We now investigate (2+1)-dimensional cubic nonlinear Schrödinger equation with the generalized method proposed in Sec. 2. According to the method, we assume that the solutions of the equation are as follows:

$$u(x,y,z)=\Big[\frac{a_0(t)+a_1(t)\Phi(\xi)+b_1(t)\Phi'(\xi)}{1+\mu\Phi(\xi)+\nu\Phi'(\xi)}\Big]\times\exp\{\mathrm{i}[\lambda_2x^2+\lambda_1x+\lambda_0+\kappa_1y+\kappa_2y^2]\}\,,\quad(19)$$

where

$$\xi=p(t)x+q(t)y+r(t)\,,\quad(20)$$

and $a_0(t)$, $a_1(t)$, $b_1(t)$, $\lambda_0=\lambda_0(t)$, $\lambda_1=\lambda_1(t)$, $\lambda_2=\lambda_2(t)$, $\kappa_1=\kappa_1(t)$, $\kappa_2=\kappa_2(t)$, $p(t)$, $q(t)$, and $r(t)$ are functions of t to be determined, $\Phi(\xi)$ satisfies Eq. (5).

Substituting Eqs. (19), (20) along with Eq. (5) into Eq. (1), removing the exponential term, collecting coefficients of monomials of $\Phi(\xi)$ and $\Phi'(\xi)$, and t of the resulting system, then separating each coefficient to the real part and imaginary part, we obtain an ordinary differential equations (ODEs) system with respect to differentiable functions $a_0(t)a_1(t)$, $b_1(t)$, $\lambda_0(t)$, $\lambda_1(t)$, $\lambda_2(t)$, $\kappa_1(t)$, $\kappa_2(t)$, $p(t)$, $q(t)$, and $r(t)$. Because the ODEs system includes 185 ODEs, for simplification, we omit them in the paper.

Solving the ODE system with symbolic computation system-Maple, we can obtain the following result.

By considering the different values of h_0, h_1, h_2, h_3, and h_4, we can derive that equation (5) admits a series of fundamental solutions. For simplicity, we only list some solitary wave solutions and Jacobi and Weierstrass doubly periodic solutions as follows:

Family 1

$$\psi_1=\frac{c_6\,\phi(\xi)}{(4\,\alpha\,t+c_7)\,(1+\mu\,\phi(\xi))}\times\exp\Big\{\mathrm{i}\Big[\frac{x^2+c_5x+c_4y+y^2}{4\alpha t+c_7}+\lambda_0\Big]\Big\}\,,$$
$$\xi=p(t)x+q(t)y+r(t)\,,\quad(21)$$

where $h_4=0$, $h_1=4\mu h_0$, $h_2=(4\mu^3h_0+h_3/2)/\mu$, and h_0,h_3 are arbitrary constants, and

$$p(t)=\frac{c_3}{4\,\alpha\,t+c_7}\,,$$
$$q(t)=\frac{\sqrt{-\alpha\,\mu\,(2\,\mu^3h_0-h_3)\,(-\alpha\,{c_3}^2\mu\,h_3+\beta\,{c_6}^2+2\,\mu^4h_0\,\alpha\,{c_3}^2)/(4\,\alpha\,t+c_7)^2}}{\alpha\mu\,(2\,\mu^3h_0-h_3)}\,,$$
$$r(t)=\frac{(c_4/2)\sqrt{-\alpha\,\mu\,(2\,\mu^3h_0-h_3)\,(-\alpha\,{c_3}^2\mu\,h_3+\beta\,{c_6}^2+2\,\mu^4h_0\,\alpha\,{c_3}^2)/(4\,\alpha\,t+c_7)^2}}{\mu\alpha\,(2\,\mu^3h_0-h_3)}+\frac{1}{2}\,\frac{c_3\,c_5}{(4\,\alpha\,t+c_7)}+c_2\,,$$
$$\lambda_0=(8\,\mu^2\alpha\,\big(2\,\mu^3h_0-h_3\big)\,(4\,\alpha\,t+c_7))^{-1}\times(\beta\,{c_6}^2h_3-4\,\mu^3h_0\,{c_6}^2\beta+4\,\mu^5h_0\,{c_5}^2\alpha+4\,\mu^5h_0\,{c_4}^2\alpha+64\,\tau\,\mu^5h_0\,t^2\alpha^2+16\,\tau\,\mu^5h_0\,t\alpha\,c_7-32\,\tau\,\mu^2h_3\,t^2\alpha^2-8\,\tau\,\mu^2h_3\,t\alpha\,c_7-2\,h_3\,\mu^2{c_4}^2\alpha-2\,h_3\,\mu^2{c_5}^2\alpha+16\,c_1\,\mu^5\alpha\,h_0\,c_7+64\,c_1\,\mu^5\alpha^2th_0-32\,c_1\,\mu^2\alpha^2th_3-8\,c_1\,\mu^2\alpha\,h_3\,c_7)\,.\quad(22)$$

If setting $h_0<0$, $h_3=-8\mu^3h_0$ from the above solution and Eq. (21) we can derive a solution as follows:

$$\psi_{11}=\frac{c_6\,\wp\Big(\sqrt{-2\,\mu^3h_0}\xi,2\,\mu^{-2},\mu^{-3}/2\Big)}{(4\,\alpha\,t+c_7)\Big(1+\mu\,\wp\Big(\sqrt{-2\,\mu^3h_0}\xi,2\,\mu^{-2},\mu^{-3}/2\Big)\Big)}\exp\Big\{\mathrm{i}\Big[\frac{(x^2+c_5x+c_4y+y^2)}{4\alpha t+c_7}+\lambda_0\Big]\Big\}\,.\quad(23)$$

Family 2

$$\psi_2=\frac{1}{4}\,\frac{c_6\,(h_3+4\,\phi(\xi)\,h_4)}{(4\,\alpha\,t+c_7)\,h_4}\exp\Big\{\mathrm{i}\Big[\frac{(x^2+c_5x+c_4y+y^2)}{4\alpha t+c_7}+\lambda_0\Big]\Big\}\,,$$
$$\xi=p(t)x+q(t)y+r(t)\,,\quad(24)$$

where h_0, h_2, h_3, and h_4 are arbitrary constants, $h_1=h_3(-h_3^2+4h_4h_2)/8h_4^2$ and

$$q(t)=\frac{c_3}{4\,\alpha\,t+c_7},\qquad q(t)=\frac{-\sqrt{-2\alpha\,h_4\,(2\,\alpha\,{c_3}^2h_4+\beta\,{c_6}^2)/(4\,\alpha\,t+c_7)^2}}{2\alpha h4}\,,$$
$$r(t)=\alpha\,\left(-\frac{1}{4}\,c_4\sqrt{-2\,\frac{\alpha\,h_4\,(2\,\alpha\,{c_3}^2h_4+\beta\,{c_6}^2)}{(4\,\alpha\,t+c_7)^2}}{h_4}^{-1}\alpha^{-2}+\frac{1}{2}\,\frac{c_3\,c_5}{(4\,\alpha\,t+c_7)\,\alpha}\right)+c_1\,,$$
$$\lambda_0=(256\,\tau\,{h_4}^2t^2\alpha^2+64\,\tau\,{h_4}^2t\alpha\,c_7-3\,h3^2\beta\,{c_6}^2+16\,{h_4}^2{c_4}^2\alpha+16\,{h_4}^2{c_5}^2\alpha+8\,\beta\,{c_6}^2h_2\,h_4+256\,c_2\,{h_4}^2\alpha^2t+64\,c_2\,{h_4}^2\alpha\,c_7)[64\,{h_4}^2(4\,\alpha\,t+c_7)\alpha]^{-1}\,,\quad(25)$$

(i) If setting $h_1 = 0$, $h_3 = 0$, $h_4 < 0$, $h_2 > 0$, $h_0 = h_2^2 m^2(1-m^2)/h_4(2m^2-1)^2$, the solution (24) is changed into

$$\psi_{21} = \frac{c_6}{4\alpha t + c_7}\sqrt{\frac{-h_2 m^2}{h_4(2m^2-1)}}\mathrm{cn}\left(\sqrt{\frac{h_2}{2m^2-1}}\xi\right)\exp\left\{\mathrm{i}\Big[\frac{(x^2+c_5x+c_4y+y^2)}{4\alpha t+c_7}+\lambda_0\Big]\right\}. \tag{26}$$

(ii) If setting $h_1 = 0$, $h_3 = 0$, $h_4 > 0$, $h_2 < 0$, $h_0 = h_2^2 m^2/h_4(m^2+1)^2$, the solution (24) is changed into

$$\psi_{22} = \frac{c_6}{4\alpha t + c_7}\sqrt{\frac{-h_2}{h_4(2-m^2)}}\mathrm{dn}\left(\sqrt{\frac{h_2}{2-m^2}}\xi\right)\exp\left\{\mathrm{i}\Big[\frac{(x^2+c_5x+c_4y+y^2)}{4\alpha t+c_7}+\lambda_0\Big]\right\}. \tag{27}$$

(iii) If setting $h_1 = 0$, $h_3 = 0$, $h_4 < 0$, $h_2 > 0$, $h_0 = h_2^2(1-m^2)/h_4(2-m^2)^2$, the solution (24) is changed into

$$\psi_{23} = \frac{c_6}{4\alpha t + c_7}\sqrt{\frac{-h_2 m^2}{h_4(1+m^2)}}\mathrm{sn}\left(\sqrt{\frac{-h_2}{1+m^2}}\xi\right)\exp\left\{\mathrm{i}\Big[\frac{(x^2+c_5x+c_4y+y^2)}{4\alpha t+c_7}+\lambda_0\Big]\right\}. \tag{28}$$

Family 3

$$\begin{aligned}\psi_3 &= \frac{c_3 + c_8\,\phi(\xi) + c_7\,\mathrm{d}\phi(\xi)/\mathrm{d}\xi}{(4\,\alpha\,t + c_9)\,(1+\nu\,\mathrm{d}\phi(\xi)/\mathrm{d}\xi)}\exp\left\{\mathrm{i}\Big[\frac{x^2+c_6x+c_5y+y^2}{4\alpha t+c_9}+\lambda_0\Big]\right\},\\ \xi &= p(t)x + q(t)y + r(t)\,,\end{aligned} \tag{29}$$

where $h_4 = 0$, and h_3 is an arbitrary constant

$$\begin{aligned}p(t) &= \frac{c_4}{4\,\alpha\,t + c_9}, \qquad q = \frac{-\sqrt{-h_3\,\alpha\,\nu\,\left(h_3\,\alpha\,c_4^2\nu/(4\,\alpha\,t+c_9)^2 - \beta\,c_8\,c_7/(4\,\alpha\,t+c_9)^2\right)}}{h_3\,\alpha\,\nu},\\ r(t) &= \alpha\left(-\frac{1}{2}\,\frac{c_5\,\sqrt{-h_3\,\alpha\,\nu\,(h_3\,\alpha\,c_4{}^2\nu - \beta\,c_8\,c_7)/(4\,\alpha\,t+c_9)^2}}{\nu h_3\alpha^2} + \frac{1}{2}\,\frac{c_4\,c_6}{(4\,\alpha\,t+c_9)\,\alpha}\right) + c_1\,,\\ \lambda_0 &= c_5{}^2\nu^2\alpha + 16\,\tau\,\nu^2t^2\alpha^2 + 4\,\tau\,\nu^2t\alpha\,c_9 + c_6{}^2\nu^2\alpha - \beta\,c_7{}^2 + 16\,c_2\,\nu^2\alpha^2 t\\ &\quad + 4\,c_2\,\nu^2\alpha\,c_9(4\,\nu^2\,(4\,\alpha\,t+c_9)\,\alpha)^{-1}\,,\\ h_2 &= \frac{(c_7 + 3\,c_3\,\nu)\,h_3}{\nu\,c_8}, \qquad h_1 = \frac{c_8{}^3\nu - 2\,h_3\,c_7{}^3 + 4\,h_3\,c_3\,\nu\,c_7{}^2 + 6\,h_3\,c_3{}^2\nu^2 c_7}{2\,c_8{}^2 c_7\,\nu^2},\\ h_0 &= \frac{c_8{}^3 c_7\,\nu - 2\,c_7{}^4 h_3 + \nu^2 c_3\,c_8{}^3 - 2\,h_3\,c_3\,\nu\,c_7{}^3 + 2\,h_3\,c_3{}^3\nu^3 c_7 + 2\,h_3\,c_3{}^2\nu^2 c_7{}^2}{2\,c_8{}^3 c_7\,\nu^3}\,.\end{aligned} \tag{30}$$

If further setting $\nu = c_7/3c_3$, and $h_3 > 0$, the solution (29) is changed into

$$\psi_{31} = \frac{3c_3(c_3 + c_8\wp(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3) + c_7(\frac{\partial}{\partial\xi}\wp(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3)))}{(4\alpha t + c_9)(-3c_3 + c_7(\frac{\partial}{\partial\xi}\wp(z, g_2, g_3)))}\exp\left\{\mathrm{i}\Big[\frac{x^2+c_6x+c_5y+y^2}{4\alpha t+c_9}+\lambda_0\Big]\right\}. \tag{31}$$

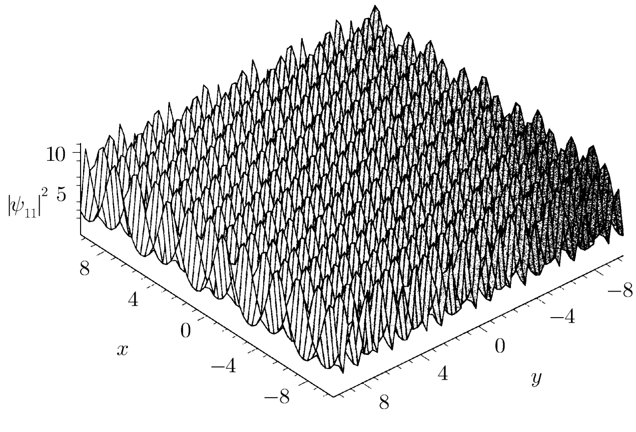

Fig. 1 The evolution plot of solution given by Eq. (23). Input parameters: $\alpha = 1$, $\beta = 3$, $\mu = 1$, $c_2 = 0$, $c_3 = 0.1$, $c_4 = 0.01$, $c_5 = 0.3$, $c_6 = 1$, $c_7 = 0.3$.

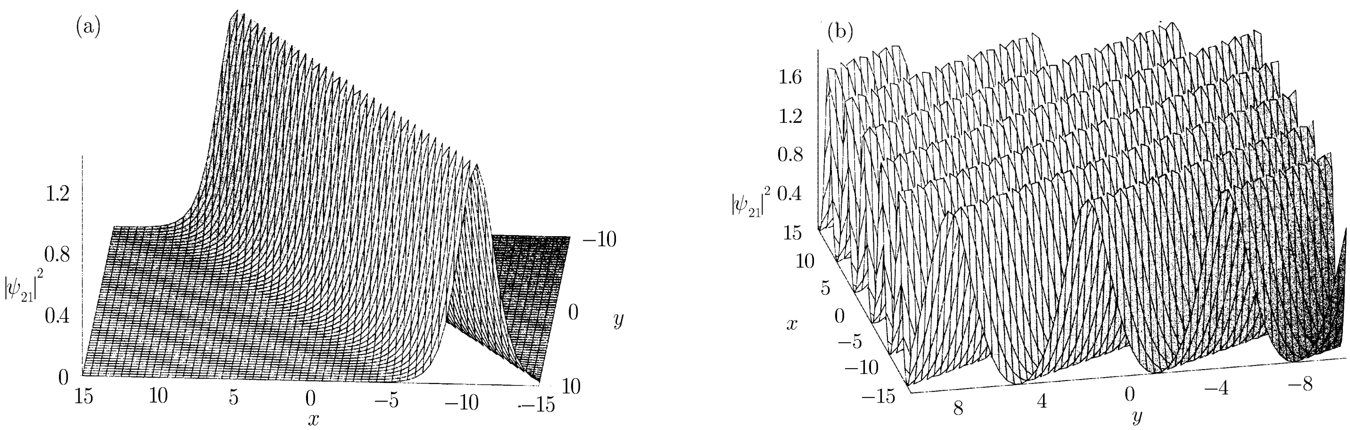

Fig. 2 The evolution plot of solution given by Eq. (26). Input parameters: $\alpha = \beta = 2$, $h_2 = 0.1$, $h_4 = -0.1$, $c_1 = 0.1$, $c_3 = 2$, $c_4 = 2$, $c_5 = 2$, $c_6 = 1.6$, $c_7 = 1$. (a) $m = 1$; (b) $m = 0.9$.

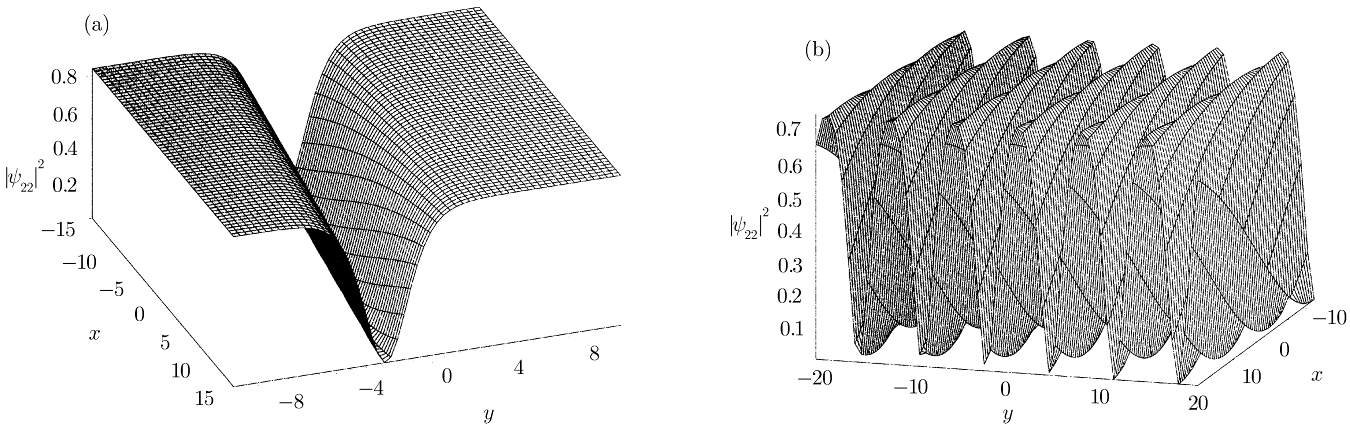

Fig. 3 The evolution plot of solution given by Eq. (27). Input parameters: $\alpha = -2$, $\beta = 2$, $h_2 = -0.1$, $h_4 = 0.1$, $c_1 = 0.1$, $c_3 = 0.2$, $c_4 = c_5 = 2$, $c_6 = 1.3$, $c_7 = 1$. (a) $m = 1$; (b) $m = 0.9$.

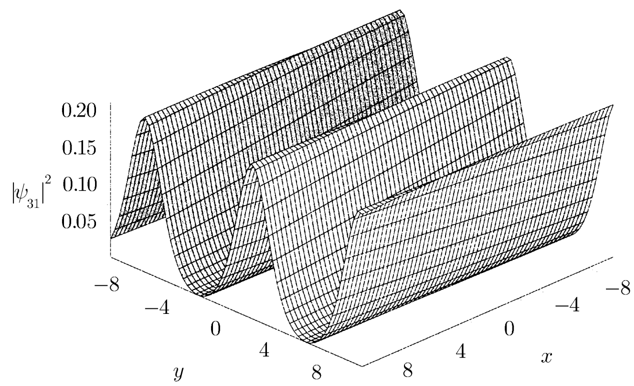

Fig. 4 The evolution plot of solution given by Eq. (31). Input parameters: $\alpha = 1$, $\beta = 3$, $c_1 = 0$, $c_2 = 0$, $c_3 = 0.1$, $c_4 = 0.01$, $c_5 = 0.3$, $c_6 = 1$, $c_7 = 0.5$, $c_8 = -1$, $c_9 = -1$.

Remark 3 From Eqs. (26) $\sim$ (28), many previous known results of the (2+1)-dimensional cubic nonlinear Schrödinger equation obtained (see, e.g., Ref. [26]) can be recovered by means of some suitable selections of the arbitrary constants. For example, if setting $m = 1$, the solutions (22) and (23) obtained in Ref. [26] can be recovered by Eqs. (26) $\sim$ (28). At the same time, the solutions obtained in Ref. [23] can be also reproduced by Eqs. (26) $\sim$ (28). But, to our knowledge, the other solutions are not reported in literature.

Remark 4 Some plots of static state solutions are given, when $t = t_0$, to show properties of solutions obtained. In order to understand the evolution of the solution (23), the main features of them were investigated by direct computer simulations with the accuracy as high as 10^{-9}. For simplification, we only consider one example for each solution under some special parameters.

From Figs. 1 and 4, we can see that the evolution of the Weierstrass elliptic doubly periodic solution presents the periodic property. With regard to Jacobi elliptic function solutions, from Figs. 2 and 3, we can see that when $m = 1$, the solutions present the solitary wave while when $0 < m < 1$, the solutions (26) $\sim$ (28) present the periodic wave.

4 Summary and Discussion

In the paper, to construct exact analytical solutions of nonlinear evolution equation, an extended subequation rational expansion method is presented. The (2+1)-dimensional cubic nonlinear Schrödinger equation is chosen to demonstrate the feasibility and affectivity of the method. With the help of symbolic computation, rich exact solutions of (2+1)-dimensional cubic nonlinear Schrödinger equation are derived. From our results, many known results of the (2+1)-dimensional cubic nonlinear Schrödinger equation can be recovered by means of some suitable selection of the arbitrary functions and arbitrary constants. With computer simulation, the main features of the analytical solutions are investigated by computer simulations. The method developed does provide a systematic way to generate various exact analytical solutions of the (2+1)-dimensional cubic nonlinear Schrödinger equation and can be also used to other PDEs and coupled ones.

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