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Explicit exact solutions for a new generalized Hamiltonian amplitude equation with nonlinear terms of any order

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Abstract. Making use of a proper transformation and a generalized ansatz, we consider a new generalized Hamiltonian amplitude equation with nonlinear terms of any order, $iu_x + u_{tt} + (\alpha|u|^p + \beta|u|^{2p})u + \delta u_{xt} = 0$. As a result, many explicit exact solutions, which include kink-shaped soliton solutions, bell-shaped soliton solutions, periodic wave solutions, the combined formal solitary wave solutions and rational solutions, are obtained.

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1. Introduction

In this paper, we firstly present a new generalized Hamiltonian amplitude equation with nonlinear terms of any order in the form

$$iu_x + u_{tt} + (\alpha|u|^p + \beta|u|^{2p})u + \delta u_{xt} = 0, \tag{1.1}$$

where α, β and δ are arbitrary constants. In one particular case ($p = 1$, $\alpha = 0$, $\beta = \pm 2$, $-\delta = \varepsilon \ll 1$), Eq. (1.1) is just the Hamiltonian amplitude equation

$$iu_x + u_{tt} + 2\sigma|u|^2 u - \epsilon u_{xt} = 0, \quad \sigma = \pm 1, \quad \epsilon \ll 1, \tag{1.2}$$

which introduced by Wadati, Segur and Ablowitz [1] in 1992, Eq. (1.2) governs certain instabilities of modulated wave trains, and the addition of term $-\epsilon u_{xt}$ overcome the ill-posedness of the unstable nonlinear Schrödinger equation. The equation is apparently not integrable, but a Hamiltonian analogue of the Kuramoto-Sivashinsky equation, which arises in dissipative systems. Under the conditions $\sigma = 1$, $v^2 + \epsilon v > 0$, $K + \Omega^2 + \epsilon K\Omega > 0$, a kind of solitary wave solution

$$u(x,t) = \mathrm{bsech}(\frac{b}{a}(x - vt - x_0)\exp[i(Kx - \Omega t)], \tag{1.3}$$

is obtained in Ref. [1], here the constants a and b are given by $a^2 = v^2 + \epsilon v$, $b^2 = K + \Omega^2 + \epsilon K\Omega$. Recently, De-Xing Kong and Wei-guo Zhang [2] obtained not only a solution as form (1.3) but also another kind of new solitary wave solution

$$u(x,t) = \pm\sqrt{\frac{K+\Omega^2+\epsilon K\Omega}{2\sigma}}\tanh\left[\frac{1}{2}\sqrt{-\frac{2(K+\Omega^2+\epsilon K\Omega)}{v^2+a\epsilon v}}\right](ax-vt)$$
$$\exp[i(Kx-\Omega t)], \tag{1.4}$$

where K and Ω satisfying $a+2\Omega v+\epsilon(Kv+a\Omega)=0$. More recently, Yan Zhen-Ya [3] further studied the solutions for Eq. (1.2) and obtained a family of soliton solution

$$u(x,t) = \pm\sqrt{\frac{M}{2\sigma}}\{\tanh[\sqrt{-\frac{M}{2N}}(x-\lambda t)]\pm i\text{sech}[\sqrt{-\frac{M}{2N}}(x-\lambda t)]\}\exp[i(Kx-\Omega t)], \tag{1.5}$$

where $M\sigma>0$, $MN<0$, $1+2\lambda\Omega+\epsilon(\lambda K+\Omega)=0$. Naturally, study of Eq. (1.1) is of important significance in explaining some physical phenomena.

Secondly, we would discuss explicit exact solutions for Eq. (1.1). As a result, new families of exact solutions for Eq. (1.1) are found by using a proper transformation and a general ansatz. The solutions obtained include not only three above mentioned solutions (1.3), (1.4) and (1.5) but also many new explicit exact solutions, which include kink-shaped soliton solutions, bell-shaped soliton solutions, periodic wave solutions, the combined formal solitary wave solutions and rational solutions.

The plan of the paper is as follows. In section 2, we describe briefly our improved tanh-method. In section 3, we apply the method to Eq. (1.1) and bring out rich solutions. In section 4, by another ansatz, we obtain the bell-shaped soliton solutions. Conclusions will be presented finally.

2. Extension of the method

In this section, we'll improve the method in [5,6]. For a given nonlinear evolution equations, say, in two variables, x,t

$$F(u,u_t,u_x,u_{xt},u_{tt},...)=0, \tag{2.1}$$

we seek the following formal travelling wave solutions

$$u(x,t)=u(\xi), \qquad \xi=x-\lambda t, \tag{2.2}$$

where λ is a constant to be determined later. Then Eq. (2.1) reduces to a nonlinear ordinary differential equation under (2.2)

$$G(u,u',u'',u'''\cdots)=0, \tag{2.3}$$

where $''{\prime}''$ denotes $\frac{d}{d\xi}$. In order to seek the travelling wave solutions of (2.3), we take the following improved transformations

$$u(\xi)=\sum_{i=1}^{m}\omega^{i-1}(\xi)[A_i\omega(\xi)+B_i\sqrt{R+\omega^2(\xi)}]+A_0, \tag{2.4}$$

and the new variable $\omega=\omega(\xi)$ satisfies

$$\omega'-(R+\omega^2)=\frac{d\omega}{d\xi}-(R+\omega^2)=0, \tag{2.5}$$

where A_0 , $A_i, B_i, (i=1,2,...,m)$ and R are constants to be determined later, and m is a positive integer. However, when we balance the highest order partial derivative term and the nonlinear term in Eq. (2.1) or Eq. (2.3), we find that the constant m needn't be a positive integer. In order to apply the method in [5,6] when m is equal to a fraction or a negative integer, we make the following transformation

(1) When $m=\frac{q}{p}$ (where $m=\frac{q}{p}$ is a fraction in lowest terms), we let

$$u(\xi)=\varphi^{q/p}(\xi), \tag{2.6}$$

then substitute Eq. (2.6) into Eq. (2.3) and return to determine the value of m by balance the highest order partial derivative term and the nonlinear term in new Eq. (2.3).

(2) When m is a negative integer, we let

$$u(\xi)=\varphi^{m}(\xi), \tag{2.7}$$

then substitute Eq. (2.7) into Eq. (2.3) and return to determine the value of m once again.

In general, the constant m can be changed into a positive integer by means of the above proper transformation. Otherwise we'll have to seek other proper transformation.

We summarize the extended method as follows:

Step 1. Determine the values of m of Eq. (2.4) by respectively balancing the highest order partial derivative term and the nonlinear term in Eq. (2.1) (or (2.3)).

(i) If m is a positive integer then *Step 2*;

(ii) If $m=\frac{q}{p}$, we make the transformation (2.6)and then return to *Step 1*;

(iii) If m is a negative integer, we make the transformation (2.7) and then return to *Step 1*.

Step 2. With the aid of Mathematica, substituting Eq. (2.4) along with the condition (2.5) into Eq. (2.3), yields a system of algebraic equations w.r.t. $\omega^i(R+\omega^2)^{j/2}$ $(j=0,1; i=0,1,2,...)$.

Step 3. Collect all terms with the same power in $\omega^i(R+\omega^2)^{j/2}$ $(j=0,1; i=0,1,2,...)$. Set the coefficients of the terms $\omega^i(R+\omega^2)^{j/2}$ $(j=0,1; i=0,1,2,...)$ to zero to get an over-determined system of nonlinear algebraic equations w.r.t. the unknown variables $\lambda, R, A_0, A_i, B_i (i=1,2,...,m)$.

Step 4. With the aid of Mathematica, we apply Wu's elimination method [7] to solve the above over-determined system of nonlinear algebraic equations obtained in *step 3*, yields the values of $\lambda, R, A_0, A_i, B_i (i = 1, 2, ..., m)$.

Step 5. It is well known that the general solutions of Eq. (2.5) are

(1) When $R < 0$,

$$\omega(\xi) = -\sqrt{-R}\tanh(\sqrt{-R}\xi), \qquad \omega(\xi) = -\sqrt{-R}\coth(\sqrt{-R}\xi) \tag{2.8}$$

(2) When $R = 0$,

$$\omega(\xi) = -\frac{1}{\xi}, \tag{2.9}$$

(3) When $R > 0$,

$$\omega(\xi) = \sqrt{R}\tan(\sqrt{R}\xi), \qquad \omega(\xi) = -\sqrt{R}\cot(\sqrt{R}\xi) \tag{2.10}$$

Because of

$$\pm\sqrt{R + [-\sqrt{-R}\tanh(\sqrt{-R}\xi)]^2} = \pm\sqrt{R}\mathrm{sech}(\sqrt{-R}\xi);$$

$$\pm\sqrt{R + [-\sqrt{-R}\coth(\sqrt{-R}\xi)]^2} = \pm\sqrt{-R}\mathrm{csch}(\sqrt{-R}\xi);$$

$$\pm\sqrt{R + [\sqrt{R}\tan\sqrt{R}\xi]^2} = \pm\sqrt{R}\sec(\sqrt{R}\xi);$$

$$\pm\sqrt{R + [-\sqrt{R}\cot\sqrt{R}\xi]^2} = \pm\sqrt{R}\csc(\sqrt{R}\xi),$$

the obtained solutions by the improved method must contain the formal solutions obtained in [1], [2], [3].

Thus according to Eqs. (2.2), (2.4), (2.6) or (2.7), (2.8), (2,9), (2.10) and the conclusions in *step 4*, we can obtain more travelling wave solutions of Eq. (2.1).

3. Explicit exact solutions for Eq. (1.1)

Let us consider Eq. (1.1). According to the above steps, we firstly make the following formal travelling wave transformation:

$$u(x,t) = \phi(\xi)\exp(i\eta), \quad \xi = x - \lambda t, \quad \eta = Kx - Qt, \tag{3.1}$$

where λ is a constant to be determined.

Substituting Eq. (3.1) into Eq. (1.1) yields a system of ODE

$$\begin{aligned} &\exp(i\eta)\{(\lambda^2 - \delta\lambda)\phi''(\xi) + i[1 + 2Q\lambda - \delta(Q + K\lambda)]\phi'(\xi) \\ &-(K + Q^2 - \delta KQ)\phi(\xi) + \alpha\phi^{p+1}(\xi) + \beta\phi^{2p+1}(\xi)\} = 0. \end{aligned} \tag{3.2}$$

Under the condition

$$1 + 2Q\lambda - \delta(Q + K\lambda) = 0, \tag{3.3}$$

Eq. (3.2) can be rewritten as follows

$$(\lambda^2 - \delta\lambda)\phi''(\xi) - (K + Q^2 - \delta KQ)\phi(\xi) + \alpha\phi^{p+1}(\xi) + \beta\phi^{2p+1}(\xi) = 0. \quad (3.4)$$

According to *Step 1* in section 2, by balancing the highest order partial derivative term and the nonlinear term in Eq. (3.4), we get the value of m, $m = 1/p$. Therefore we make the following transformation

$$\phi(\xi) = \varphi^{\frac{1}{p}}(\xi), \quad (3.5)$$

then substituting (3.5) into Eq. (3.4) reads

$$N[p\varphi(\xi)\varphi''(\xi) + (1-p)\varphi'^2(\xi)] + p^2[M\varphi^2(\xi) + \alpha\varphi^3(\xi) + \beta\varphi^4(\xi)] = 0, \quad (3.6)$$

where $N = \lambda^2 - \delta\lambda$, $M = -(K + Q^2 - \delta KQ)$. According to *Step 1* in section 2, we suppose that Eq. (3.6) has the following formal solutions

$$\varphi(\xi) = A_0 + A_1\omega + B_1\sqrt{R + \omega^2} \quad (3.7)$$

and $\omega = \omega(\xi)$ satisfies Eq. (2.5), where A_0, A_1, B_1 are constants to be determined later.

With the aid of *Mathematica*, substituting Eq. (3.7) into Eq. (3.6) along with Eq. (2.5) and collecting all terms with the same power in $\omega^i(R+\omega^2)^{j/2}$ $(j = 0, 1; i = 0, 1, 2, 3, 4)$, yield a system of equations w.r.t. $\omega^i(R+\omega^2)^{j/2}$. Setting the coefficients of $\omega^i(R+\omega^2)^{j/2}$ $(j = 0, 1; i = 0, 1, 2, 3, 4)$ in the obtained system of equations to zero, we can deduce the following set of over-determined algebraic polynomial with the unknowns $\lambda, R, A_0, A_1, B_1$ namely:

$$R^2(N(-{A_1}^2(-1+p) + {B_1}^2 p) + {B_1}^4 p^2\beta) + {A_0}^2 p^2(M + A_0(\alpha + A_0\beta)) + {B_1}^2 p^2 R(M + 3A_0(\alpha + 2A_0\beta)) = 0, \quad (3.8)$$

$$B_1 p(R(A_0 N + {B_1}^2 p(\alpha + 4A_0\beta)) + A_0 p(2M + A_0(3\alpha + 4A_0\beta))) = 0, \quad (3.9)$$

$$A_1 p(R(2A_0 N + 3{B_1}^2 p(\alpha + 4A_0\beta)) + A_0 p(2M + A_0(3\alpha + 4A_0\beta))) = 0, \quad (3.10)$$

$$A_1 B_1(2NR + p(NR + 2p(M + 2{B_1}^2 R\beta + 3A_0(\alpha + 2A_0\beta)))) = 0, \quad (3.11)$$

$$R(N(2{A_1}^2 + {B_1}^2(1+2p)) + 2{B_1}^2(3{A_1}^2 + {B_1}^2)p^2\beta) + ({A_1}^2 + {B_1}^2)p^2(M + 3A_0(\alpha + 2A_0\beta)) = 0, \quad (3.12)$$

$$B_1 p(2A_0 N + (3{A_1}^2 + {B_1}^2)p(\alpha + 4A_0\beta)) = 0, \quad (3.13)$$

$$A_1 p(2A_0 N + ({A_1}^2 + 3{B_1}^2)p(\alpha + 4A_0\beta)) = 0, \quad (3.14)$$

$$2A_1 B_1(N + Np + 2({A_1}^2 + {B_1}^2)p^2\beta) = 0, \quad (3.15)$$

$$N({A_1}^2 + {B_1}^2)(1+p) + ({A_1}^4 + 6{A_1}^2{B_1}^2 + {B_1}^4)p^2\beta = 0. \quad (3.16)$$

We have found the following seven cases of solution for Eqs. (3.8)-(3.16) with the aid of *Mathematica* and Wu's elimination method:

Case 1.

$$A_0 = B_1 = \alpha = 0, \quad p = 1, \quad A_1 = \pm\sqrt{\frac{M}{\beta R}}, \quad R = -\frac{M}{2N}. \quad (3.17)$$

Case 2.

$$A_0 = A_1 = \alpha = 0, \quad B_1 = \pm\sqrt{-\frac{M(1+P)}{\beta R}}, \quad R = \frac{MP^2}{N}. \tag{3.18}$$

Case 3.

$$A_0 = M = \alpha = R = 0, \quad A_1 = \pm B_1 = \pm\sqrt{-\frac{N(1+P)}{4\beta p^2}}. \tag{3.19}$$

Case 4.

$$A_0 = \alpha = 0, \quad p = 1, \quad A_1 = \pm B1 = \pm\sqrt{\frac{M}{\beta R}}, \quad R = -\frac{2M}{N}. \tag{3.20}$$

Case 5.

$$B_1 = 0, \quad A_0 = \pm\sqrt{\frac{M(1+p)}{4\beta}}, \quad A_1 = \pm\sqrt{-\frac{M(1+p)}{4\beta R}},$$

$$R = \frac{Mp^2}{4N}, \quad \alpha^2 = \frac{M\beta(2+p)^2}{1+p}. \tag{3.21}$$

Case 6.

$$A_1 = 0, \quad p = 1, \quad A_0 = \pm\sqrt{\frac{M}{2\beta}}, \quad B_1 = \pm\sqrt{\frac{M}{\beta R}},$$

$$R = -\frac{M}{2N}, \quad \alpha^2 = \frac{9M\beta}{2}. \tag{3.22}$$

Case 7.

$$A_0 = \pm\sqrt{\frac{M(1+p)}{4\beta}}, \quad A_1 = \pm B_1 = \pm\sqrt{-\frac{M(1+p)}{4\beta R}},$$

$$R = \frac{Mp^2}{N}, \quad \alpha^2 = \frac{M\beta(2+p)^2}{1+p}. \tag{3.23}$$

Therefore, according to *step 5*, combining Eqs. (2.8), (2.9), (2.10), (3.1), (3.3), (3.5), (3.7) along with Cases 1–7, seven families explicit and exact travelling solutions , which contain solitary wave solutions, periodic wave solutions and new travelling wave solution, rational solutions and singular solutions, are found as follows for Eq. (1.1):
Case 1.
When $R < 0$, i.e., $NM > 0$,

$$u_{11} = \pm\sqrt{\frac{-M}{\beta}}\tanh[\sqrt{\frac{M}{2N}}(x - \lambda t + \xi_0)]\exp[i(kx - Qt)], \tag{3.24}$$

$$u_{12} = \pm\sqrt{\frac{-M}{\beta}}\coth[\sqrt{\frac{M}{2N}}(x - \lambda t + \xi_0)]\exp[i(kx - Qt)], \tag{3.25}$$

When $R > 0$, i.e., $NM < 0$

$$u_{13} = \pm\sqrt{\frac{M}{\beta}}\tan[\sqrt{-\frac{M}{2N}}(x-\lambda t+\xi_0)]\exp[i(kx-Qt)], \tag{3.26}$$

$$u_{14} = \pm\sqrt{\frac{M}{\beta}}\cot[\sqrt{-\frac{M}{2N}}(x-\lambda t+\xi_0)]\exp[i(kx-Qt)], \tag{3.27}$$

where $\beta \neq 0$.
Case 2.
when $R < 0$, i.e., $NM < 0$,

$$u_{21} = \{\pm\sqrt{-\frac{M(1+p)}{\beta}}\mathrm{sech}[\sqrt{-\frac{Mp^2}{N}}(x-\lambda t+\xi_0)]\}^{\frac{1}{p}}\exp[i(Kx-Qt)], \tag{3.28}$$

$$u_{22} = \{\pm\sqrt{\frac{M(1+p)}{\beta}}\mathrm{csch}[\sqrt{-\frac{Mp^2}{N}}(x-\lambda t+\xi_0)]\}^{\frac{1}{p}}\exp[i(Kx-Qt)], \tag{3.29}$$

when $R > 0$, i.e., $NM > 0$,

$$u_{23} = \{\pm\sqrt{-\frac{M(1+p)}{\beta}}\sec[\sqrt{\frac{Mp^2}{N}}(x-\lambda t+\xi_0)]\}^{\frac{1}{p}}\exp[i(Kx-Qt)], \tag{3.30}$$

$$u_{24} = \{\pm\sqrt{-\frac{M(1+p)}{\beta}}\csc[\sqrt{\frac{Mp^2}{N}}(x-\lambda t+\xi_0)]\}^{\frac{1}{p}}\exp[i(Kx-Qt)], \tag{3.31}$$

where $\beta \neq 0$.
Case 3.
When $A_0 = M = \alpha = R = 0$,

$$u_3 = [\pm\sqrt{-\frac{N(1+p)}{\beta p^2}}(\frac{1}{x-\lambda t+\xi_0})]^{\frac{1}{p}}\exp[i(Kx-Qt)], \tag{3.32}$$

where λ is arbitrary constant, $\beta \neq 0$.
Case 4.
When $A_0 = \alpha = 0, \quad p = 1, \quad R < 0 (i.e., NM > 0)$,

$$u_{41} = \pm\sqrt{-\frac{M}{\beta}}[\tanh[\sqrt{-R}(x-\lambda t+\xi_0)] \pm \mathrm{isech}[\sqrt{-R}(x-\lambda t+\xi_0)]]\exp(i\eta), \tag{3.33}$$

$$u_{42} = \pm\sqrt{-\frac{M}{\beta}}[\coth[\sqrt{-R}(x-\lambda t+\xi_0)] \pm csch[\sqrt{-R}(x-\lambda t+\xi_0)]]\exp(i\eta), \tag{3.34}$$

When $R > 0$, i.e., $NM < 0$,

$$u_{43} = \pm\sqrt{\frac{M}{\beta}}[\tan[\sqrt{R}(x-\lambda t+\xi_0)] \pm \sec[\sqrt{R}(x-\lambda t+\xi_0)]]\exp(i\eta), \tag{3.35}$$

$$u_{44}=\pm\sqrt{\frac{M}{\beta}}[\cot[\sqrt{R}(x-\lambda t+\xi_0)]\pm csc[\sqrt{R}(x-\lambda t+\xi_0)]]\exp(i\eta). \tag{3.36}$$

Where $R=-\frac{2M}{N}$, $\eta=Kx-Qt$, $\beta\neq 0$.
Case 5.
When $R<0$, i.e., $NM<0$

$$u_{51}=\{\pm\sqrt{\frac{M(1+p)}{4\beta}}[1\pm\tanh[\sqrt{-\frac{Mp^2}{4N}}(x-\lambda t+\xi_0)]]\}^{\frac{1}{p}}\exp[i(Kx-Qt)], \tag{3.37}$$

$$u_{52}=\{\pm\sqrt{\frac{M(1+p)}{4\beta}}[1\pm\coth[\sqrt{-\frac{Mp^2}{4N}}(x-\lambda t+\xi_0)]]\}^{\frac{1}{p}}\exp[i(Kx-Qt)], \tag{3.38}$$

when $R>0$, i.e., $NM>0$

$$u_{53}=\{\pm\sqrt{\frac{M(1+p)}{4\beta}}[1\pm \mathrm{i}\tan[\sqrt{\frac{Mp^2}{4N}}(x-\lambda t+\xi_0)]]\}^{\frac{1}{p}}\exp[i(Kx-Qt)], \tag{3.39}$$

$$u_{54}=\{\pm\sqrt{\frac{M(1+p)}{4\beta}}[1\pm \mathrm{i}\cot[\sqrt{\frac{Mp^2}{4N}}(x-\lambda t+\xi_0)]]\}^{\frac{1}{p}}\exp[i(Kx-Qt)], \tag{3.40}$$

where $\alpha^2=\frac{Ma_4(2+p)}{1+p}$, $\beta\neq 0$.
Case 6.
When $A_1=0$, $p=1$, $R<0$, i.e., $NM>0$

$$u_{61}=\{\pm\sqrt{\frac{M}{2\beta}}\pm\sqrt{-\frac{M}{\beta}}\mathrm{sech}[\sqrt{\frac{M}{2N}}(x-\lambda t+\xi_0)]\}\exp[i(Kx-Qt)], \tag{3.41}$$

$$u_{62}=\{\pm\sqrt{\frac{M}{2\beta}}\pm\sqrt{\frac{M}{\beta}}csch[\sqrt{\frac{M}{2N}}(x-\lambda t+\xi_0)]\}\exp[i(Kx-Qt)], \tag{3.42}$$

when $R>0$, i.e., $NM<0$,

$$u_{63}=\{\pm\sqrt{\frac{M}{2\beta}}\pm\sqrt{\frac{M}{\beta}}\sec[\sqrt{-\frac{M}{2N}}(x-\lambda t+\xi_0)]\}\exp[i(Kx-Qt)], \tag{3.43}$$

$$u_{64}=\{\pm\sqrt{\frac{M}{2\beta}}\pm\sqrt{\frac{M}{\beta}}csc[\sqrt{-\frac{M}{2N}}(x-\lambda t+\xi_0)]\}\exp[i(Kx-Qt)], \tag{3.44}$$

where $\alpha^2=\frac{9M\beta}{2}$, $\beta\neq 0$.
Case 7.
When $R<0$, i.e., $NM<0$

$$u_{71}=\{A_0[1\pm\tanh(\sqrt{-R}(x-\lambda t+\xi_0))\pm \mathrm{i}\,\mathrm{sech}(\sqrt{-R}(x-\lambda t+\xi_0))]\}^{\frac{1}{p}}\exp[i(Kx-Qt)], \tag{3.45}$$

$$u_{72} = \{A_0[1 \pm \coth(\sqrt{-R}(x-\lambda t+\xi_0)) \pm \mathrm{csch}(\sqrt{-R}(x-\lambda t+\xi_0))]\}^{\frac{1}{p}} \exp[i(Kx-Qt)], \tag{3.46}$$

When $R > 0$, i.e., $NM > 0$,

$$u_{73} = \{A_0[1 \pm \mathrm{i}\tan(\sqrt{R}(x-\lambda t+\xi_0)) \pm \mathrm{i}\sec(\sqrt{R}(x-\lambda t+\xi_0))]\}^{\frac{1}{p}} \exp[i(Kx-Qt)], \tag{3.47}$$

$$u_{74} = \{A_0[1 \pm \mathrm{i}\cot(\sqrt{R}(x-\lambda t+\xi_0)) \pm \mathrm{i}\csc(\sqrt{R}(x-\lambda t+\xi_0))]\}^{\frac{1}{p}} \exp[i(Kx-Qt)], \tag{3.48}$$

where $\beta \neq 0$, $R = \frac{Mp^2}{N}$, $A_0 = \pm\sqrt{\frac{M(1+p)}{4\beta}}$.

Remark.
(1) It is easy to see that our obtained solutions for Eq. (1.1) completely include the solutions in [1], [2] and [3]: u_{21}, u_{11} and u_{41}, are just the solutions (1.3), (1.4) and (1.5).
(2) To our knowledge, the rest obtained solutions of a new class of Eq. (1.1) were not be found before.

4. The bell-shaped soliton solutions to generalized Hamiltonian amplitude equation

In section 3, when $\alpha = 0$ in Eq. (1.1), the bell-shaped soliton solution (3.28) are obtained. In this section, we would consider the bell-shaped soliton solution for Eq. (1.1) under the condition $\alpha \neq 0$. Now, we assume that the solution of Eq. (3.6) has the following form:

$$\varphi(\xi) = \frac{Ae^{\rho(\xi+\xi_0)}}{(1+e^{\rho(\xi+\xi_0)})^2 + Be^{\rho(\xi+\xi_0)}} = \frac{\mathrm{Asech}^2[\frac{\rho}{2}(\xi+\xi_0)]}{4+\mathrm{Bsech}^2[\frac{\rho}{2}(\xi+\xi_0)]}, \tag{4.1}$$

where A, B and ρ are constants to be determined, and ξ_0 is an arbitrary phase shift.

With the aid of *Mathematica*, substuting Eq. (4.1) into Eq. (3.6), we obtain

$$A^2(a2p^2 + a1\rho^2) = 0, \tag{4.2}$$

$$A^2p(2a2(2+B)p + Ap\alpha - a1(2+B)\rho^2) = 0, \tag{4.3}$$

$$A^2(a2(6+4B+B^2)p^2 + A(2+B)p^2\alpha + A^2p^2\beta - 2a1(1+2p)\rho^2) = 0, \tag{4.4}$$

$$A^2p(2a2(2+B)p + Ap\alpha - a1(2+B)\rho^2) = 0, \tag{4.5}$$

$$A^2(a2p^2 + a1\rho^2) = 0. \tag{4.6}$$

By solving Eqs. (4.2)-(4.7), we get the following conclusions:

$$\rho = \pm\sqrt{-\frac{Mp^2}{N}}, \quad A_{1,2} = \mp\frac{2(2+p)M\sqrt{1+p}}{\sqrt{(1+p)\alpha^2 - M(2+p)^2\beta}},$$

$$B_{1,2} = -2 \pm \frac{2\alpha\sqrt{1+p}}{\sqrt{(1+p)\alpha^2 - M(2+p)^2\beta}}.$$

Therefore, there are two solutions of the form (4.2) for Eq. (3.6):

$$\varphi_1(\xi) = \frac{\frac{(2+p)M\sqrt{1+p}}{\sqrt{(1+p)\alpha^2-M(2+p)^2\beta}}\operatorname{sech}^2[\pm\sqrt{-\frac{Mp^2}{4N}}(\xi+\xi_0)]}{2+(-1-\frac{\alpha\sqrt{1+p}}{\sqrt{(1+p)\alpha^2-M(2+p)^2\beta}})\operatorname{sech}^2[\pm\sqrt{-\frac{Mp^2}{4N}}(\xi+\xi_0)]}, \tag{4.7}$$

$$\varphi_2(\xi) = \frac{-\frac{(2+p)M\sqrt{1+p}}{\sqrt{(1+p)\alpha^2-M(2+p)^2\beta}}\operatorname{sech}^2[\pm\sqrt{-\frac{Mp^2}{4N}}(\xi+\xi_0)]}{2+(-1+\frac{\alpha\sqrt{1+p}}{\sqrt{(1+p)\alpha^2-M(2+p)^2\beta}})\operatorname{sech}^2[\pm\sqrt{-\frac{Mp^2}{4N}}(\xi+\xi_0)]}. \tag{4.8}$$

Then Eq. (1.1) has the bell-shaped solitary-wave solutions

$$u_1(x,t) = [\varphi_1(\xi)]^{\frac{1}{p}} \exp[i(Kx - Qt)], \tag{4.9}$$

$$u_2(x,t) = [\varphi_2(\xi)]^{\frac{1}{p}} \exp[i(Kx - Qt)], \tag{4.10}$$

where $\varphi_1(\xi)$ and $\varphi_2(\xi)$ are given by (4.7), (4.8) respectively.

Remark. It should be pointed out that there is no method to find all solutions of nonlinear partial different equations. Although our method can be used to find more solutions of nonlinear partial differential equations, we cannot obtain the formal solutions (4.9) and (4.10) by our method. It is easily to see that the solutions (4.9) and (4.10), when $\alpha = 0$, are just the solution (3.28).

5. Conclusions

In this paper, firstly we present a new generalized Hamiltonian amplitude equation with nonlinear terms of any order Eq. (1.1), which is of important significance in explaining some physical phenomena. Secondly, the tanh-method is improved by means of a new general ansatz, therefore we make the method much more lucid and straightforward to write solutions. Thirdly, we apply the improved method to Eq. (1.1) and obtain seven families of solution, which include solutions in [1], [2], [3] and more new explicit exact solutions. Forthly, by another ansatz (4.1), new bell-shaped soliton solutions are found. As a result, rich explicit exact solutions, which contain new kink-profile solitary-wave solutions, periodic wave solutions and combined formal solitary-wave solutions, are obtained. In addition, we also can derive rational solutions for Eq. (1.1). Presently we are making the method computerizable, which allow us to perform complicated and tedious algebraic calculation on a computer.

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