Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT®

ELSEVIER Applied Mathematics and Computation 152 (2004) 581–595

APPLIED MATHEMATICS AND COMPUTATION

www.elsevier.com/locate/amc

Generalized Riccati equation expansion method and its application to the (3 + 1)-dimensional Jumbo–Miwa equation

Biao Li [a,*], Yong Chen [a], Hengnong Xuan [b], Hongqing Zhang [a]

[a] *Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, People's Republic of China*

[b] *Department of Computer, Nanjing University of Economics, Nanjing 210003, People's Republic of China*

Abstract

Based on the computerized symbolic system *Maple* and a Riccati equation, a generalized Riccati equation expansion method for constructing soliton-like solutions of non-linear evolution equations (NEEs) is presented by a new general ansätz. The proposed method is more powerful than most of the existing tanh methods, the extended tanh-function method, the modified extended tanh-function method and generalized hyperbolic-function method. By use of the method, we not only can successfully recover the previously known formal solutions but also construct new and more general formal solutions for some NEEs. Making use of the method, we study the the (3 + 1)-dimensional Jumbo–Miwa equation and obtain rich new families of the exact solutions, including the non-travelling wave and coefficient functions' soliton-like solutions, singular soliton-like solutions, periodic form solutions.

Keywords: Generalized Riccati equation expansion method; (3 + 1)-dimensional Jumbo–Miwa equation; Soliton-like solutions; Symbolic computation

* Corresponding author.
E-mail address: libiao@dlut.edu.cn (B. Li).

0096-3003/$ - see front matter
doi:10.1016/S0096-3003(03)00578-2

1. Introduction

In recently years, directly searching for exact solutions of non-linear evolution equations (NEEs) has become more and more attractive, on the one hand, due to their occurrence in many fields of science, in physics as well as in chemistry or biology and the interesting features and rich variety of their solutions, on the other hand, due to the availability of computer systems like *Maple* or *Mathematica* which allow us to perform some complicated and tedious algebraic calculation and differential calculation on a computer, at the same time help us to find new exact solution of NEEs. In order to obtain exact solutions of NEEs, many effective methods have been presented, such as, inverse scattering method, Bäcklund transformation, Darboux transformation, Cole–Hopf transformation, Hirota method, Painlevé method [1,2], tanh method [3–8], extended tanh-function method [9–11], modified extended tanh-function method [17], generalized hyperbolic-function method [18,19]. Particularly, various ansätz have been proposed in order to obtain new form of solutions.

One of most effectively straightforward methods to construct exact solutions of NEEs is tanh method [3–8]. In [9–11], Fan proposed an extended tanh-function method. Fan [12], Yan [13] and Li and co-workers [14–16] further developed this idea and made it much more lucid and straightforward for a class of NEEs. Recently, Elwakil et al. [17] modified extended tanh-function method and obtained some new exact solutions. Gao and Tian [18,19] presented a generalized hyperbolic-function method by introducing coefficient functions and non-travelling transformation. As we known, when applying direct method, the choice of an appropriate ansätz is of great importance. In this paper, based on the above work [3–19], by introducing a more general ansätz than the ansätz in the above methods, we present the generalized Riccati equation expansion method. Then we choose the (3 + 1)-dimensional Jumbo–Miwa equation to illustrate our algorithm and obtain rich new families of the exact solutions, including the non-travelling wave and coefficient functions' soliton-like solutions, singular soliton-like solutions, periodic form solutions.

2. Generalized Riccati equation expansion mehtod

We now establish the generalized Riccati equation expansion method as follows:

(A) For a given NEEs with one physical field $u(x, y, t)$ in three variables x, y, t

$$H(u, u_t, u_x, u_y, u_{xx}, u_{xt}, u_{xy}, u_{yt}, \dots,) = 0. \tag{2.1}$$

We express the solutions of the NEEs by the more general ansätz

$$u(x, y, t) = f + \sum_{i=1}^{m} \left[g_i \phi^i(\xi) + h_i \phi^{i-1}(\xi)\sqrt{R + \phi^2(\xi)} + k_i \phi^{-i}(\xi) \right], \tag{2.2}$$

where m is an integer to be determined by balancing the highest order derivative terms with the non-linear terms in (2.1), R is a real constant, while $f = f(x,y,t),\ g_i = g_i(x,y,t),\ h_i = h_i(x,y,t),\ k_i = k_i(x,y,t),\ (i = 1,\dots,m),\ \xi = \xi(x,y,t)$ are all arbitrary differentiable functions and $\phi(\xi)$ satisfies

$$\frac{\mathrm{d}\phi(\xi)}{\mathrm{d}\xi} = R + \phi^2(\xi). \tag{2.3}$$

(B) Substituting (2.2) along with (2.3) into (2.1), multiplying the most simplify common denominator in the obtained system, setting the coefficients of $\phi^j(\xi)\left(\sqrt{R+\phi^2(\xi)}\right)^n$ $(j = 0,1,\dots;\ n = 0,1)$ (*Note:* where $\phi^i(\xi)$ denotes i power of $\phi(\xi)$ and $\left(\sqrt{R+\phi^2(\xi)}\right)^n$ denotes n power of $\sqrt{R+\phi^2(\xi)}$) to zero, we obtain a set of over-determined partial differential equations with respect to differentiable functions f, g_i, h_i, k_i $(i = 1,\dots,m)$ and ξ.

(C) Solving the over-determined partial differential equations by use of the PDEtools package of *Maple*, we would end up with the explicit expressions for f, g_i, h_i, k_i $(i = 1,\dots,m)$ and ξ or the constrains among them.

(D) It is well-known that the general solutions of Riccati equation (2.3) are

$$\phi(\xi) = \begin{cases} -\sqrt{-R}\tanh(\sqrt{-R}\xi), & R < 0, \\ -\sqrt{-R}\coth(\sqrt{-R}\xi), & R < 0, \\ -\dfrac{1}{\xi}, & R = 0, \\ \sqrt{R}\tan(\sqrt{R}\xi), & R > 0, \\ -\sqrt{R}\cot(\sqrt{R}\xi), & R > 0. \end{cases} \tag{2.4}$$

Thus according to (2.2), (2.4) and the conclusions in (C), we obtain many general solutions of (2.1). At the same time, it is necessary to point out that, when $R = 0$, some solutions are missed using the above method. So under this condition, we take the following steps to deal with it: (1) set $h_i = 0$; (2) set the coefficients of the same power of $\phi^j(\xi)$ $(i = 0,1,\dots)$ to zero and obtain a set of over-determined partial differential equations with respect to functions f, g_i, k_i $(i = 1,\dots,m)$; (3) solve the set of over-determined partial differential equations; (4) write the solutions by use of Eq. (2.4).

Remarks

1. Generalization

The method proposed here is more general than generalized hyperbolic-function method [18,19], tanh method [3–8], extended tanh-function method [9–12], modified extended tanh-function method [17]. Firstly, compared with the tanh method, extended tanh-function, as well as the modified extended tanh-function method, the restriction on $\xi(x,y,t)$ as merely a linear function of

x, y, t and the restriction on the coefficients f, g_i, h_i, k_i $(i = 1, \ldots, m)$ as constants are removed. Secondly, compared with the generalized hyperbolic-function method, setting $k_i = 0$ $(i = 1, \ldots, m)$, we can not only recover the exact soliton-like solutions obtained by generalized hyperbolic-function method for a given NEEs but also we can, with no extra efforts, find singular soliton-like solutions and periodic form solutions. When $k_i \neq 0$ $(i = 1, \ldots, m)$, some new formal solutions would be expected for some NEEs.

2. Feasibility

For the generalization of the ansätz, naturally more complicated computation is expected than ever before. Even if the availability of computer symbolic systems like *Maple* or *Mathematica* allows us to perform the complicated and tedious algebraic calculation and differential calculation on a computer, in general, it is very difficult, sometime impossible, to solve the set of over-determined partial differential equations in step (B). As the calculation goes on, in order to drastically simplify the work or make the work feasible, we often choose special function forms for f, g_i, h_i, k_i $(i = 1, \ldots, m)$ and ξ, on a trial-and-error basis.

3. The (3 + 1)-dimensional Jumbo–Miwa equation

In this section, by use of the generalized Riccati equation expansion method, we investigate (3 + 1)-dimensional Jumbo–Miwa equation [20–24]

$$u_{xxxy} + 3u_{xy}u_x + 3u_y u_{xx} + 2u_{yt} - 3u_{xz} = 0, \tag{3.1}$$

which comes from the second member of a KP hierarchy. Eq. (3.1) does not pass the Painlevé test and hence it is expected to be non-integrable [24]. Also it does not lead to Kac–Moody–Virasoro type subalgebras which typically exist in many of the integrable higher-dimensional equations [21–23]. Some travelling wave solutions for Eq. (3.1) are obtained in [20].

By balancing the highest-order contributions from both the linear and nonlinear terms in Eq. (3.1), we obtain $m = 1$ in (2.2). Therefore we assume the solutions of Eq. (3.1) in the following special form

$$u(x, y, z, t) = f + g\phi(\xi) + h\sqrt{R + \phi^2(\xi)} + k\phi^{-1}(\xi), \tag{3.2}$$

where $f = f(y, z, t)$, $g = g(y, z, t)$, $h = h(y, z, t)$, $k = k(y, z, t)$ and $\xi = xp(y, z, t) + q(y, z, t)$ are all differentiable functions and $\phi(\xi)$ satisfies (2.3).

Substituting (3.2) along with (2.3) into (3.1), multiplying $\phi(\xi)^5\sqrt{R + \phi(\xi)^2}$ in the obtained system, collecting coefficients of monomials of $\phi(\xi)$, $\sqrt{R + \phi(w)^2}$ and x (notice that f, g, h, k, p and q are independent of x) with the aid of Maple, then setting each coefficients to zero, we can deduce the following set of over-determined partial differential equations with respect to the unknown functions f, g, h, k, p and q

$$2R(h_yp_t + hp_{ty} + h_tp_y) = 0, \tag{3.3}$$

$$6p^2h_ykR^3 = 0, \tag{3.4}$$

$$3p(2p^2h_y + 3pg_yh + 6pp_yh + 3ph_yg + 2p_ygh) = 0, \tag{3.5}$$

$$3R^2pk(3ph_y - 2p_yh) = 0, \tag{3.6}$$

$$3p(2g_yp^2 + 3pg_yg + 6pp_yg + 3phh_y + p_yg^2 + h^2p_y) = 0, \tag{3.7}$$

$$36p^2gR^2q_yh + 9p^2f_yhR - 9phq_zR + 6hq_yq_tR + 2h_{ty} - 18p^2hq_ykR + 33p^3hR^2q_y = 0, \tag{3.8}$$

$$3Rh(-6p^2p_yk - 3pp_z + 2p_yq_t + 12p^2p_ygR + 11Rp^3p_y + 2p_tq_y) = 0, \tag{3.9}$$

$$2hp_{ty} + 2h_tp_y + 2h_yp_t = 0, \tag{3.10}$$

$$6hp_yp_tR = 0, \tag{3.11}$$

$$12R^3p^2kq_y(2Rp - k) = 0, \tag{3.12}$$

$$12R^3p^2p_yk(2Rp - k) = 0, \tag{3.13}$$

$$R(2hq_yq_tR + 2h_{ty} + 3p^2f_yhR + 6p^2gR^2q_yh - 3phq_zR + 5p^3hR^2q_y) = 0, \tag{3.14}$$

$$R^2h(6p^2p_ygR + 2p_yq_t - 3pp_z + 2p_tq_y + 5Rp^3p_y) = 0, \tag{3.15}$$

$$2hp_yp_tR^2 = 0, \tag{3.16}$$

$$R(5p^3h_yR + 3p^2k_yh + 2hq_{ty} - 3p_zh + 2h_yq_t - 12pp_yhk + 15p^2p_yhR + 6pp_ygRh - 3ph_z + 9p^2h_ygR + 6p^2g_yhR + 2h_tq_y) = 0, \tag{3.17}$$

$$2h_yq_t - 3ph_z + 11p^3h_yR + 3p^2k_yh + 15p^2g_yhR - 3p_zh + 2hq_{ty} - 3p^2h_yk + 33p^2p_yhR + 12pp_ygRh + 18p^2h_ygR + 2h_tq_y - 6pp_yhk = 0, \tag{3.18}$$

$$2h(2p_tq_y + 26p^3Rp_y - 3pp_z - 6p^2p_yk + 27p^2p_ygR + 2p_yq_t) = 0, \tag{3.19}$$

$$2h(27p^2gRq_y + 26p^3Rq_y - 3pq_z + 2q_yq_t - 6p^2q_yk + 3p^2f_y) = 0, \tag{3.20}$$

$$4hp_yp_t = 0, \tag{3.21}$$

$$2R^2k(-12p^2p_yk + 2p_yq_t + 6p^2p_ygR + 2p_tq_y + 20p^3Rp_y - 3pp_z) = 0, \tag{3.22}$$

$$2R^2k(-12p^2q_yk + 2q_yq_t + 6p^2gRq_y + 20p^3Rq_y - 3pq_z + 3p^2f_y) = 0, \tag{3.23}$$

$$4kR^2p_yp_t = 0, \tag{3.24}$$

$$- 6p^2p_ykR + 3p^2g_yR^2g - 12pp_ygRk + 3p^2h_yhR^2 - 3p_zgR + 3pp_yk^2 + 3p^2k_yk + 3pp_yg^2R^2 + 6p^2p_ygR^2 - 3pg_zR + 2p^3g_yR^2 - 2p^3k_yR + 2g_tRq_y + 2f_{ty} + 2gRq_{ty} - 2kq_{ty} + 3pk_z - 2k_yq_t + 2g_yRq_t + 3p_zk - 2k_tq_y = 0, \quad (3.25)$$

$$-2k_tp_y - 2kp_{ty} + 2g_tRp_y + 2gRp_{ty} - 2k_yp_t + 2g_yRp_t = 0, \quad (3.26)$$

$$-3R^2p(2k_yRp^2 + 6pp_ykR - 3pk_yk - p_yk^2) = 0, \quad (3.27)$$

$$- 12p^2k^2Rq_y - 6pkRq_z + 6p^2f_ykR + 16p^3kR^2q_y + 2k_{ty} + 12p^2gR^2q_yk + 4kRq_yq_t = 0, \quad (3.28)$$

$$2Rk(-6p^2p_yk + 2p_yq_t - 3pp_z + 6p^2p_ygR + 2p_tq_y + 8p^3Rp_y) = 0, \quad (3.29)$$

$$4kRp_yp_t = 0, \quad (3.30)$$

$$12p^2h_yhR - 3pg_z + 2g_yq_t - 6pp_ygk + 6pp_yg^2R + 2g_tq_y + 3p^2k_yg + 3pp_yh^2R + 24p^2p_ygR + 2gq_{ty} - 3p_zg - 3p^2g_yk + 12p^2g_yRg + 8p^3g_yR = 0, \quad (3.31)$$

$$-2R(k_tp_y + kp_{ty} + k_yp_t) = 0, \quad (3.32)$$

$$- R(-12p^2k_yk + 6pp_ygRk - 3p_zk + 8p^3k_yR + 2k_yq_t - 6pp_yk^2 - 3pk_z + 24p^2p_ykR - 3Rp^2g_yk + 2k_tq_y + 3Rp^2k_yk_yg + 2kq_{ty}) = 0, \quad (3.33)$$

$$16p^3gR^2q_y + 12p^2g^2R^2q_y - 6pgRq_z + 6p^2f_ygR + 2g_{ty} + 4gRq_yq_t - 12p^2gRq_yk + 6p^2h^2q_yR^2 = 0, \quad (3.34)$$

$$2R(8p^3gRp_y - 6p^2gp_yk + 6p^2g^2Rp_y + 3p^2h^2p_yR - 3pgp_z + 2gq_yp_t + 2gp_yq_t) = 0, \quad (3.35)$$

$$4gRp_yp_t = 0, \quad (3.36)$$

$$24p^2g^2Rp_y - 12p^2gp_yk - 6pgp_z + 4gq_yp_t + 18p^2h^2p_yR + 4gp_yq_t + 40p^3gRp_y = 0, \quad (3.37)$$

$$4gp_yp_t = 0, \quad (3.38)$$

$$24p^2g^2Rq_y + 18p^2h^2q_yR + 6p^2f_yg - 6pgq_z + 4gq_yq_t - 12p^2gq_yk + 40p^3gRq_t = 0, \quad (3.39)$$

$$6p^2hq_ykR^3 = 0, \quad (3.40)$$

$$6p^2hp_ykp^3 = 0, \quad (3.41)$$

$$24p^2hq_y(p + g) = 0, \quad (3.42)$$

$$24p^2p_yh(p + g) = 0, \quad (3.43)$$

$$2gp_{ty} + 2g_yp_t + 2g_tp_y = 0, \tag{3.44}$$

$$12p^2p_y(g^2 + 2pg + h^2) = 0, \tag{3.45}$$

$$12p^2q_y(g^2 + 2pg + h^2) = 0. \tag{3.46}$$

Using the powerful PDEtools soft package of *Maple*, solving the set of partial differential equations (3.3)–(3.46), we can obtain the following non-trivial results. (*Note*: in the rest of this paper, c_1, C_i $(i = 1, \ldots, 5)$ are arbitrary constants and $F_i(t)$ $(i = 1, \ldots, 5)$ denote arbitrary function of t, and so on.)

Case 1

$$\begin{aligned} &g = -2C_1, \quad h = 0, \quad k = 0, \quad p = C_1, \quad q = F_2(y,z) + F_4(z) + C_2t + C_3,\\ &f = \int \frac{1}{3C_1^2}\left[4C_1^3R\frac{\partial F_2(y,z)}{\partial y} + 3C_1\frac{\partial F_2(y,z)}{\partial z} - 2\frac{\partial F_2(y,z)}{\partial y}C_2)\right]\mathrm{d}y\\ &\qquad + \frac{1}{C_1}\frac{\partial F_4(z)}{\partial z}y + F_5(z,t). \end{aligned} \tag{3.47}$$

Case 2

$$\begin{aligned} &g = -2C_1, \quad h = 0, \quad k = 0, \quad p = C_1,\\ &q = F_3(z)y + F_4(z) + F_5\left(t + \frac{2}{3C_1}\int F_3(z)\,\mathrm{d}z\right),\\ &f = \frac{1}{2C_1}\frac{\partial F_3(z)}{\partial z}y^2 + \frac{4}{3}yC_1RF_3(z) + \frac{1}{C_1}\frac{\partial F_4(z)}{\partial z}y + F_6(z,t). \end{aligned} \tag{3.48}$$

Case 3

$$\begin{aligned} &g = -2\sqrt{2c_1z + C_1}\,\mathrm{e}^{C_2t}C_3, \quad h = 0, \quad k = 0, \quad p = \sqrt{2c_1z + C_1}\,C_2\mathrm{e}^{C_2t}C_3,\\ &q = \frac{1}{\sqrt{2c_1z + C_1}\,C_2}\left[\sqrt{2c_1z + C_1}\,F_2\left[t + \frac{1}{2C_2}\ln(2c_1z + C_1)\right]C_2\right.\\ &\qquad - 4\mathrm{e}^{3C_2t}\left(c_1z + \frac{1}{2}C_1\right)^2RC_3^3\ln(2c_1z + C_1)\\ &\qquad \left. + \left[\frac{3}{2}yc_1C_3 + \sqrt{2c_1z + C_1}\,F_1(z)C_2\right]\mathrm{e}^{C_2t}\right],\\ &f = \frac{1}{C_2(2c_1z + C_1)^{5/2}C_3}\left[4\frac{\partial F_1(z)}{\partial z}C_2c_1^2z^2y - 2zc_1^2F_1(z)C_2y\right.\\ &\qquad + 4z\frac{\partial F_1(z)}{\partial z}C_2C_1c_1y - \frac{3}{2}y^2C_3\sqrt{2c_1z + C_1}\,c_1^2\\ &\qquad \left. - F_1(z)C_2C_1c_1y + \frac{\partial F_1(z)}{\partial z}C_1^2C_2y\right] + F_3(z,t). \end{aligned} \tag{3.49}$$

Case 4

$$g=-2p,\quad h=0,\quad k=0,\quad q=\int \frac{1}{2}\frac{p(4p^2Rp_y+3p_z)}{p_y}\,\mathrm{d}t+F_1(y,z),$$
$$f=\int \frac{1}{2p_yp}\left[3\int \frac{4p_zp_y^2p^2R+p_z^2p_y+pp_yp_{zz}-pp_{yz}p_z}{p_y^2}\,\mathrm{d}tp_y+2p_y\frac{\partial F_1(y,z)}{\partial z}\right.$$
$$\left.-3p_z\int \frac{4p^2Rp_y^3+p_zp_y^2+pp_yp_{yz}-pp_{yy}p_z}{p_y^2}\,\mathrm{d}t-2p_z\frac{\partial F_1(y,z)}{\partial y}\right]\mathrm{d}y+F_2(z,t), \tag{3.50}$$

where $p=p(y,z,t)$ satisfies

$$p_{yz}=-\frac{-4p_y^2pp_zR-p_{zz}p_y}{p_z},\quad p_{yy}=-\frac{-8p_zRpp_y^3-p_{zz}p_y^2}{p_z^2},\quad p_t=0.$$

Case 5

$$g=-C_1,\quad h=C_1,\quad k=0,\quad p=C_1,\quad q=F_6(y,z)+F_8(z)+C_4t+C_5,$$
$$f=\int \frac{1}{3C_1^2}\left[C_1^3R\frac{\partial F_6(y,z)}{\partial y}+3C_1\frac{\partial F_6(y,z)}{\partial z}-2\frac{\partial F_6(y,z)}{\partial y}C_4\right]\mathrm{d}y$$
$$+\frac{1}{C_1}\frac{\partial F_8(z)}{\partial z}y+F_9(z,t). \tag{3.51}$$

Case 6

$$g=-C_1,\quad h=-C_1,\quad k=0,\quad p=C_1,\quad q=F_2(y,z)+F_4(z)+C_2t+C_3,$$
$$f=\int \frac{1}{3C_1^2}\left[C_1^3R\frac{\partial F_2(y,z)}{\partial y}+3C_1\frac{\partial F_2(y,z)}{\partial z}-2\frac{\partial F_2(y,z)}{\partial y}C_2\right]\mathrm{d}y$$
$$+\frac{1}{C_1}\frac{\partial F_4(z)}{\partial z}y+F_9(z,t). \tag{3.52}$$

Case 7

$$g=-C_1,\quad h=C_1,\quad k=0,\quad p=C_1,$$
$$q=F_8(z)y+F_9(z)+F_{10}\left[t+\frac{2}{3C_1}\int F_8(z)\,\mathrm{d}z\right], \tag{3.53}$$
$$f=\frac{1}{2C_1}\frac{\partial}{\partial z}(F_8(z))y^2+\frac{1}{3}yC_1RF_8(z)+\frac{1}{C_1}y\frac{\partial}{\partial z}(F_9(z))+F_{11}(z,t).$$

Case 8

$$g=-C_1,\quad h=-C_1,\quad k=0,\quad p=C_1,$$
$$q=F_3(z)y+F_4(z)+F_5\left[t+\frac{2}{3C_1}\int F_3(z)\,\mathrm{d}z\right],\tag{3.54}$$
$$f=\frac{1}{2C_1}\frac{\partial F_3(z)}{\partial z}y^2+\frac{1}{3}yC_1RF_3(z)+\frac{1}{C_1}\frac{\partial F_4(z)}{\partial z}y+F_{11}(z,t).$$

Case 9

$$g=-\sqrt{2c_1z+C_1}\mathrm{e}^{C_2t}C_3,\quad h=-\sqrt{2c_1z+C_1}\mathrm{e}^{C_2t}C_3,\quad k=0,$$
$$p=\sqrt{2c_1z+C_1}\mathrm{e}^{C_2t}C_3,$$
$$q=\frac{1}{\sqrt{2c_1z+C_1}C_2}\left\{\sqrt{2c_1z+C_1}F_4\left[t+\frac{1}{2C_2}\ln(2c_1z+C_1)\right]C_2\right.$$
$$-\frac{1}{4}C_3^3R\mathrm{e}^{3C_2t}(2c_1z+C_1)^2\ln(2c_1z+C_1)$$
$$\left.+\mathrm{e}^{C_2t}\left[\frac{3}{2}yc_1C_3+\sqrt{2c_1z+C_1}F_3(z)C_2\right]\right\},$$
$$f=-\frac{3c_1^2y^2}{2C_2(2c_1z+C_1)^2}+\frac{y\frac{\partial F_3(z)}{\partial z}}{C_3(2c_1z+C_1)^{1/2}}-\frac{yc_1F_3(z)}{C_3(2c_1z+C_1)^{3/2}}+F_5(z,t).\tag{3.55}$$

Case 10

$$g=-\sqrt{2c_1z+C_1}\mathrm{e}^{C_2t}C_3,\quad h=\sqrt{2c_1z+C_1}\mathrm{e}^{C_2t}C_3,\quad k=0,$$
$$p=\sqrt{2c_1z+C_1}\mathrm{e}^{C_2t}C_3,$$
$$q=-\frac{1}{4\sqrt{2c_1z+C_1}C_2}\left\{-4\sqrt{2c_1z+C_1}F_2\left[t+\frac{1}{2C_2}\ln(2c_1z+C_1)\right]C_2\right.$$
$$+C_3^3R\mathrm{e}^{3C_2t}(2c_1z+C_1)^2\ln(2c_1z+C_1)$$
$$\left.+\left[-6yc_1C_3-4\sqrt{2c_1z+C_1}F_1(z)C_2\right]\mathrm{e}^{C_2t}\right\},$$
$$f=-\frac{3c_1^2y^2}{2C_2(2c_1z+C_1)^2}+\left\{\frac{2c_1^2D(F_2)\left[t+\frac{1}{2C_2}\ln(2c_1z+C_1)\right]z}{\mathrm{e}^{C_2t}C_3(2c_1z+C1)^{5/2}C_2}\right.$$
$$+\frac{c_1D(F_2)\left[t+\frac{1}{2C_2}\ln(2c_1z+C_1)\right]C_1}{\mathrm{e}^{C_2t}C_3(2c_1z+C_1)^{5/2}C_2}-\frac{c_1D(F_2)\left[t+\frac{1}{2C_2}\ln(2c_1z+C_1)\right]}{\mathrm{e}^{C_2t}C_3(2c_1z+C_1)^{3/2}C_2}$$
$$\left.+\frac{1}{C_3(2c_1z+C_1)^{1/2}}\frac{\partial F_1(z)}{\partial z}-\frac{1}{C_3(2c_1z+C_1)^{3/2}c_1F_1(z)}\right\}y+F_5(z,t),\tag{3.56}$$

where $D(F_2)[t+(1/2C_2)\ln(2c_1z+C_1)]$ denotes the derivative of function $F_2[t+(1/2C_2)\ln(2c_1z+C_1)]$ with respect to variable $[t+(1/2C_2)\ln(2c_1z+C_1)]$.

Case 11

$$g=-p,\quad h=p,\quad k=0,\quad q=\int\frac{1}{2}\frac{p(p^2Rp_y+3p_z)}{p_y}\,\mathrm{d}t+F_2(y,z),$$
$$f=\int\frac{1}{2pp_y}\left[3\int\frac{p_zp_y^2p^2R+p_z^2p_y+pp_yp_{zz}-pp_{yz}p_z}{p_y^2}\,\mathrm{d}tp_y+2p_y\frac{\partial F_2(y,z)}{\partial z}\right.$$
$$\left.-3p_z\int\frac{p^2Rp_y^3+p_zp_y^2+pp_yp_{yz}-pp_{yy}p_z}{p_y^2}\,\mathrm{d}t-2p_z\frac{\partial F_2(y,z)}{\partial y}\right]\mathrm{d}y+F_3(z,t),\tag{3.57}$$

where $p=p(y,z,t)$ satisfies

$$p_{yy}=-\frac{-2p_zRpp_y^3-p_{zz}p_y^2}{p_z^2},\quad p_{yz}=\frac{p_y^2pp_zR+p_{zz}p_y}{p_z},\quad p_t=0.$$

Case 12

$$g=-p,\quad h=-p,\quad q=\int\frac{1}{2p}\frac{p^2Rp_y+3p_z}{p_y}\,\mathrm{d}t+F_1(y,z),\quad k=0,$$
$$f=\int\frac{1}{2pp_y}\left[3\int\frac{p_zp_y^2p^2R+p_z^2p_y+pp_yp_{zz}-pp_{yz}p_z}{p_y^2}\,\mathrm{d}tp_y+2p_y\frac{\partial F_1(y,z)}{\partial z}\right.$$
$$\left.-3p_z\int\frac{p^2Rp_y^3+p_zp_y^2+pp_yp_{yz}-pp_{yy}p_z}{p_y^2}\,\mathrm{d}t-2p_z\frac{\partial F_1(y,z)}{\partial y}\right]\mathrm{d}y+F_3(z,t),\tag{3.58}$$

where $p=p(y,z,t)$ satisfies

$$p_{yy}=-\frac{-2p_zRpp_y^3-p_{zz}p_y^2}{p_z^2},\quad p_{yz}=\frac{p_y^2pp_zR+p_{zz}p_y}{p_z},\quad p_t=0.$$

Case 13

$$g=-2C_1,\quad k=2C_1R,\quad h=0,\quad p=C_1,$$
$$q=F_2(y,z)+F_4(z)+C_2t+C_3,$$
$$f=\int\frac{1}{3C_1^2}\left[16C_1^3R\frac{\partial F_2(y,z)}{\partial y}+3C_1\frac{\partial F_2(y,z)}{\partial z}-2\frac{\partial F_2(y,z)}{\partial y}C_2\right]\mathrm{d}y\tag{3.59}$$
$$+\frac{1}{C_1}\frac{\partial F_4(z)}{\partial z}y+F_5(z,t).$$

Case 14

$$g=-2C_1,\quad k=2C_1R,\quad h=0,\quad p=C_1,$$
$$q=F_3(z)y+F_4(z)+F_5\left[t+\frac{2}{3C_1}\int F_3(z)\,\mathrm{d}z\right],$$
$$f=\frac{1}{2C_1}\frac{\partial F_3(z)}{\partial z}y^2+\frac{16}{3}yC_1RF_3(z)+\frac{1}{C_1}\frac{\partial F_4(z)}{\partial z}y+F_6(z,t). \tag{3.60}$$

Case 15

$$g=-2\sqrt{2c_1z+C_1}\mathrm{e}^{C_2t}C_3,\quad k=2\sqrt{2c_1z+C_1}\mathrm{e}^{C_2t}C_3R,\quad h=0,$$
$$p=\sqrt{2c_1z+C_1}\mathrm{e}^{C_2t}C_3,$$
$$q=\frac{1}{(2c_1z+C_1)^{1/2}C_2}\left\{\sqrt{2c_1z+C_1}F_2\left[t+\frac{1}{2C_2}\ln(2c_1z+C_1)\right]C_2\right.$$
$$-16\mathrm{e}^{3C_2t}\left(c_1z+\frac{1}{2}C_1\right)^2RC_3^3\ln(2c_1z+C_1)$$
$$\left.+\left[\frac{3}{2}yc_1C_3+\sqrt{2c_1z+C_1}F_1(z)C_2\right]\mathrm{e}^{C_2t}\right\}, \tag{3.61}$$
$$f=\frac{1}{C_2(2c_1z+C_1)^{5/2}C_3}\left[4\frac{\partial F_1(z)}{\partial z}C_2c_1^2z^2y-2zc_1^2F_1(z)C_2y\right.$$
$$+4z\frac{\partial F_1(z)}{\partial z}C_2C_1c_1y-\frac{3}{2}y^2C_3\sqrt{2c_1z+C_1}c_1^2$$
$$\left.-F_1(z)C_2C_1c_1y+\frac{\partial F_1(z)}{\partial z}C_1^2C_2y\right]+F_3(z,t).$$

Case 16

$$g=-2p,\quad k=2pR,\quad h=0,\quad q=\int\frac{1}{2}\frac{p(16p^2Rp_y+3p_z)}{p_y}\,\mathrm{d}t+F_1(y,z),$$
$$f=\int-\frac{1}{2pp_y}\left[-3\int\frac{16p_zp_y^2p^2R+p_z^2p_y+pp_yp_{zz}-pp_{yz}p_z}{p_y^2}\,\mathrm{d}tp_y\right.$$
$$-2p_y\frac{\partial F_1(y,z)}{\partial z}+3p_z\int\frac{16p^2Rp_y^3+p_zp_y^2+pp_yp_{yz}-pp_{yy}p_z}{p_y^2}\,\mathrm{d}t$$
$$\left.+2p_z\frac{\partial F_1(y,z)}{\partial y}\right]\mathrm{d}y+F_2(z,t), \tag{3.62}$$

where $p=p(y,z,t)$ satisfies

$$p_{yz}=-\frac{-16p_y^2pp_zR-p_{zz}p_y}{p_z},\quad p_{yy}=-\frac{-32p_zRpp_y^3-p_{zz}p_y^2}{p_z^2},\quad p_t=0.$$

Case 17

$$R=0,\quad h=0,\quad g=0,\quad k=F_2(y,z)+F_1(z,t),\quad p=F_3(z),\quad q=q,$$

$$\begin{aligned} f=&F_5(y,z)+F_4(z,t)+\int\frac{\partial F_1(z,t)}{\partial t}q\,\mathrm{d}t+\int\int q_{ty}F_2(y,z)\,\mathrm{d}y\,\mathrm{d}t\\ &+\int F_1(z,t)q_t\,\mathrm{d}t+\int\int\frac{\partial F_2(y,z)}{\partial y}q_t\,\mathrm{d}y\,\mathrm{d}t-\frac{3}{2}F_3(z)^2F_2(y,z)\int F_1(z,t)\,\mathrm{d}t\\ &+\left\{-\frac{3}{2}F_3(z)\int\frac{\partial F_1(z,t)}{\partial z}\mathrm{d}t-\frac{3}{2}\frac{\partial F_3(z)}{\partial z}\int F_1(z,t)\,\mathrm{d}t\right\}y\\ &+\left\{-\frac{3}{2}F_3(z)\int\frac{\partial F_2(y,z)}{\partial z}\mathrm{d}y-\frac{3}{2}\frac{\partial F_3(z)}{\partial z}\int F_2(y,z)\,\mathrm{d}y-\frac{3}{4}F_3(z)^2F_2(y,z)^2\right\}t. \end{aligned} \tag{3.63}$$

Case 18

$$\begin{aligned} &R=0,\quad h=0,\quad g=0,\quad k=F_1(z,t),\quad p=F_2(z,t),\quad q=q,\\ &f=F_4(y,z)+F_3(z,t)+\int F_1(z,t)q_t\,\mathrm{d}t+\int\frac{\partial F_1(z,t)}{\partial t}q\,\mathrm{d}t\\ &\quad+\left\{-\frac{3}{2}\int\frac{\partial F_2(z,t)}{\partial z}F_1(z,t)\,\mathrm{d}t-\frac{3}{2}\int F_2(z,t)\frac{\partial F_1(z,t)}{\partial z}\,\mathrm{d}t\right\}y. \end{aligned} \tag{3.64}$$

Case 19

$$\begin{aligned} &R=0,\quad h=0,\quad g=C_1,\quad k=F_2(y,z)+F_4(z)+F_3(t),\quad p=-\frac{1}{2}C_1,\\ &f=\int-\frac{1}{24C_1^2}\Bigg[-24C_1^2F_3(t)\frac{\partial F_6(y,z)}{\partial y}-72C_1^2F_2(y,z)\frac{\partial F_6(y,z)}{\partial y}\\ &\qquad-72C_1^2F_4(z)\frac{\partial F_6(y,z)}{\partial y}+64_C2\frac{\partial F_6(y,z)}{\partial y}+9C_1^4F_3(t)t\frac{\partial F_2(y,z)}{\partial y}\\ &\qquad+27C_1^4F_2(y,z)t\frac{\partial F_2(y,z)}{\partial y}+27C_1^4F_4(z)t\frac{\partial F_2(y,z)}{\partial y}-24_C2tC_1^2\frac{\partial F_2(y,z)}{\partial y}\\ &\qquad+48C_1\frac{\partial F_6(y,z)}{\partial z}-18tC_1^3\frac{\partial F_2(y,z)}{\partial z}\Bigg]\mathrm{d}y+\frac{3}{4}C_1t\frac{\partial F_4(z)}{\partial z}y+F_7(z,t),\\ &q=\int\frac{3}{8}C_1^2F_3(t)\,\mathrm{d}t-\frac{3}{8}tC_1^2F_2(y,z)-\frac{3}{8}tC_1^2F_4(z)+tC_2+F_6(y,z). \end{aligned} \tag{3.65}$$

Case 20

$$R = 0, \quad h = 0, \quad p = -\frac{1}{2}\frac{C_1}{F_1(y)}, \quad q = \frac{3C_1^2 t}{4F_1(y)} + F_2(y,z), \quad k = F_1(y),$$
$$f = \int -\frac{2}{C_1}\frac{\partial F_2(y,z)}{\partial z} F_1(y), y + F_3(z,t), \quad g = \frac{C_1}{F_1(y)}. \tag{3.66}$$

It is necessary to exclaim that the Cases 1–16 is directly obtained by solving the (3.3)–(3.46), the Cases 17–20 is obtained under the condition: $R = h = 0$ (here we omitted the trivial solutions and very complicated solutions). From (3.2), (2.4) and Cases 1–20, we can obtain the following five types of solutions for $(3+1)$-dimensional Jumbo–Miwa equation.

Type I: From Cases 1–4, we can obtain the following types of solutions.

$$u_{11} = f - g\sqrt{-R}\tanh\left[\sqrt{-R}(xp+q)\right], \quad R < 0, \tag{3.67}$$

$$u_{12} = f - g\sqrt{-R}\coth\left[\sqrt{-R}(xp+q)\right], \quad R < 0, \tag{3.68}$$

$$u_{13} = f + g\sqrt{R}\tan\left[\sqrt{R}(xp+q)\right], \quad R > 0, \tag{3.69}$$

$$u_{14} = f - g\sqrt{R}\cot\left[\sqrt{R}(xp+q)\right], \quad R > 0, \tag{3.70}$$

where f, g, p, q are determined by (3.47)–(3.50) respectively.

Type II: From Cases 5–12, we can obtain the following types of solutions

$$u_{21} = f - g\sqrt{-R}\left\{\tanh\left[\sqrt{-R}(xp+q)\right] + \mathrm{i}\,\mathrm{sech}\left[\sqrt{-R}(xp+q)\right]\right\}, \quad R < 0, \tag{3.71}$$

$$u_{22} = f - g\sqrt{-R}\left\{\coth\left[\sqrt{-R}(xp+q)\right] + \mathrm{csch}\left[\sqrt{-R}(xp+q)\right]\right\}, \quad R < 0, \tag{3.72}$$

$$u_{23} = f + g\sqrt{R}\left\{\tan\left[\sqrt{R}(xp+q)\right] + \sec\left[\sqrt{R}(xp+q)\right]\right\}, \quad R > 0, \tag{3.73}$$

$$u_{24} = f - g\sqrt{R}\left\{\tan\left[\sqrt{R}(xp+q)\right] + \cot\left[\sqrt{R}(xp+q)\right]\right\}, \quad R > 0, \tag{3.74}$$

where f, g, h, p, q are determined by (3.51)–(3.58) respectively.

Type III: From Cases 12–16, we can obtain the following types of solutions

$$u_{31} = f - g\sqrt{-R}\{\tanh[\sqrt{-R}(xp+q)] + \coth[\sqrt{-R}(xp+q)]\}, \quad R < 0, \tag{3.75}$$

$$u_{32} = f + g\sqrt{R}\{\tan[\sqrt{R}(xp+q)] + \cot[\sqrt{R}(xp+q)]\}, \quad R > 0, \tag{3.76}$$

where f, g, p, q are determined by (3.58)–(3.62) respectively.

Type IV: From Cases 17 and 18, we can obtain the following types of solutions

$$u_4 = f - k(xp+q), \tag{3.77}$$

where f, k, p and q are determined by (3.63) and (3.64) respectively.

Type V: From Cases 19 and 20, the following types of solutions is obtained

$$u_5 = f - \frac{g}{xp+q} - k(xp+q), \tag{3.78}$$

where f, g, k, p, q are determined by (3.65) and (3.66) respectively.

Remarks. When setting the arbitrary functions to be equal to special functions or special constants, the travelling wave solutions for the (3+1)-dimensional Jumbo–Miwa equation can be recovered. For example, if setting $F_4(z) = 0$, $F_5(z,t) = a_0$, $F_2(y,z) = \beta y + \gamma z$, $R = -k^2$, $C_2 = -2C_1^3K^2 + \frac{2C_1\gamma}{2\beta}$ in the solution (3.67) determined by (3.47), the solutions obtained in [20] can be reproduced. But to our knowledge, the other soliton-like solutions, singular soliton-like solutions and periodic form solutions are not found before.

4. Conclusions

In summary, based on the computerized symbolic computation, by introducing a more general ansätz than the ansätz in the extended tanh-function method, modified extended tanh-function method, and generalized hyperbolic-function method, we have proposed a generalized Riccati equation expansion method for searching for exact soliton-like solutions of NEEs and implemented in computer symbolic systems. Making use of our method and with the aid of maple, we study the $(3+1)$-dimensional Jumbo–Miwa equation and obtain new families of the exact solutions. In our obtained exact solutions the restriction on $\xi(x,y,z,t)$ as merely a linear function x, y, z, t and the restriction on the coefficients f, g_i, h_i, k_i $(i = 1,\ldots,m)$ as constants are removed and, with no extra effects, the singular soliton-like solution and periodic form solutions, even rational solutions could be obtained. To make the work feasible, how to choose the forms of f, g_i, h_i, k_i $(i = 1,\ldots,m)$ and ξ in the ansätz would be the key step in the computation of our method. The method, proposed in this paper for single equation, may be extended to find exact soliton-like solutions of coupled equations.

Acknowledgements

The author (B. Li) would express his sincerely thanks to Dr. Z.-s. Lü for his help and guidance in symbolic computation. The work is supported by the National Natural Science Foundation of China under the Grant no. 10072013, the National Key Basic Research Development Project Program under the Grant no. G1998030600.

References

[1] M.J. Ablowitz, P.A. Clarkson, Soliton, Nonlinear Evolution Equations and Inverse Scatting, Cambridge University Press, New York, 1991.
[2] C.H. Gu et al., Soliton Theory ans Its Application, Zhejiang Science and Technology Press, Zhejiang, 1990.
[3] E.J. Parkes, B.R. Duffy, Comput. Phys. Commun. 98 (1996) 288.
[4] E.J. Parkes, B.R. Duffy, Phys. Lett. A 229 (1997) 217.
[5] B.R. Duffy, E.J. Parkes, Phys. Lett. A 214 (1996) 271.
[6] Z.B. Li, M.L. Wang, J. Phys. A: Math. Gen. 26 (1993) 6027.
[7] S. Lou, G. Huang, H. Ruan, J. Phys. A: Math. Gen. 24 (1991) L584.
[8] A.H. Khater, W. Malfiet, D.K. Callebaut, E.S. Kamel, Chaos Soliton. Fract. 14 (2002) 513.
[9] E. Fan, Phys. Lett. A 277 (2000) 212.
[10] E. Fan, Comput. Math. Appl. 43 (2002) 671.
[11] E. Fan, Nuovi Cimento B 116 (2001) 1385.
[12] E. Fan, J. Zhang, Y.C. Hon Benny, Phys. Lett. A 291 (2001) 376.
[13] Z.Y. Yan, Phys. Lett. A 292 (2001) 100.
[14] Y. Chen, B. Li, H.Q. Zhang, Commun. Theor. Phys. 38 (2002) 261.
[15] B. Li, Y. Chen, H.Q. Zhang, J Phys A: Math. Gen. 35 (2002) 8253.
[16] B. Li, Y. Chen, H.Q. Zhang, Chaos Soliton. Fract. 15 (2003) 647.
[17] S.A. Elwakil, S.K. El-labany, M.A. Zahran, R. Sabry, Phys. Lett. A 299 (2002) 179.
[18] Y.T. Gao, B. Tian, Comput. Phys. Commun. 133 (2001) 158.
[19] B. Tian, Y.T. Gao, Z. Naturforsch. A 57 (2002) 39.
[20] M. Senthilvelan, Appl. Math. Comput. 123 (2001) 381.
[21] G. Paquin, P. Winternitz, Physica D 46 (1990) 122.
[22] M. Faucher, P. Winternitz, Phys. Rev. E 48 (1993) 3066.
[23] M. Senthilvelan, M. Winternitz, J. Nonlinear Math. Phys. 5 (1998) 190.
[24] B. Dorrizi, B. Grammaticos, A. Ramani, P. Winternitz, J. Math. Phys. 27 (1986) 2848.