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Communications in Nonlinear Science and Numerical Simulation 10 (2005) 133–138
Communications in Nonlinear Science and Numerical Simulation
www.elsevier.com/locate/cnsns

Exact solutions for two nonlinear wave equations with nonlinear terms of any order

Yong Chen *, Biao Li, Hongqing Zhang

Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China
Key Laboratory of Mathematics Mechanization, Chinese Academy of Sciences, Beijing 100080, China

Received 27 June 2002; received in revised form 27 December 2002; accepted 30 April 2003
Available online 28 August 2003

Abstract

In this paper, based on a variable-coefficient balancing-act method, by means of an appropriate transformation and with the help of *Mathematica*, we obtain some new types of solitary-wave solutions to the generalized Benjamin–Bona–Mahony (BBM) equation and the generalized Burgers–Fisher (BF) equation with nonlinear terms of any order. These solutions fully cover the various solitary waves of BBM equation and BF equation previously reported.

PACS: 05.45.Yv; 02.30.Jr
Keywords: Benjamin–Bona–Mahony equation; Burgers–Fisher equation; Solitary-wave solution

1. Introduction

In recent years, the homogeneous balance method has been widely applied to derive the nonlinear transformation and exact solutions (especially the solitary-wave solutions) [1–8]. Direct searching for exact solutions of nonlinear partial differential equations (NPDEs) has become more and more attractive partly due to the availability of symbolic computing system such as *Maple* or *Mathematica*. Tian et al. [8–12], based on the idea of homogeneous balance, proposed a variable-coefficient balancing-act method, in which the beginning point is a generic transformation for a given PDE

* Corresponding author. Address: Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China. Fax: +86-41-14707204.
E-mail addresses: chenyong@dlut.edu.c (Y. Chen), libiao@dlut.edu.cn (B. Li).

1007-5704/$ - see front matter
doi:10.1016/S1007-5704(03)00121-7

$$u(x,t) = \Omega(\partial_x, \partial_t)\{f[w(x,t)]\}, \tag{1}$$

where Ω and $w(x,t)$ are an operator and a function to be determined respectively. Substituting ansätz (1) into the original PDE, we hope to establish an ordinary differential equation (ODE) for $f(w)$. Thus, imposing conditions upon w and Ω, Ω might be determined first and foremost, which would largely reduce the amount of computations. Notice that sometimes the effects of different mechanisms that change wave forms may exactly be cancelled out, resulting in solitary waves. thus (see Ref. [12] for detail), ansätz (1) should have the following special form:

$$u(x,t) = \partial_x^m \partial_t^n \{f[w(x,t)]\}, \tag{2}$$

where m and n are integers which can be determined by balancing the linear term of highest order with nonlinear term in PDE. We find that m and n may not be a positive integer. It is necessary to seek for an appropriate transformation to utilize the above method.

Consider the generalized Benjamin–Bona–Mahony (BBM) equation with nonlinear terms of any order

$$u_t + au_x + bu^p u_x - \delta u_{xxt} = 0, \tag{3}$$

and the generalized Burgers–Fisher (BF) equation with nonlinear terms of any order [18]

$$u_t + pu^r u_x - u_{xx} - qu(1 - u^r) = 0. \tag{4}$$

In the study of many mechanical and physical problems, various of BBM equations (or regularized long-wave equations) have been proposed [13–17]. In [12], Zhang et al. presented Eq. (3) with $p = 1$:

$$u_t + au_x + buu_x - \delta u_{xxt} = 0, \tag{5}$$

and obtained its solitary-wave solutions. By means of an extended tanh-function method, Fan [18] obtained a travelling solution to the generalized BF equation. In this paper, we obtain the travelling solutions to Eqs. (3) and (4) by the method mentioned above.

2. The generalized BBM equation

According to the variable-coefficient balancing-act method [8–12], by balancing the highest order partial derivative term u_{xxt} and the nonlinear term $u^p u_x$ in Eq. (3), we obtain balance constants $m = \frac{1}{p}$, $n = \frac{1}{p}$. Make the following transformation:

$$u(x,t) = v^{1/p}(x,t). \tag{6}$$

Substituting transformation (6) into Eq. (3) yields

$$bp^2 v^3 v_x + (-1 + 3p - 2p^2)\delta v_t v_x^2 + (-1 + p)p\delta v(2v_x v_{xt} + v_t v_{xx}) + p^2 v^2 (v_t + av_x - \delta v_{xxt}) = 0. \tag{7}$$

Balancing $v^3 v_x$ and $v^2 v_{xxt}$ in Eq. (7), we get the value of the balance constant $m = 1$, $n = 1$. Therefore we seek for the solution of Eq. (7) in the form

$$v = f'' w_x w_t + f' w_{xt}, \tag{8}$$

where $f = f(w)$ and $w = w(x,t)$ are functions to be determined.

With the help of *Mathematica*, substituting (8) into (7) yields (because the formula is so long, just part of it is shown here)

$$[bp^2(f'')^3f^{(3)} + (-1+3p-2p^2)\delta(f^{(3)})^3 + 3(-1+p)\delta f''f^{(3)}f^{(4)} - p^2\delta(f'')^2f^{(5)}]w_x^5w_t^4 + \cdots = 0. \tag{9}$$

To simplify Eq. (9), setting the coefficient of $w_x^5w_t^4$ to zero yields an ordinary differential equation for f

$$bp^2(f'')^3f^{(3)} + (-1+3p-2p^2)\delta(f^{(3)})^3 + 3(-1+p)\delta f''f^{(3)}f^{(4)} - p^2\delta(f'')^2f^{(5)} = 0. \tag{10}$$

Solving (10) we obtain a solution

$$f = -\frac{2\delta(1+p)(2+p)}{bp^2}\log w. \tag{11}$$

Substituting (11) into (9), we can simplify formula (9) to a linear polynomial of $\frac{1}{w}$. Thus, setting the coefficients of $\frac{1}{w^i}$ $(i = 0,\ldots,8)$ to zero yields a set of partial differential equations for $w(x,t)$

$$(-1+3p-2p^2)\delta w_{xtt}w_{xxt}^2 + (-1+p)p\delta w_{xt}(2w_{xxt}w_{xxtt} + w_{xtt}w_{xxxt}) + p^2w_{xt}^2(w_{xtt} + aw_{xxt} - \delta w_{xxxtt}) = 0, \tag{12}$$

$$\begin{aligned}&w_{xt}(\delta w_{xt}(-3pw_{xtt}w_{xx} + 2(2+p)w_{xt}w_{xxt}) + w_x(2ap^2w_{xt}^2 - (4-5p+p^2)\delta w_{xtt}w_{xxt}\\&\quad + (-4+p)p\delta w_{xt}w_{xxtt})) + w_{tt}(-p\delta w_{xt}(-2(-1+p)w_{xx}w_{xxt} + pw_{xt}w_{xxx}) + w_x(p^2w_{xt}^2\\&\quad + (-1+3p-2p^2)\delta w_{xxt}^2 + (-1+p)p\delta w_{xt}w_{xxxt})) + w_t(2p^2w_{xt}^3 + pw_{xt}^2(apw_{xx} - 2\delta w_{xxxt})\\&\quad - (-1+p)\delta(-2pw_xw_{xxt}w_{xxtt} + w_{xtt}(2(-1+2p)w_{xx}w_{xxt} - pw_xw_{xxxt}))\\&\quad + w_{xt}((-1+p)\delta(2w_{xxt}^2 + 2pw_{xx}w_{xxtt} + pw_{xtt}w_{xxx}) + 2p^2w_x(w_{xtt} + aw_{xxt} - \delta w_{xxxtt}))) = 0,\end{aligned} \tag{13}$$

$$\begin{aligned}&\delta w_xw_{xt}(-4(2+p)w_{xt}^3 + w_{xt}(2(2+p)w_xw_{xtt} - (-7+p)pw_{tt}w_{xx}) + (4-5p+p^2)w_{tt}w_xw_{xxt})\\&\quad + w_t(4(-1+p)\delta w_{xt}^3w_{xx} + \delta w_xw_{xt}((4+p+p^2)w_{xtt}w_{xx} - 4(1+2p)w_{xt}w_{xxt})\\&\quad + w_x^2(-6ap^2w_{xt}^2 + (4-5p+p^2)\delta w_{xtt}w_{xxt} - 2(-4+p)p\delta w_{xt}w_{xxtt})\\&\quad + w_{tt}(-2(-1+p)p\delta w_{xt}w_{xx}^2 + \delta w_x(2(-1+p)^2w_{xx}w_{xxt} + p(1+p)w_{xt}w_{xxx})\\&\quad + pw_x^2(-2pw_{xt} - (-1+p)\delta w_{xxxt}))) + w_t^2(\delta((1-3p+2p^2)w_{xtt}w_{xx}^2\\&\quad + 2w_{xt}(-2(-1+p)w_{xx}w_{xxt} + pw_{xt}w_{xxx})) + w_x(-6p^2w_{xt}^2 - (-1+p)\delta(2w_{xxt}^2 + 2pw_{xx}w_{xxtt}\\&\quad + pw_{xtt}w_{xxx}) - 2pw_{xt}(apw_{xx} - 2\delta w_{xxxt})) - p^2w_x^2(w_{xtt} + aw_{xxt} - \delta w_{xxxtt})) = 0,\end{aligned} \tag{14}$$

$$\begin{aligned}&2(2+p)\delta w_{tt}w_x^3w_{xt}^2 + \delta w_tw_x^2(-12(2+p)w_{xt}^3 + w_{xt}(4(2+p)w_xw_{xtt} + (4+9p-p^2)w_{tt}w_{xx})\\&\quad + (4-5p+p^2)w_{tt}w_xw_{xxt}) + w_t^2w_x(2(-2+5p)\delta w_{xt}^2w_{xx} + \delta w_x((4-2p+p^2)w_{xtt}w_{xx}\\&\quad + 2(2-5p)w_{xt}w_{xxt}) - pw_x^2(6apw_{xt} + (-4+p)\delta w_{xxtt}) + w_{tt}(-p^2w_x^2 - (-1+p)\delta w_{xx}^2\\&\quad + p\delta w_xw_{xxx})) + w_t^3(-2(-1+p)\delta w_{xt}w_{xx}^2 + 4\delta w_x(-(-1+p)w_{xx}w_{xxt} + pw_{xt}w_{xxx})\\&\quad - pw_x^2(6pw_{xt} + apw_{xx} - 2\delta w_{xxxt})) = 0,\end{aligned} \tag{15}$$

$$w_t w_x(2(2+p)\delta w_{tt} w_x^3 w_{xt} + (2+p)\delta w_t w_x^2(-6w_{xt}^2 + w_x w_{xtt} + w_{tt} w_{xx}) + w_t^2 w_x(-ap^2 w_x^2 + 2(1+2p)\delta w_{xt} w_{xx} - 2(-1+p)\delta w_x w_{xxt}) + w_t^3(-p^2 w_x^2 - (-1+p)\delta w_{xx}^2 + p\delta w_x w_{xxx})) = 0, \tag{16}$$

$$w_t^2 w_x^3 (w_{tt} w_x^2 + w_t(-2w_x w_{xt} + w_t w_{xx})) = 0. \tag{17}$$

Setting

$$w(x,t) = 1 + \exp(Ax + Bt + C_0), \tag{18}$$

where A, B, C_0 are constants, then substituting (18) into (12)–(17) yields

$$-A^3 B^3 \exp(3(Ax + bt + C_0))(-aAp^2 - BP^2 + A^2 B\delta) = 0, \tag{19}$$

$$5A^3 B^3 \exp(4(Ax + bt + C_0))(-aAp^2 - BP^2 + A^2 B\delta) = 0, \tag{20}$$

$$9A^3 B^3 \exp(5(Ax + bt + C_0))(-aAp^2 - BP^2 + A^2 B\delta) = 0, \tag{21}$$

$$7A^3 B^3 \exp(6(Ax + bt + C_0))(-aAp^2 - BP^2 + A^2 B\delta) = 0, \tag{22}$$

$$A^3 B^3 \exp(7(Ax + bt + C_0))(-aAp^2 - BP^2 + A^2 B\delta) = 0. \tag{23}$$

Therefore,

$$w(x,t) = \begin{cases} 1 + \exp\left[A\left(x + \frac{aAp^2}{-p^2 + A^2\delta}\right) + C_0\right] & \text{if } a \neq 0,\ -p^2 + A^2\delta \neq 0, \\ 1 + \exp\left(\pm\sqrt{\frac{p^2}{\delta}}x + Bt + C_0\right) & \text{otherwise.} \end{cases} \tag{24}$$

From (6), (8), (11) and (24), the set of the exact solutions of the generalized BBM equation (1) can be obtained as follows:

$$u(x,t) = \begin{cases} \left\{\frac{aA^2(1+p)(2+p)\delta}{2b(p^2 - A^2\delta)}\operatorname{sech}^2\left[\frac{1}{2}A\left(x + \frac{ap^2}{-p^2 + A^2\delta}t\right) + C_0\right]\right\}^{1/p} & \text{if } a \neq 0,\ A^2\delta - p^2 \neq 0, \\ \left\{\pm\frac{B(1+p)(2+p)\sqrt{\delta}}{2bp}\operatorname{sech}^2\left[\frac{1}{2}\left(\pm\sqrt{\frac{p^2}{\delta}}x + Bt + C_0\right)\right]\right\}^{1/p} & \text{otherwise.} \end{cases} \tag{25}$$

It is easy to see that the solutions obtained in [12] are special cases of our solutions with $p = 1$.

3. The generalized BF equation

According to the above method, balancing the highest order partial derivative term u_{xx} and the nonlinear term $u^r u_x$ in Eq. (4), we obtain balance constants $m = \frac{1}{r}$, $n = 0$. Make the following transformation:

$$u(x,t) = v^{1/r}(x,t). \tag{26}$$

Substituting transformation (26) into Eq. (4) yields

$$(-1+r)v_x^2 + rv^2[qr(-1+v) + pv_x] + rv(v_t - v_{xx}) = 0. \tag{27}$$

Balancing $v^2 v_x$ and vv_{xx} in Eq. (27), we get the value of the balance constant $m = 1$, $n = 0$. Therefore we seek for the solutions of Eq. (27) in the form

$$v = f' w_x, \tag{28}$$

where $f = f(w)$ and $w = w(x, t)$ are functions to be determined.

With the help of *Mathematica*, substituting (28) into (27) yields (because the formula is so long, just part of it is shown here)

$$[prf'^2 f'' + (r-1)(f'')^2 - rf' f^{(3)}]w_x^4 + \cdots = 0. \tag{29}$$

Setting the coefficient of w_x^4 to zero yields an ordinary differential equation for f

$$prf'^2 f'' + (r-1)(f'')^2 - rf' f^{(3)} = 0. \tag{30}$$

Solving (30) we obtain a solution

$$f = -\frac{1+r}{pr}\log w. \tag{31}$$

Substituting (31) into (29), we can simplify formula (29) to a linear polynomial of $\frac{1}{w}$. Thus, setting the coefficients of $\frac{1}{w^i}$ $(i = 0, \ldots, 3)$ to zero yields a set of partial differential equations for $w(x, t)$

$$qr^2 w_x^2 - (-1+r)w_{xx}^2 + rw_x(-w_{xt} + w_{xxx}) = 0, \tag{32}$$

$$w_x^2[prw_x + qr(1+r)w_x - pw_{xx}] = 0. \tag{33}$$

Setting

$$w(x, t) = 1 + \exp(Ax + Bt + C_0), \tag{34}$$

where A, B, C_0 are constants, then substituting (34) into (32) and (33) yields

$$A^2 + r(-B + qr) = 0, \tag{35}$$

$$A^2 p - Bpr - Aqr(1+r) = 0. \tag{36}$$

Therefore,

$$w(x, t) = 1 + \exp\left(-\frac{pr}{1+r}x + \frac{(1+r)^2 qr + p^2 r}{(1+r)^2}t + C_0\right). \tag{37}$$

From (26), (28), (31), (37), the solution of Eq. (4) is as follows:

$$u(x, t) = \left\{\frac{1}{2} + \frac{1}{2}\tanh\left[\frac{1}{2}\left(-\frac{pr}{1+r}x + \frac{qr(1+r)^2 + p^2 r}{(1+r)^2}t\right)\right]\right\}^{1/r}. \tag{38}$$

4. Summary

In summary, using the variable-coefficient balancing-act method, the appropriate transformation and *Mathematica*, we have derived some new types of exact travelling solutions of two nonlinear partial differential equation, the generalized BBM equation and BF equation with nonlinear terms of any order. These solutions fully cover the solitary-wave solutions of various forms of BBM equations and BF equations previously reported. We introduce two appropriate transformations to these two equations because m and n in Eq. (2) may not be positive integers. This method can also be applied to other PDEs.

Acknowledgements

The work is supported by the National Natural Science Foundation of China under the Grant No. 10072013, the National Key Basic Research Development Project Program under the Grant No. G1998030600 and Doctoral Foundation of China under the Grant No. 98014119.

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