A Method to Construct the Nonlocal Symmetries of Nonlinear Evolution Equations *

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A method is proposed to seek the nonlocal symmetries of nonlinear evolution equations. The validity and advantages of the proposed method are illustrated by the applications to the Boussinesq equation, the coupled Korteweg-de Vries system, the Kadomtsev–Petviashvili equation, the Ablowitz–Kaup–Newell–Segur equation and the potential Korteweg-de Vries equation. The facts show that this method can obtain not only the nonlocal symmetries but also the general Lie point symmetries of the given equations.

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Since the Lie group theory was introduced by Sophus Lie,^[1] the study of Lie group has always been an important subject in mathematics and physics. Using both classical and non-classical Lie group approaches,^[2–9] one can reduce the dimensions and construct the analytical solutions of the given partial differential equations (PDEs). In addition to the classical and non-classical Lie symmetries, there exist the so-called nonlocal symmetries reported in the literature in the 1980s largely through the work of Olver.^[10] To search for nonlocal symmetries of the nonlinear systems is an interesting work because the nonlocal symmetries^[11–14] can enlarge the class of symmetries and they are connected with integrable models.

However, it is difficult to find the nonlocal symmetries of nonlinear PDEs. Usually, the nonlocal symmetries may be obtained with the help of a recursion operator.^[15] However, sometimes seeking the recursion operators is a difficult work. The concept of potential symmetry^[11] was explicitly formulated first by Bluman et al. and was subsequently applied in investigations of important classes of PDEs. Galas^[13] obtained the nonlocal Lie-Bäcklund symmetries by introducing the pseudo-potentials as an auxiliary system. Recently, Lou et al.^[16,17] obtained explicit analytic interaction solutions between cnoidal waves and solitary wave through the localization procedure of nonlocal symmetries which are related to the Darboux transformation (DT) for the well-known Korteweg-de Vries (KdV) equation.

In some cases, one can obtain nonlocal symmetries through Lie point symmetries or Lie–Bäcklund symmetries of the extended systems which include the original equation and auxiliary systems. Nevertheless, these methods may lose some important results such as integral terms or high order derivative terms of nonlocal variables in the symmetries. Thus it is necessary to improve the previous methods to avoid missing the above important terms. In this Letter, we present a systemic method to find the high order nonlocal symmetries.

We consider a system \mathcal{F} of *n*th order differential equations in *p* independent and *q* dependent variables, expressed by

$$\Delta_v(x, u^{(n)}) = 0, \quad v = 1, 2, \dots, l, \tag{1}$$

involving $x = (x^1, x^2, ..., x^p)$, $u = (u^1, u^2, ..., u^q)$, and the derivatives of u with respect to xup to order n. The function $\Delta_v(x, u^{(n)}) =$ $(\Delta_1(x, u^{(n)}), ..., \Delta_l(x, u^{(n)}))$ will be assumed to be smooth in the arguments. Let $X = \mathbb{R}^p$ be the space representing the independent variables.

Let

$$V = \xi^p(x, u)\frac{\partial}{\partial x^p} + \eta^q(x, u)\frac{\partial}{\partial u^q},$$
 (2)

be the infinitesimal generator of the Lie group of point transformations $\tilde{x} = F(x, u, \varepsilon), \ \tilde{u} = G(x, u, \varepsilon).$

Next, we describe the method of constructing the nonlocal symmetries as follows. For simplicity, we consider the case p = 2, q = 1, i.e. $(x^1, x^2) = (t, x)$.

Step 1. Choose the proper auxiliary systems. Usually, one can use the Lax pair, Bäcklund transformation, potential system, pseudo-potential, etc. with the following forms

$$F_{\alpha}(x, t, u, u_x, u_t, \dots, \psi_x, \psi_t, \psi_{xx}, \psi_{xt}, \psi_{tt}, \dots, \psi_{\lambda x}, \psi_{\mu t}) = 0, \alpha \in \mathbb{Z}^+,$$
(3)

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where $\psi = (\psi^1, \psi^2, \dots, \psi^\beta)$ denotes the β auxiliary variables and $\psi_{\lambda x}$ denotes the λ th-order partial derivatives with respect to $x, \psi_{\mu t}$ denotes the μ th-order partial derivatives with respect to t.

Let $U \simeq \mathbb{R}$ be the space representing the single coordinate u, the space U_1 is isomorphic to \mathbb{R}^2 with coordinates (u_x, u_t) . Similarly, $U_2 \simeq \mathbb{R}^3$ has the coordinates representing the second order partial derivatives of u, and in general, $U_k \simeq \mathbb{R}^{k+1}$, since there are k+1 distinct kth order partial derivatives of u. Finally, the space $U^{(k)} = U \times U_1 \times \cdots \times U_k$ with coordinates $U^{(k)} = (u; u_x, u_t; u_{xx}, u_{xt}, u_{tt}; \ldots)$.

Step 2. In this step, we prolong the basic space $X \times U$ to the space $X \times U^{(n)}$, with coordinates $(x, t, u, u, u_x, u_t, \ldots)$. The *n*th prolongation of *V*, denoted by $\tilde{V}^{(n)}$, will be a vector field on the *n*-jet space $X \times U^{(n)}$. The vector field in general takes the form

$$\tilde{V}^{(n)} = \sum_{i=1}^{2} \xi^{i} \frac{\partial}{\partial x^{i}} + \sum_{L} \eta^{L} \frac{\partial}{\partial u_{L}}.$$
 (4)

Here we give a different definition of coefficients, i.e. the coefficients ξ^i and η^L all depend on the variables $(x, t, u, \ldots, \psi, \psi, \psi_x, \psi_t, \ldots, \psi_{\lambda x}, \psi_{\mu t})$. Here $\eta^0 = \eta$ and η^L have the form

$$\eta^{L} = D_{L}u - \sum_{i=1}^{2} u_{L} D_{L} \xi^{i}.$$
 (5)

Remark 1: The prolongation of vector fields show that this kind of symmetry is neither classical Lie point symmetries nor Lie–Bäcklund symmetries because it depends on the auxiliary variables and the high order partial derivatives. More results may be obtained if we assume the coefficients ξ^i and η^L have integral terms of the auxiliary variable, i.e., they are the functions of $(x, t, u, \ldots, \int \psi dx, \ldots)$.

Step 3. In order to seek the nonlocal symmetries, we should solve the following equations

$$\tilde{V}^{(n)} \Delta_v(x, u^{(n)}) \Big|_{\substack{\Delta_v(x, u^{(n)}) = 0 \\ \text{Eq. (3)}}} = 0.$$
(6)

Using the above equation, one can obtain a large number of elementary determining equations for the coefficient functions. Those determining equations can be solved and the general solution will determine the most general symmetry of the system.

Example 1: The well-known Boussinesq equation^[18-20] is

$$u_{tt} + (u^2)_{xx} + \frac{1}{3}u_{xxxx} = 0, (7)$$

and the corresponding Lax pair of Eq. (7) has the form

$$\psi_{xxx} = -\frac{3}{2}u\psi_x - \left(\frac{3}{4}u_x + \frac{3}{4}\partial_x^{-1}u_t\right)\psi,$$

$$\psi_t = -\psi_{xx} - u\psi,$$
 (8)

and its adjoint version is

$$\phi_{xxx} = -\frac{3}{2}u\phi_x - \left(\frac{3}{4}u_x - \frac{3}{4}\partial_x^{-1}u_t\right)\phi,$$

$$\phi_t = \phi_{xx} + u\phi.$$
 (9)

That is to say, the integrable conditions of Eqs. (8) and (9), $\psi_{xxxt} = \psi_{txxx}$ and $\phi_{xxxt} = \phi_{txxx}$ are just the Boussinesq equation (7).

Apply the Lax pair and its adjoint Lax pair of the Boussinesq equation as the auxiliary systems. Then, the vector field takes the form

$$V = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}.$$
 (10)

One can prolong the basic space V to the space $X \times U^{(4)}$ and obtain the prolongation of \tilde{V} ,

$$\tilde{V} = \xi^{1} \frac{\partial}{\partial x} + \xi^{2} \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u} + \eta^{x} \frac{\partial}{\partial u_{x}} + \eta^{xx} \frac{\partial}{\partial u_{xx}} + \eta^{tt} \frac{\partial}{\partial u_{tt}} + \eta^{xxxx} \frac{\partial}{\partial u_{xxxx}}, \quad (11)$$

where the coefficients of \tilde{V} all depend on the variables $(x, t, u, u_x, u_{xx}, \psi, \phi, \psi_x, \phi_x, \psi_{xx}, \phi_{xx})$, and

$$\eta^{t} = D_{t}(\eta - \xi^{1}u_{x} - \xi^{2}u_{t}) + \xi^{1}u_{xt} + \xi^{2}u_{tt},$$

$$\eta^{x} = D_{x}(\eta - \xi^{1}u_{x} - \xi^{2}u_{t}) + \xi^{1}u_{xx} + \xi^{2}u_{tx},$$

$$\dots$$

$$\eta^{xxxx} = D_{xxxx}(\eta - \xi^{1}u_{x} - \xi^{2}u_{t}) + \xi^{1}u_{xxxxx}$$

$$+ \xi^{2}u_{txxxx}.$$
(12)

Applying \tilde{V} to Eq. (7), one can obtain the infinitesimal criterion (6) to be

$$\eta^{tt} + 4u_x\eta^x + 2\eta u_{xx} + 2u\eta^{xx} + \frac{1}{3}\eta^{xxxx} = 0.$$
 (13)

Substituting the general formulae (12) into (13) and replacing $u_{tt}, \psi_{xxx}, \psi_t, \phi_{xxx}, \phi_t$ by Eqs. (7), (8) and (9), we obtain the determining equations for the functions ξ^1, ξ^2, η . Calculated by computer algebra, the general solutions of them take the form

$$V = \left(\frac{1}{2}c_1x + c_3\right)\frac{\partial}{\partial x} + (c_1t + c_2)\frac{\partial}{\partial t} - (c_1u - c_4(\psi\phi)_x)\frac{\partial}{\partial u}$$
(14)

where c_1 , c_2 , c_3 , c_4 are arbitrary constants.

Remark 2: The vector field (14) contains two parts, $V^1 = (\frac{1}{2}c_1x + c_3)\frac{\partial}{\partial x} + (c_1t + c_2)\frac{\partial}{\partial t} - c_1u\frac{\partial}{\partial u}$ and $V^2 = c_4(\psi\phi)_x\frac{\partial}{\partial u}$. One can see that the first part is the classical Lie point symmetry, and the second part is the nonlocal symmetry. Therefore, both the general local symmetries and nonlocal symmetries can be obtained by this method.

Example 2: The coupled KdV system^[22,23] has the form

$$u_t = -6vv_x + 6uu_x - u_{xxx}, v_t = 6uv_x + 6vu_x - v_{xxx},$$
(15)

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and the Lax pair for Eqs. (15) is as follows:

$$\begin{split} \phi_{1xx} &= v\phi_2 + u\phi_1 - \lambda\phi_1, \\ \phi_{2xx} &= u\phi_2 - v\phi_1 - \lambda\phi_2, \\ \phi_{1t} &= -4\phi_{1xxx} + 6v\phi_{2x} + 6u\phi_{1x} + 3v_x\phi_2 + 3u_x\phi_1 \\ \phi_{2t} &= -4\phi_{2xxx} + 6u\phi_{2x} - 6v\phi_{1x} + 3u_x\phi_2 - 3v_x\phi_1 \end{split}$$

The vector field has the form

$$V = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial t} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v}.$$
 (16)

Using the formula (4), one can prolong the space V to the space $X \times U^{(3)} \times V^{(3)}$, which here we omit. Applying the prolonged vector field and following step 3, one can obtain the general solutions

$$V = \left(\frac{c_1 x}{3} + c_3 t + c_4\right) \frac{\partial}{\partial x} + (c_1 t + c_2) \frac{\partial}{\partial t} + \left[\frac{c_6}{2} (\phi_1^2 - \phi_2^2)_x - c_5 (\phi_1 \phi_2)_x + \frac{2c_1 u}{3} + \frac{c_3}{6}\right] \frac{\partial}{\partial u} - \left[c_6 (\phi_1 \phi_2)_x - \frac{c_5}{2} (\phi_2^2 - \phi_1^2)_x - \frac{2c_1 v}{3}\right] \frac{\partial}{\partial v}.$$
 (17)

where c_1 , c_2 , c_3 , c_4 , c_5 , c_6 are arbitrary constants.

Example 3: The Kadomtsev–Petviashvili (KP) equation^[24,25] has the following form

$$u_{xt} - 6u_x^2 - 6uu_{xx} + u_{xxxx} + 3u_{yy} = 0.$$
 (18)

It is well known that the KP equation possesses the Lax pair and the adjoint Lax pair

$$\psi_{xx} = u\psi - \psi_y,$$

$$\psi_t = -4\psi_{xxx} + 6u\psi_x + 3(u_x - \int u_y dx)\psi,$$

$$\phi_{xx} = u\phi + \phi_y,$$

$$\phi_t = -4\phi_{xxx} + 6u\phi_x + 3(u_x + \int u_y dx)\phi.$$
(19)

Let the vector field of Eq. (18) take the form

$$V = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \xi^3 \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}.$$
 (20)

Using the prolonged vector field and following step 3, one can obtain the solution

$$V = \left(\frac{x}{3}F_{1t} - \frac{y^2}{18}F_{1tt} - \frac{y}{6}F_{2t} - 6F_3 + c_2\right)\frac{\partial}{\partial x} + \left(\frac{2y}{3}F_{1t} + F_2\right)\frac{\partial}{\partial y} + F_1\frac{\partial}{\partial t} + \left(\frac{2u}{3}F_{1t} + \frac{x}{18}F_{1tt} - c_1(\psi\phi)_x - \frac{y^2}{108}F_{1ttt} - \frac{y}{36}F_{2t} - F_{3t}\right)\frac{\partial}{\partial u},$$
(21)

where F_1 , F_2 and F_3 are arbitrary functions of t, and c_1 and c_2 are arbitrary constants.

Example 4: The Ablowitz–Kaup–Newell–Segur (AKNS) equations^[21,26]

$$u_t = -iu_{xx} + 2iu^2v,$$

$$v_t = iv_{xx} - 2iv^2u$$
(22)

have the following Lax pair

$$\begin{pmatrix} \phi_{1x} \\ \phi_{2x} \end{pmatrix} = \begin{pmatrix} -i\lambda & u \\ v & i\lambda \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},$$

$$\begin{pmatrix} \phi_{1t} \\ \phi_{2t} \end{pmatrix} = \begin{pmatrix} 2i\lambda^2 + iuv & -2\lambda u - iu_x \\ -2\lambda v + iv_x & -2i\lambda^2 - iuv \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

$$(23)$$

Using the same method we can obtain the nonlocal symmetries taking the form

$$V = (c_1 t + c_3 x + c_2) \frac{\partial}{\partial x} + (2c_3 t + c_4) \frac{\partial}{\partial t}$$
$$- \left(\frac{4c_3 + 2c_6 + c_1 ix}{2}u - c_5 \phi_1^2\right) \frac{\partial}{\partial u}$$
$$+ \left(\frac{2c_6 + c_1 ix}{2}v + c_5 \phi_2^2\right) \frac{\partial}{\partial v}, \qquad (24)$$

where c_1 , c_2 , c_3 , c_4 , c_5 and c_6 are arbitrary constants.

Example 5: Here we construct the nonlocal symmetries by using the Bäcklund transformation (BT) of the potential Korteweg-de Vries (pKdV) equation. The pKdV equation^[27] has the form

$$u_t = -u_{xxx} + 3u_x^2. (25)$$

For Eq. (25), there exists the following $BT^{[28]}$

$$u_{1,x} = -u_x - 2\lambda + \frac{(u - u_1)^2}{2},$$

$$u_{1,t} = -u_t + 2u_x^2 + 2u_{1,x}^2 + 2u_x u_x$$

$$-(u - u_1)(u_{xx} - u_{1,xx}),$$
(26)

with λ being the arbitrary parameter.

Equations (26) show that if u is a solution to Eq. (25), so is u_1 , that is to say, they represent a finite symmetry transformation between two exact solutions of Eq. (25). Hence u_1 satisfies the following form

$$u_{1,t} = -u_{1,xxx} + 3u_{1,x}^2. \tag{27}$$

In order to obtain the nonlocal symmetry of Eq. (25), we first give a transformation

$$v = u - u_1, \quad w = u + u_1,$$
 (28)

where v and w are functions of x and t. Using Eq. (28), we can obtain

$$u = \frac{v}{2} + \frac{w}{2}, \quad u_1 = \frac{w}{2} - \frac{v}{2},$$
 (29)

then, the vector field of u, u_1 and v, w have the following relations

$$V_1 = \frac{\tilde{V}_1}{2} + \frac{\tilde{V}_2}{2}, \quad V_2 = \frac{\tilde{V}_2}{2} - \frac{\tilde{V}_1}{2},$$
 (30)

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where V_1 , V_2 , \tilde{V}_1 , \tilde{V}_2 are vector field of u, u_1 , v, w, respectively, and the coefficients of \tilde{V}_1 , \tilde{V}_2 all depend on the variables $(x, t, v, w, \int v dx, \int w dx)$.

Substituting transformations (29) into Eqs. (25)–(27), one can obtain

$$v_t = -v_{xxx} + 3v_x w_x, \quad w_t = -w_{xxx} + \frac{3}{2}v_x^2 + \frac{3}{2}w_x^2,$$
(31)

$$w_x = -2\lambda + \frac{v^2}{2}, \quad w_t = \frac{1}{2}v_x^2 + \frac{3}{2}w_x^2 - vv_{xx}.$$
 (32)

Using the same method we can obtain the nonlocal symmetries of Eqs. (31) with Eqs. (32) having the forms

$$\tilde{V}_{1} = \left(\frac{c_{1}x}{3} + c_{6}t + c_{7}\right)\frac{\partial}{\partial x} + (c_{1}t + c_{2})\frac{\partial}{\partial t} + \left(c_{4}e^{-\int v dx} + c_{5}e^{\int v dx} - \frac{c_{1}v}{3} + c_{3}\right)\frac{\partial}{\partial v},$$
$$\tilde{V}_{2} = \left(\frac{c_{1}x}{3} + c_{6}t + c_{7}\right)\frac{\partial}{\partial x} + (c_{1}t + c_{2})\frac{\partial}{\partial t} - \left(\frac{c_{1}w}{3} + \frac{c_{6}x}{3} + c_{4}e^{-\int v dx} - c_{5}e^{\int v dx} - c_{8}\right)\frac{\partial}{\partial w}.$$
(33)

Finally, substituting the above results and Eqs. (28) into Eqs. (30), we can obtain the nonlocal symmetries of Eqs. (25) and (27) with BT (26),

$$V_{1} = \left(\frac{c_{1}x}{3} + c_{6}t + c_{7}\right)\frac{\partial}{\partial x} + (c_{1}t + c_{2})\frac{\partial}{\partial t} + \left(c_{5}e^{\int (u-u_{1})dx} - \frac{c_{1}u}{3} - \frac{c_{6}x}{6} + \frac{c_{3}+c_{8}}{2}\right)\frac{\partial}{\partial u},$$

$$V_{2} = \left(\frac{c_{1}x}{3} + c_{6}t + c_{7}\right)\frac{\partial}{\partial x} + (c_{1}t + c_{2})\frac{\partial}{\partial t} - \left(c_{4}e^{-\int (u-u_{1})dx} + \frac{c_{1}u_{1}}{3} + \frac{c_{6}x}{6} + \frac{c_{3}-c_{8}}{2}\right)\frac{\partial}{\partial u_{1}}.$$
(34)

where c_1 , c_2 , c_3 , c_4 , c_5 , c_6 , c_7 and c_8 are arbitrary constants. If we set $c_1 = c_2 = c_3 = c_4 = c_6 = c_7 = c_8 = 0$, $c_5 = 1$, the same result will be found in Ref. [16] but their result is only a special case of ours.

In summary, a systemic method to find the nonlocal symmetry of nonlinear evolution equation has been presented. Through several classical examples, we can observe that this method can obtain nonlocal symmetries effectively.

Moreover, how to use the nonlocal symmetries to build similarity solutions is another important work. In Ref. [17], the authors proposed a new method. The idea is to incorporate the original equation(s) in an extended related system by introducing other auxiliary dependent variables. In this case, the primary non-local symmetry is equivalent to Lie point symmetries of prolonged systems, then one can find the nonlocal group as well as the explicit similarity solutions. Following this idea, seeking new explicit analytic solutions using the nonlocal symmetries of these nonlinear systems is worthy of further study.

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