Novel exact solutions of coupled nonlinear Schödinger equations with time–space modulation*

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We construct various novel exact solutions of two coupled dynamical nonlinear Schödinger equations. Based on the similarity transformation, we reduce the coupled nonlinear Schödinger equations with time- and space-dependent potentials, nonlinearities, and gain or loss to the coupled dynamical nonlinear Schödinger equations. Some special types of nontravelling wave solutions, such as periodic, resonant, and quasiperiodically oscillating solitons, are used to exhibit the wave propagations by choosing some arbitrary functions. Our results show that the number of the localized wave of one component is always twice that of the other one. In addition, the stability analysis of the solutions is discussed numerically.

Keywords: coupled dynamical nonlinear Schödinger equations, coupled nonlinear Schödinger equations with time–space modulation, exact solutions

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1. Introduction

The nonlinear Schödinger (NLS) equation and its various generalizations have been used to describe a large number of physical problems in modern science, including the nonlinear optical systems,[1] dilute atomic gas Bose–Einstein condensates (BECs),[2,3] biomolecular dynamics,[4] and others.[5,6] In recent years, there has been much interest on the study of NLS equations with nonlinear coefficients depending either on space (inhomogeneous), time (nonautonomous), or both. The motivation comes from the applications of the model to the fields of BECs and nonlinear optics. In BEC, the nonlinear interaction can be easily tuned by an external magnetic field, namely, the Feshbach resonance management.[3,7] In the optical soliton communication, the dispersion management has been explored extensively to improve the communication.[1,8] For those variable-coefficient equations, various methods have been applied to investigate explicitly the exact solutions in the literature, particularly soliton-like solutions. The Lax pair analysis[9–14] and the Painlevé analysis[10,15–18] are very useful in discussing integrability conditions, by which some special solutions similar to the ones of the NLS equation with constant coefficients can be derived directly.[19] However, the governing equations of fundamental theoretical interest are usually not complete integrable, as the varying coefficients do not satisfy the corresponding integrability conditions. These peculiar cases stimulate the researchers to find other powerful techniques that can be applied to deal with both the integrable and nonintegrable nonlinear wave equations, such as the similarity transformation method,[20–34] the Lie group symmetry method,[35–38] the Hirota bilinear method,[39,40] the subequation expansion method,[41–44] the F-expansion technique,[45,46] and the direct method.[47–50] In particular, the similarity transformation allows us to find self-similar solutions of equations by connecting the given variable-coefficient equation to the corresponding equation with constant coefficients. The coupled NLS (CNLS) equations in (1+1)-dimension are an important model for a variety of physical problems.

In this work, we provide some novel solutions of two coupled dynamical NLS equations by means of the special rational form expansion method and then present a detailed study on the exact solution of CNLS equations with variable coefficients using the similarity transformation method. Some special types of nontravelling wave solutions, such as periodic, resonant, and quasiperiodically oscillating solitons, are used to exhibit the wave propagations by choosing some arbitrary functions. Furthermore, the localized wave propagation and interaction scenario are discussed and simulated. In particular, for the two CNLS equations with time–space modulation, it is shown that the number of the localized wave of one component is always twice that of the other one. Finally, numerical simulations are used to show the stability of our analytical solutions.

This paper is organized as follows. In Section 2, we

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present some novel solutions of two coupled dynamical NLS equations using the special rational form expansion method. In Section 3, using the similarity transformation, we reduce the CNLS equations with time- and space-dependent potentials, nonlinearities, and gain or loss to coupled dynamical NLS equations. Some special types of nontravelling wave solutions, such as periodic, resonant, and quasiperiodically oscillating solitons, are used to exhibit the wave propagations by choosing some arbitrary functions. In addition, the stability analysis of the solutions is discussed numerically. Finally, discussion and conclusion are given in Section 4.

2. Novel solutions of two coupled dynamical NLS equations

The original N coupled nonlinear Schödinger-like equations can be written as

$$\mathrm{i}q_{j\tau}+q_{j\xi\xi}+\vartheta_j q_j+\left(\sum_{j=1}^{k}\mu_{jk}|q_j|^2\right)q_j+\left(\sum_{j=1}^{k}\iota_{jk}|q_j|^2\right)q_j^*=0,\ \ j=1,\cdots,N, \tag{1}$$

where $q_j(\xi,\tau)$ denotes the complex amplitude of the j-th electric field envelope in the nonlinear optics theory or the j-th polarization component in the BEC theory, ϑ_j, μ_{jk}, and ι_{jk} are the parameters of the medium and interaction, and the subscripts in ξ and τ denote the derivatives with respect to ξ and τ as opposed to the subscript j for different components. Based on an ansatz of Lamé functions, Hioe has developed a general algorithm to study analytical solutions for the coupled nonlinear Schödinger-like equations.[51,52]

To obtain the stationary-wave solution for Eq. (29), we substitute $q_j(\xi,\tau)=\Phi_j(\xi)\exp(\mathrm{i}\Omega\tau)$ in Eq. (29), then equation (29) reduces to the following equations:

$$\frac{\mathrm{d}^2\Phi_j}{\mathrm{d}\xi^2}+v_j\Phi_j+\left(\sum_{j=1}^{k}g_{jk}\Phi_k^2\right)\Phi_j=0,\ \ j=1,\ldots,N, \tag{2}$$

where $v_j=\vartheta_j-\Omega$ and $g_{jk}=\mu_{jk}+\iota_{jk}$. Equation (2) can be called as the associated dynamical coupled NLS equations. In this work, we only consider the case of $N=2$ for Eq. (29), and rewrite the two coupled dynamical NLS equations as

$$\frac{\mathrm{d}^2\Phi_1}{\mathrm{d}\xi^2}+v_1\Phi_1+g_{11}\Phi_1^3+g_{12}\Phi_2^2\Phi_1=0, \tag{3a}$$

$$\frac{\mathrm{d}^2\Phi_2}{\mathrm{d}\xi^2}+v_2\Phi_2+g_{21}\Phi_1^2\Phi_2+g_{22}\Phi_2^3=0. \tag{3b}$$

For two coupled dynamical NLS equations (3), although abundant families of elliptic function solutions have been given in Refs. [51] and [52], there are other solutions with different forms. In the following, with the help of one kind of the special rational form expansion method, some nontrivial solutions of Eq. (3) can be derived directly. We assume the solution of Eq. (3) in the form

$$\Phi_1=\frac{a_0+a_1\phi+a_2\phi'+a_3\phi^2+a_4\phi\phi'\pm\sqrt{a_5+a_6\phi+a_7\phi^2}}{\sqrt{c_0+c_1\phi+c_2\phi^2}}, \tag{4a}$$

$$\Phi_2=\frac{b_0+b_1\phi+b_2\phi'+b_3\phi^2+b_4\phi\phi'\pm\sqrt{b_5+b_6\phi+b_7\phi^2}}{\sqrt{d_0+d_1\phi+d_2\phi^2}}, \tag{4b}$$

where ϕ is the Jacobi elliptic function or rational combination of Jacobi elliptic functions such as $\mathrm{sn}(\mu\xi,k)$, $\mathrm{cn}(\mu\xi,k)$, $\mathrm{dn}(\mu\xi,k)$, $\mathrm{sn}(\mu\xi,k)[A+\mathrm{dn}(\mu\xi,k)]^{-1}$, and $\mathrm{cn}(\mu\xi,k)[A+\mathrm{dn}(\mu\xi,k)]^{-1}$ $(A\geq 0)$. The parameters a_m, b_m $(m=0,1,\ldots,7)$, c_n, and d_n $(n=0,1,2)$ are real constants to be determined. Substituting Eq. (4) into Eq. (3) yields various nontrivial solutions as follow.

Family 1 When $\Phi_1=a_0+a_1\phi$ and $\Phi_2=b_0+b_1\phi$,

case 1

$$\Phi_1=-\frac{\lambda a_1}{k}+a_1\mathrm{sn}(\mu\xi,k),$$

$$\Phi_2=\frac{\lambda b_1}{k}+b_1\mathrm{sn}(\mu\xi,k),\quad \lambda=\pm\frac{\sqrt{2k^2+2}}{2}, \tag{5a}$$

$$v_1=v_2=\mu^2(1+k^2),$$

$$3g_{11}=g_{21}=-\frac{3\mu^2k^2}{2a_1^2},\ \ g_{12}=3g_{22}=-\frac{3\mu^2k^2}{2b_1^2}; \tag{5b}$$

case 2

$$\Phi_1=-\frac{\lambda a_1}{k}+a_1\mathrm{cn}(\mu\xi,k),$$

$$\Phi_2=\frac{\lambda b_1}{k}+b_1\mathrm{cn}(\mu\xi,k),\quad \lambda=\pm\frac{\sqrt{4k^2-2}}{2}, \tag{6a}$$

$$v_1=v_2=\mu^2(1-2k^2),$$

$$3g_{11}=g_{21}=\frac{3\mu^2k^2}{2a_1^2},\ \ g_{12}=3g_{22}=\frac{3\mu^2k^2}{2b_1^2}; \tag{6b}$$

case 3

$$\Phi_1=-\lambda a_1+a_1\mathrm{dn}(\mu\xi,k),$$

$$\Phi_2=\lambda b_1+b_1\mathrm{dn}(\mu\xi,k),\quad \lambda=\pm\frac{\sqrt{4-2k^2}}{2}, \tag{7a}$$

$$v_1=v_2=\mu^2(k^2-2),$$

$$3g_{11}=g_{21}=\frac{3\mu^2}{2a_1^2},\ \ g_{12}=3g_{22}=\frac{3\mu^2}{2b_1^2}; \tag{7b}$$

case 4

(i)

$$\Phi_1=a_1\mathrm{cd}(\mu\xi,k),\ \ \Phi_2=b_1\mathrm{cd}(\mu\xi,k),$$

$$v_1=v_2=\mu^2(1+k^2), \tag{8a}$$

$$g_{12}=-\frac{g_{11}a_1^2+2\mu^2k^2}{b_1^2},\ \ g_{22}=-\frac{g_{21}a_1^2+2\mu^2k^2}{b_1^2}; \tag{8b}$$

(ii)

$$\Phi_1 = -\frac{\lambda a_1}{k} + a_1 \mathrm{cd}(\mu\xi, k),$$
$$\Phi_2 = \frac{\lambda b_1}{k} + b_1 \mathrm{cd}(\mu\xi, k), \quad \lambda = \pm\frac{\sqrt{2k^2+2}}{2}, \tag{9a}$$
$$v_1 = v_2 = \mu^2(1+k^2),$$
$$3g_{11} = g_{21} = -\frac{3\mu^2k^2}{2a_1^2}, \; g_{12} = 3g_{22} = -\frac{3\mu^2k^2}{2b_1^2}; \tag{9b}$$

(iii)

$$\Phi_1 = \frac{a_1 \mathrm{cn}(\mu\xi,k)}{\sqrt{1-k^2}+\mathrm{dn}(\mu\xi,k)}, \; \Phi_2 = \frac{b_1 \mathrm{cn}(\mu\xi,k)}{\sqrt{1-k^2}+\mathrm{dn}(\mu\xi,k)}, \tag{10a}$$
$$v_1 = v_2 = \frac{\mu^2(2-k^2)}{2},$$
$$g_{11} = -\frac{2g_{12}b_1^2+\mu^2k^4}{2a_1^2}, \; g_{22} = -\frac{2g_{21}a_1^2+\mu^2k^4}{2b_1^2}; \tag{10b}$$

case 5

(i)

$$\Phi_1 = a_1 \mathrm{sd}(\mu\xi,k), \; \Phi_2 = b_1 \mathrm{sd}(\mu\xi,k),$$
$$v_1 = v_2 = \mu^2(1-2k^2), \tag{11a}$$
$$g_{12} = \frac{2\mu^2k^2-2\mu^2k^4-g_{11}a_1^2}{b_1^2},$$
$$g_{22} = \frac{2\mu^2k^2-2\mu^2k^4-g_{21}a_1^2}{b_1^2}; \tag{11b}$$

(ii)

$$\Phi_1 = -\frac{\lambda a_1}{k} + a_1 \mathrm{sd}(\mu\xi,k), \quad \Phi_2 = \frac{\lambda b_1}{k} + b_1 \mathrm{sd}(\mu\xi,k),$$
$$\lambda = \pm\frac{\sqrt{2(k^2-2)(1-2k^2)}}{2(k^2-1)}, \tag{12a}$$
$$v_1 = v_2 = \mu^2(1-2k^2),$$
$$3g_{11} = g_{21} = \frac{3\mu^2k^2(1-k^2)}{2a_1^2},$$
$$g_{12} = 3g_{22} = \frac{3\mu^2k^2(1-k^2)}{2b_1^2}; \tag{12b}$$

(iii)

$$\Phi_1 = \frac{a_1 \mathrm{sn}(\mu\xi,k)}{1+\mathrm{dn}(\mu\xi,k)}, \; \Phi_2 = \frac{b_1 \mathrm{sn}(\mu\xi,k)}{1+\mathrm{dn}(\mu\xi,k)},$$
$$v_1 = \mu^2(1-\frac{k^2}{2}), \tag{13a}$$
$$v_1 = v_2 = \mu^2\left(1-\frac{k^2}{2}\right),$$
$$g_{12} = -\frac{\mu^2k^4+2g_{11}a_1^2}{2b_1^2}, \; g_{22} = -\frac{\mu^2k^4+2g_{21}a_1^2}{2b_1^2}; \tag{13b}$$

(iv)

$$\Phi_1 = \frac{\lambda a_1}{k^2} + \frac{a_1 \mathrm{sn}(\mu\xi,k)}{1+\mathrm{dn}(\mu\xi,k)},$$
$$\Phi_2 = -\frac{\lambda b_1}{k^2} + \frac{b_1 \mathrm{sn}(\mu\xi,k)}{1+\mathrm{dn}(\mu\xi,k)}, \quad \lambda = \sqrt{2-k^2} \tag{14a}$$
$$v_1 = v_2 = \mu^2\left(1-\frac{k^2}{2}\right),$$
$$3g_{11} = g_{21} = -\frac{3\mu^2k^4}{8a_1^2}, \; g_{12} = 3g_{22} = -\frac{3\mu^2k^4}{8b_1^2}. \tag{14b}$$

Family 2 When $\Phi_1 = a_0 + a_3\phi^2$ and $\Phi_2 = b_0 + b_3\phi^2$,

$$\Phi_1 = -\frac{a_3b_3}{3b_0k^2} + a_3 \mathrm{cd}^2(\mu\xi,k),$$
$$\Phi_1 = b_0 + b_3 \mathrm{cd}^2(\mu\xi,k), \tag{15a}$$
$$k^2 = -\frac{b_3(b_3+2b_0)}{b_0(3b_0+2b_3)},$$
$$v_1 = -v_2 = \frac{2\mu^2(3b_0^2+3b_0b_3+b_3^2)}{b_0(3b_0+2b_3)}, \tag{15b}$$
$$g_{12} = g_{22} = -\frac{a_3^2g_{21}}{b_3^2},$$
$$g_{11} = g_{21} = \frac{-9\mu^2b_3^2(b_3+2b_0)^2}{2b_0a_3^2(3b_0+2b_3)(3b_0^2+3b_0b_3+b_3^2)}. \tag{15c}$$

Family 3 When $\Phi_1 = a_0 + a_1\phi$ and $\Phi_2 = \pm\sqrt{b_5+b_6\phi}$,

case 1

$$\Phi_1 = \frac{a_1\lambda_1}{k} + a_1 \mathrm{cd}(\mu\xi,k),$$
$$\Phi_2 = \pm[b_5 \pm b_5 k \mathrm{cd}(\mu\xi,k)]^{1/2}, \tag{16a}$$
$$\lambda = \pm\frac{\sqrt{2(k^2+1)}}{2}, \; g_{12} = \pm\frac{6\lambda\mu^2}{b_5},$$
$$g_{22} = \frac{\mu^2(1\pm3\lambda)}{2b_5}, \tag{16b}$$
$$g_{11} = \frac{8g_{21}}{3} = -\frac{2\mu^2k^2}{a_1^2},$$
$$v_1 = \mu^2(k^2\mp6\lambda+1), \; v_2 = \frac{\mu^2(5k^2\mp12\lambda-1)}{8}; \tag{16c}$$

case 2

$$\Phi_1 = a_1\lambda_1 + a_1 \mathrm{dn}(\mu\xi,k),$$
$$\Phi_2 = \pm\left[\frac{\mu^2(3k^2-4)(4-\lambda_2)}{8g_{22}(\lambda_2-2)} - \frac{6\mu^2k\lambda_1 \mathrm{dn}(\mu\xi,k)}{g_{22}\lambda_2}\right]^{1/2}, \tag{17a}$$
$$\lambda_1 = \pm\frac{\sqrt{2(2-k^2)}}{2},$$
$$\lambda_2 = \frac{12[3k^2-6\pm\sqrt{2(2-k^2)}]}{9k^2-16},$$
$$g_{11} = \frac{8g_{21}}{3} = \frac{2\mu^2}{a_1^2}, \tag{17b}$$
$$g_{12} = \lambda_2 g_{22},$$
$$v_1 = \frac{\mu^2(13\lambda_2k^2-24\lambda_2-40k^2+80)}{2(\lambda_2-2)},$$
$$v_2 = \frac{\mu^2(5\lambda_2k^2-12\lambda_2-17k^2+40)}{4(\lambda_2-2)}; \tag{17c}$$

case 3

$$\Phi_1 = \frac{a_1\lambda_1}{2k(k^2-1)} + a_1\mathrm{sd}(\mu\xi,k),$$

$$\Phi_2 = \pm\left[\frac{\lambda_2 k\mu^2}{g_{12}} + \frac{3\lambda_1 k\mu^2\mathrm{sd}(\mu\xi,k)}{g_{12}}\right]^{1/2}, \tag{18a}$$

$$\lambda_1 = \pm\sqrt{2(2k^2-1)(1-k^2)},\ \ \lambda_2 = \pm 3\sqrt{2(2k^2-1)},$$

$$g_{22} = \frac{g_{12}(18k^2-\lambda_2 k-9)}{36(2k^2-1)}, \tag{18b}$$

$$g_{11} = \frac{8g_{21}}{3} = \frac{3\mu^2k^2(1-k^2)}{4a_1^2},$$

$$v_1 = -\mu^2(2k^2+\lambda_2 k-1),\quad v_2 = -\frac{\mu^2(4k^2+\lambda_2 k-5)}{8} \tag{18c}$$

Family 4 When $\Phi_1 = a_2\phi'$ and $\Phi_2 = \frac{b_4\phi\phi'}{\sqrt{1+d_2\phi^2}}$,

case 1

$$\Phi_1 = \frac{a_2\mu\mathrm{cn}(\mu\xi,k)}{\mathrm{dn}^2(\mu\xi,k)},\ \ \Phi_2 = \frac{b_4\mu\mathrm{sn}(\mu\xi,k)}{\mathrm{dn}^2(\mu\xi,k)}$$

$$v_1 = \mu^2(4k^2-5), \tag{19a}$$

$$v_2 = \mu^2(k^2-5),$$

$$g_{11} = g_{21} = -\frac{6(k^2-1)}{a_2^2},\ \ g_{12} = g_{22} = \frac{6(k^2-1)^2}{b_4^2}; \tag{19b}$$

case 2

$$\Phi_1 = \frac{a_2\mu\mathrm{cn}(\mu\xi,k)}{\mathrm{dn}^2(\mu\xi,k)},$$

$$\Phi_2 = \frac{b_4\mu\mathrm{sn}(\mu\xi,k)\mathrm{cn}(\mu\xi,k)}{\mathrm{dn}^2(\mu\xi,k)},$$

$$v_1 = \mu^2(4k^2+1), \tag{20a}$$

$$v_2 = \mu^2(k^2+4),\ \ g_{11} = g_{21} = -\frac{6k^2}{a_2^2},$$

$$g_{12} = g_{22} = \frac{6k^4}{b_4^2}; \tag{20b}$$

case 3

$$\Phi_1 = -\frac{a_2\mu(1-k^2)\mathrm{sn}(\mu\xi,k)}{\mathrm{dn}^2(\mu\xi,k)},$$

$$\Phi_2 = -\frac{b_4\mu\sqrt{1-k^2}\mathrm{sn}(\mu\xi,k)\mathrm{cn}(\mu\xi,k)}{\mathrm{dn}^2(\mu\xi,k)}, \tag{21a}$$

$$v_1 = -\mu^2(5k^2-1),\ \ v_2 = -\mu^2(5k^2-4),$$

$$g_{11} = g_{21} = \frac{6k^2}{a_2^2},\ \ g_{12} = g_{22} = \frac{6k^4}{b_4^2}. \tag{21b}$$

3. Similarity transformation and the solution to variable-coefficient CNLS equations

To show the significance of the special solutions obtained in the above section, we will utilize them to study the novel features of the two CNLS equations with varying coefficients. The two CNLS equations with time- and space-dependent potentials and nonlinearities can be written in a dimensionless form

$$\mathrm{i}\partial_t\psi_j = \left(-\partial_x^2 + V_j(x,t) + \sum_{k=1}^{2} G_{jk}(x,t)|\psi_k|^2 - \varGamma_j(x,t)\right)\psi_j,$$
$$j = 1,2, \tag{22}$$

where $\psi_j(x,t)$ are complex functions with external potential $V_j(x,t)$ and nonlinearities $G_{jk}(x,t)$ $(j,k=1,2)$, and $\varGamma_j(x,t)$ are the gain or loss coefficients. Here, we mainly focus on the spatially localized solutions for which $\lim_{|x|\to\infty}\psi_j = 0$ $(j=1,2)$. Our first objective is to reduce Eq. (22) to two coupled dynamical NLS equations (3) in Section 2. Therefore, we search for the similarity transformation

$$\psi_j = \rho_j(x,t)\,\mathrm{e}^{\varphi_j(x,t)}\Phi_j(\xi(x,t)),\ \ j=1,2, \tag{23}$$

where $\xi \equiv \xi(x,t)$, and $\Phi_j(\xi)$ $(j=1,2)$ satisfy Eq. (3). Then the following set of equations is found:

$$2\rho_{jx}\xi_x + \rho_j\xi_{xx} = 0,\ \ \xi_t + 2\xi_x\varphi_{jx} = 0, \tag{24a}$$

$$G_{jk}(x,t)\rho_k^2 + g_{jk}\xi_x^2 = 0,$$

$$\rho_{jt} + 2\rho_{jx}\varphi_{jx} + \rho_j\varphi_{jxx} + \rho_j\varGamma_j(x,t) = 0, \tag{24b}$$

$$\rho_{jx} - \rho_j[V_j(x,t) + v_j\xi_x^2 + \varphi_{jt} + \varphi_{jx}^2] = 0,$$
$$j,k = 1,2. \tag{24c}$$

Next, we introduce a new function $\xi(x,t)$ such that $\xi(x,t) = F(X)$, where $X = \gamma(t)\chi(x) + \delta(t)$. In this case, we can obtain

$$V_j(x,t) = \frac{\rho_{jxx}}{\rho_j} - \varphi_{jt} - \varphi_{jx}^2 - v_j\xi_x^2,$$

$$G_{jk}(x,t) = -\frac{g_{jk}\xi_x^2}{\rho_k^2}, \tag{25a}$$

$$\varGamma_j(x,t) = -\varphi_{jxx} - \frac{\rho_{jt}}{\rho_j} - \frac{2\rho_{jt}\varphi_{jx}}{\rho_j},$$

$$\varphi_j = -\frac{1}{2}\int\frac{\chi(x)\gamma'(t)+\delta'(t)}{\gamma(t)\chi'(x)}\,\mathrm{d}x + \alpha_j(t), \tag{25b}$$

$$\rho_j(x,t) = \frac{\beta_j(t)\exp[-\gamma_1(t)\int\chi(x)U\,\mathrm{d}x/2]}{\sqrt{\chi'(x)}},\ \ U = \frac{F_{\xi\xi}}{F_\xi}, \tag{26}$$

where $\alpha_j(t)$ and $\beta_j(t)$ $(j=1,2)$ are arbitrary functions of t. It follows from Eqs. (25) and (26) that if $\gamma(t), \chi(x)$, $\delta(t)$, and $F(X)$ are given, one can generate potential $V_j(x,t)$, nonlinearities $G_{jk}(x,t)$, and gain or loss coefficients $\varGamma_j(x,t)$ $(j,k=1,2)$ for which the solutions of Eq. (22) can be constructed naturally from Eq. (3) using transformation (23). To illustrate the procedure, we only focus our attention to some specific cases. Actually, there are many different choices of the free functions. Here, we only consider that X is a linear function of x, namely, $\chi(x) = x$, $\varGamma_j(x,t) = 0$. And we select the Gaussian shaped nonlinearities

$$G_{jk}(x,t) = -\frac{g_{jk}\gamma(t)\exp(-3X^2)}{R_k^2}, \tag{27}$$

which can be obtained by the application of modulated Gaussian laser beams on a BEC, as experimentally demonstrated in Ref. [25] to realize optically controlled interactions via optical Feshbach resonance. Therefore, the potentials including a combination of harmonic trap and Gaussian barrier read

$$V_j(x,t) = w^2(t)x^2 + f(t)x + h_j(t) - v_j\gamma(t)^2\exp(-2X^2), \quad (j=1,2), \tag{28}$$

where

$$w^2(t) = \frac{\gamma''(t)}{4\gamma(t)} - \frac{\gamma'(t)^2}{2\gamma(t)^2} + \gamma(t)^4, \tag{29a}$$

$$f(t) = \frac{\delta''(t)}{2\gamma(t)} - \frac{\gamma'(t)\delta'(t)}{\gamma(t)^2} + 2\gamma(t)^3\delta(t), \tag{29b}$$

$$h_j(t) = \gamma(t)^2[1+\delta(t)^2] - \frac{\delta'(t)^2}{4\gamma(t)^2} - \alpha_j'(t), \quad (j=1,2). \tag{29c}$$

Furthermore, if setting $f(t)=0$, $h_j(t)=0$, and $\tau(t) = \int\gamma(t)^2\mathrm{d}t$, one can obtain the potentials

$$V_j(x,t) = w^2(t)x^2 - v_j\gamma(t)^2\exp(-2X^2), \quad (j=1,2), \tag{30}$$

under the conditions

$$\delta(t) = A\cos(2\tau(t)), \quad \alpha_j(t) = A^2\sin[4\tau(t)] + \tau(t) + c_{j0}, \quad (j=1,2). \tag{31}$$

From Eq. (29a), which is equivalent to the Ermakov–Pinney equation $z_{tt} + 4w^2(t)z = 4/z^3$ with $z = 1/\gamma(t)$, we known that $\gamma(t)$ has the form $\gamma(t) = [2s_1^2(t) + 2s_2^2(t)/\Delta^2]^{-1/2}$, where Δ is the Wronskian of two linearly independent solutions $\{s_1(t), s_2(t)\}$ of the Mathieu equation $s_{tt} + 4w^2(t)s = 0$. To discuss the explicit solution, we choose

$$w^2(t) = 1 + \varepsilon\cos(w_0 t), \tag{32}$$

where $\varepsilon \in (-1,1)$ and $w_0 \neq 0$ ($\in \mathbf{R}$). The different choices of parameters ε and w_0 imply that one can single out three different types of behaviors: periodic, resonant, and quasiperiodic. In the following, we present some special examples corresponding to the three different cases.

Now, we select a solution of two coupled dynamical NLS equations (3) of the following form:

$$\Phi_1 = -\frac{a_2\mu(1-k^2)\mathrm{sn}(\mu\xi,k)}{\mathrm{dn}^2(\mu\xi,k)},$$

$$\Phi_2 = -\frac{b_4\mu\sqrt{1-k^2}\mathrm{sn}(\mu\xi,k)\mathrm{cn}(\mu\xi,k)}{\mathrm{dn}^2(\mu\xi,k)}, \tag{33a}$$

$$v_1 = -\mu^2(5k^2-1), \quad v_2 = -\mu^2(5k^2-4),$$

$$g_{11} = g_{21} = \frac{6k^2}{a_2^2}, \quad g_{12} = g_{22} = \frac{6k^4}{b_4^2}. \tag{33b}$$

Meanwhile, we choose $\xi = \sqrt{\pi}\mathrm{erf}(X)/2 + \sqrt{\pi}/2$ in the particular case $\delta(t) = 0$ ($A = 0$), which takes values in $(0, \sqrt{\pi})$. So $\mu = 2nK(k)/\sqrt{\pi}$, where $K(k)$ is the elliptic integral, can be derived when the requirement that the zero boundary condition for $x \to \infty$ must be meet. This leads to the family of solutions

$$\psi_{1n} = -\frac{2a_2R_1nK(k)(1-k^2)\sqrt{\gamma(t)}\,\mathrm{e}^{\mathrm{i}\varphi_1(x,t)}\mathrm{sn}[\theta_n(x,t),k]}{\sqrt{\pi}\,\mathrm{e}^{-[\gamma(t)+\delta(t)]^2/2}\mathrm{dn}^2[\theta_n(x,t),k]}, \tag{34a}$$

$$\psi_{2n} = -2b_4R_2nK(k)\sqrt{(1-k^2)\gamma(t)}\,\mathrm{e}^{\mathrm{i}\varphi_2(x,t)}\mathrm{sn}[\theta_n(x,t),k] \times \mathrm{cn}[\theta_n(x,t),k]\{\sqrt{\pi}\,\mathrm{e}^{-[\gamma(t)+\delta(t)]^2/2}\mathrm{dn}^2[\theta_n(x,t),k]\}^{-1}, \tag{34b}$$

with $\theta_n(x,t) = nK(k)\mathrm{erf}[\gamma(t)x] + nK(k)$.

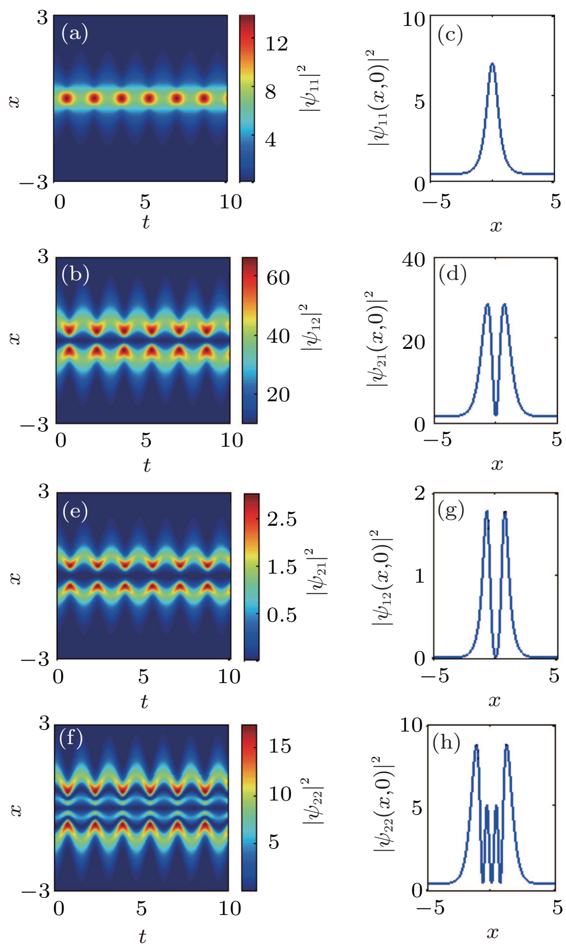

Fig. 1. (color online) The density plots of $|\psi_1(x,t)|^2$ and $|\psi_2(x,t)|^2$ given by Eq. (3) with parameters $R_1 = R_2 = a_2 = b_4 = 1$, $k = 1/2$, and $n = 1$ in panels (c) and (d); $n = 2$ in panels (g) and (h). The plots of $|\psi_1(x,t)|^2$ and $|\psi_2(x,t)|^2$ at time $t = 0$ are shown in panels (a), (b) and panels (e), (f), respectively.

For the first example, we consider the periodic behavior of solutions (34), for which $\gamma(t)$ occurs when $\varepsilon = 0$ or in the frontiers between the stability and instability regions of Eq. (32). So we set $\varepsilon = 0$ and require that the initial data for the Ermakov–Pinney equation are $z(0) = \sqrt{2}$ and $z_t(0) = 0$.

In Fig. 1, the plots of the periodic soliton are exhibited corresponding to $n = 1, 2$. When $\varepsilon \neq 0$ and $s_{1,2}(t)$ belong to the instability region of Eq. (32), the resonant phenomenon can be observed. In Fig. 2, in the case of $\varepsilon = 0.5$ and $w_0 = 2$ and with the same initial data as the periodic case, we show an example of such a group of solutions for $n - 1$, which displays an increasing resonant behavior. In fact, in this case, few papers pay attention to the search for some exact solutions to describe the resonant behavior. Our results provide the explicit resonant solution in a parametrically modulated one-dimensional BEC. In the quasiperiodic case, we still choose $\varepsilon = 0.5$ and use the same initial data as the periodic case but $w_0 = \sqrt{2}$ to ensure that the solutions $s_{1,2}(t)$ of Eq. (32) belong to the stability region. In this way, $\gamma(t)$ is a quasiperiodic solution and the solutions (34) show a quasiperiodic behavior. An example of this behavior is displayed in Fig. 3. For the above three cases, multisoliton solutions can be constructed when n is larger in Eq. (34).

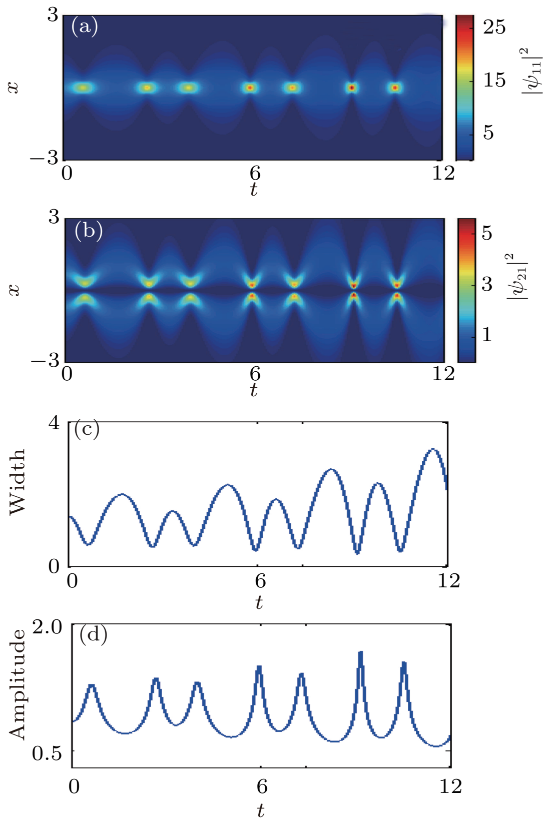

Fig. 2. (color online) The density plots of (a) $|\psi_1(x,t)|^2$ and (b) $|\psi_2(x,t)|^2$ given by Eq. (34) with the same parameters as those in Fig. 1. (c) The $z(t)$ and (d) amplitude $\sqrt{\gamma(t)}$ for ψ_1 and ψ_2 versus t.

It is necessary to point out such a fact that the number of the localized wave of ψ_2 is always twice that of ψ_1 (or exchange of both) in the expression of solutions (34). This differs from the situation in Refs. [24], [29], and [37], in which the propagation of ψ_2 is deemed to be similar to ψ_1 in coupled system (22). In addition, the center of mass of the soliton moves with zero velocity $\delta(t) = 0$ $(A = 0)$, as shown in the previous discussion. Similarly, if one set $\delta(t) \neq 0$, a moving solution will be constructed, in which the center of mass of the soliton moves in a complex way according to the first equation in Eq. (31).

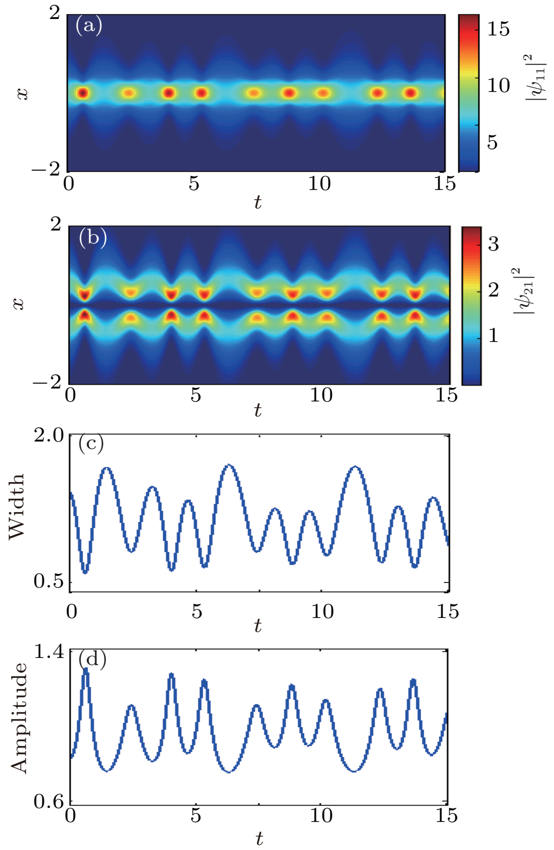

Fig. 3. (color online) The density plots of (a) $|\psi_1(x,t)|^2$ and (b) $|\psi_2(x,t)|^2$ given by Eq. (34) with the same parameters as those in Fig. 1. (c) The $z(t)$ and (d) amplitude $\sqrt{\gamma(t)}$ for ψ_1 and ψ_2 versus t.

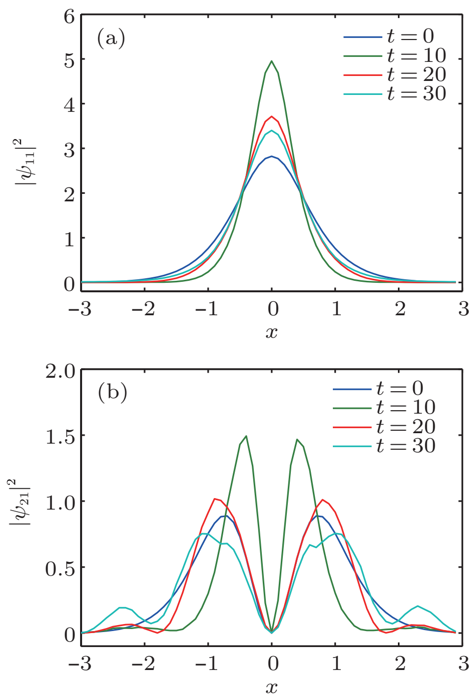

Fig. 4. (color online) The profiles of (a) $|\psi_1(x,t)|^2$ and (b) $|\psi_2(x,t)|^2$ corresponding to the waves in Figs. 1(a) and 1(b) at different time.

To verify the stability of the solutions, we carry out numerical simulations of a solution with a small perturbation initially implanted to see whether the propagation is stationary or not. Here we just present a simulation of the solutions in Figs. 1(a) and 1(b), where about five percent of the amplitudes are initially added as the perturbations. The profiles of $|\psi_1(x,t)|^2$ and $|\psi_2(x,t)|^2$ at different time are plotted in Fig. 4. It is observed that during their propagations, ψ_1 is quite stable, whereas ψ_2 becomes unstable because the oscillation appears at both ends of two main wave peaks after a longer time.

4. Discussion and summary

In this paper, some novel solutions of two coupled dynamical NLS equations are derived using one special rational form of solution. Based on the similarity transformation, the CNLS equations with time- and space-dependent potentials, nonlinearities, and gain or loss are reduced to the coupled dynamical NLS equations. Some special types of nontravelling wave solutions, such as periodic, resonant, and quasiperiodically oscillating solitons, are used to exhibit the wave propagations by choosing some arbitrary functions. Furthermore, the localized wave propagation and interaction scenario are discussed and simulated. In particular, our results show that the number of the localized wave of one component is always twice that of the other one. Finally, numerical simulations are used to show the stability of our analytical solutions. These results may provide more information for the nonlinear physical system and should be readily verified experimentally.

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