Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Chaos, Solitons and Fractals 22 (2004) 675–682
CHAOS
SOLITONS & FRACTALS
www.elsevier.com/locate/chaos

A generalized method and general form solutions to the Whitham–Broer–Kaup equation

Yong Chen [a,b,c,*], Qi Wang [b,c], Biao Li [b,c]

[a] *Department of Physics, Shanghai Jiao Tong University, Shanghai 200030, China*
[b] *Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China*
[c] *MM Key Lab, Chinese Academy of Sciences, Beijing 100080, China*

Accepted 25 February 2004

Communicated by Prof. M. Wadati

Abstract

Based on a more general transformation presented in this paper, a generalized method for finding more types travelling wave solutions of nonlinear evolution equations (NLEEs) is presented and implemented in a computer algebraic system. As an application of the method, Whitham–Broer–Kaup (WBK) equation is studied to illustrate the method. As a result, We cannot only successfully recover the previously known travelling wave solutions found by Fan's method [J. Phys. A 35 (2002) 6853; Comput. Phys. Commun. 53 (2003) 17], but also obtain some new formal solutions. The solutions obtained in this paper include polynomial, exponential, solitary wave, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions.

1. Introduction

It is well known that the investigation of the travelling wave solutions of nonlinear evolution equations (NLEEs), which describe many important phenomena and dynamical processes in physics, chemistry, biology, etc., plays an important role in the study of soliton theory. In the past several decades, both mathematicians and physicists have made many attempts in this direction. Many effective methods have been presented, such as inverse scattering transform method [1,2], Backlund transformation [3,4], Darboux transformation [5,6], Hirota bilinear method [7,8], the variable separation method [9], Homogeneous balance method [10] and similarity reduction method [11] etc. The tanh method [12] is considered to be one of most straightforward and effective algorithm to obtain solitary wave solutions for a large NLEEs. Much work [13–16] has been concentrated on the various extensions and applications of the tanh method, such as the extended tanh method by Fan [13], improved tanh method by Yan [15], Generalized extended tanh-function method by Chen et al. [16]. The basic purpose of above work is to simplify the routine calculation of the method or obtain more general solutions. In [17], Fan develop a new algebraic method with symbolic computation for obtaining the above-mentioned various travelling wave solutions in a unified way. Compared with most of the existing methods, the proposed method not only gives a unified formulation to construct various travelling wave solutions, but also provides a guideline to classify the various types of the travelling wave solutions according to the values of some parameters. The present work is motivated by the desire to extend the transformation in [17] to more general transformations and use symbolic computation to solve WBK equation [18–20]

* Corresponding author.
E-mail addresses: chenyong@dlut.edu.cn (Y. Chen), libiao@dlut.edu.cn (B. Li).

0960-0779/$ - see front matter © 2004 Elsevier Ltd. All rights reserved.
doi:10.1016/j.chaos.2004.02.024

$$\begin{cases} u_t + uu_x + v_x + \beta u_{xx} = 0, \\ v_t + u_x v + uv_x + \alpha u_{xxx} - \beta, v_{xx} = 0, \end{cases} \tag{1.1}$$

where $\alpha, \beta \neq 0$ are all constants. Under Boussinesq approximation, Whitham [18], Broer [19] and Kaup [20] obtained nonlinear WBK equation. It is not difficult to see that when parameters α and β take different constants, system (1.1) includes many important mathematical and physical equations, such as when $\alpha = 0$, $\beta \neq 0$ system (1.1) becomes classical long wave equation that describe shallow water with dispersive [21], and when $\alpha = 1$, $\beta = 0$ system becomes variant Boussinesq equation [1]. Many mathematicians and physicists have devoted consider effort to the study on WBK equation and make new developments in the regards (see, e.g., [22–24] for detail).

2. Summary of the generalized method

In the following we would like to outline the main steps of our general method:

Step 1. For a given of nonlinear evolution equations (NLEEs) with some physical fields $u_i(x, y, t)$ in three variables x, y, t,

$$F_i(u_i, u_{i,t}, u_{i,x}, u_{i,y}, u_{i,tt}, u_{i,xt}, u_{i,yt}, u_{i,xx}, u_{i,yy}, u_{i,xy}, \ldots) = 0 \quad (i = 1, 2, \ldots, n) \tag{2.1}$$

by using the wave transformation

$$u_i(x, y, t) = u_i(\xi), \quad \xi = k(x + ly - \lambda t), \tag{2.2}$$

where k, l and λ are constants to be determined later. Then the nonlinear partial differential equation (2.1) is reduced to a nonlinear ordinary differential equation (ODE)

$$G_m(u_i, u_i', u_i'', \ldots) = 0. \tag{2.3}$$

Step 2. We introduce a new and more general ansates in the forms:

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \left\{ a_{ij}\phi^j + b_{ij}\phi^{-j} + f_{ij}\phi^{j-1}\sqrt{\sum_{l=0}^{r} c_l\phi^l} + k_{ij}\frac{\sqrt{\sum_{l=0}^{r} c_l\phi^l}}{\phi^j} \right\}, \tag{2.4}$$

where the new variable $\phi = \phi(\xi)$ satisfying

$$\phi' = \frac{d\phi}{d\xi} = \sqrt{\sum_{l=0}^{r} c_l\phi^l}, \tag{2.5}$$

and $a_{i0}, a_{ij}, b_{ij}, f_{ij}, k_{ij}$ $(i = 1, 2, \ldots;\ j = 1, 2, \ldots, m_i)$ are constants to be determined later.

Step 3. The underlying mechanism for a series of fundamental solutions such as polynomial, exponential, solitary wave, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions to occur is that differ effects that act to change wave forms in many nonlinear equations, i.e. dispersion, dissipation and nonlinearity, either separately or various combination are able to balance out. We define the degree of $u_i(\xi)$ as $D[u_i(\xi)] = n_i$, which gives rise to the degrees of other expressions as

$$D[u_i^{(q)}] = n_i + q, \quad D[u_i^p(u_j^{(q)})^s] = n_i p + (q + n_j)s. \tag{2.6}$$

Therefore we can get the value of m_i in Eq. (2.4). If n_i is a nonnegative integer, then we first make the transformation $u_i = \omega_i^{n_i}$.

Step 4. Substitute Eq. (2.4) into Eq. (2.3) along with Eq. (2.5) and then set all coefficients of $\phi^p\left(\sqrt{\sum_{l=0}^{r} c_l\phi^l}\right)^q$ $(q = 0, 1;\ p = 0, 1, 2, \ldots)$ to be zero to get an over-determined system of nonlinear algebraic equations with respect to λ, l, k, a_{i0}, a_{ij}, b_{ij}, f_{ij} and k_{ij} $(i = 1, 2, \ldots;\ j = 1, 2, \ldots, m_i)$.

Step 5. Solving the over-determined system of nonlinear algebraic equations by use of Maple, we would end up with the explicit expressions for λ, l, k, a_{i0}, a_{ij}, b_{ij}, f_{ij} and k_{ij} $(i = 1, 2, \ldots;\ j = 1, 2, \ldots, m_i)$.

Step 6. By using the results obtained in the above step, we can derive a series of fundamental solutions such as polynomial, exponential, solitary wave, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions. Because we interested in solitary wave, Jacobi and Weierstrass doubly periodic solutions. On the other hand, tan and cot type solutions appear in pairs with tanh and coth type solutions respectively, polynomial, rational triangular periodic solutions are omitted in this paper. By considering the different values of c_0, c_1, c_2, c_3 and c_4 (in this paper we only consider the case $l = 4$ in Eqs. (2.4) and (2.5)), Eq. (2.5) has many kinds of solitary wave, Jacobi and Weierstrass doubly periodic solutions which are listed as follows:

(i) Solitary wave solutions

(a) Bell-shaped solitary wave solutions

$$\phi = \sqrt{-\frac{c_2}{c_4}}\operatorname{sech}(\sqrt{c_2}\xi), \quad c_0 = c_1 = c_3 = 0,\ c_2 > 0,\ c_4 < 0, \tag{2.7}$$

$$\phi = -\frac{c_2}{c_3}\operatorname{sech}^2\left(\frac{\sqrt{c_2}}{2}\xi\right), \quad c_0 = c_1 = c_4 = 0,\ c_2 > 0. \tag{2.8}$$

(b) Kink-shaped solitary wave solutions

$$\phi = k\sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right), \quad c_0 = \frac{c_2^2}{4c_4},\ c_1 = c_3 = 0, \quad c_2 < 0,\ c_4 > 0. \tag{2.9}$$

(c) Solitary wave solutions

$$\phi = \frac{c_2\operatorname{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3}, \quad c_0 = c_1 = 0,\ c_2 > 0. \tag{2.10}$$

(ii) Jacobi and Weierstrass doubly periodic solutions

$$\phi = \sqrt{\frac{-c_2m^2}{c_4(2m^2-1)}}\operatorname{cn}\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right), \quad c_4 < 0,\ c_2 > 0,\ c_0 = \frac{c_2^2m^2(1-m^2)}{c_4(2m^2-1)^2}, \tag{2.11}$$

$$\phi = \sqrt{\frac{-m^2}{c_4(2-m^2)}}\operatorname{dn}\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right), \quad c_4 < 0,\ c_2 > 0,\ c_0 = \frac{c_2^2(1-m^2)}{c_4(2-m^2)^2}, \tag{2.12}$$

$$\phi = \sqrt{\frac{-c_2m^2}{c_4(m^2+1)}}\operatorname{sn}\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right), \quad c_4 > 0,\ c_2 < 0,\ c_0 = \frac{c_2^2m^2}{c_4(m^2+1)^2}, \tag{2.13}$$

where m is a modulus.

$$\phi = \wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right), \quad c_2 = 0,\ c_3 > 0, \tag{2.14}$$

where $g_2 = -4\frac{c_1}{c_3}$ and $g_3 = -4\frac{c_0}{c_3}$ are called invariants of Weierstrass elliptic function. The Jacobi elliptic functions are doubly periodic and possess properties of triangular functions:

$$\operatorname{sn}^2\xi + \operatorname{cn}^2\xi = 1, \quad \operatorname{dn}^2\xi = 1 - m^2\operatorname{sn}^2\xi, \tag{2.15}$$
$$(\operatorname{sn}\xi)' = \operatorname{cn}\xi\operatorname{dn}\xi, \quad (\operatorname{cn}\xi)'' = -\operatorname{sn}\xi, \quad (\operatorname{dn}\xi)'' = -m^2\operatorname{sn}\xi\operatorname{cn}\xi.$$

When $m \to 1$, the Jacobi functions degenerate to the hyperbolic functions, i.e.

$$\operatorname{sn}\xi \to \tanh\xi, \quad \operatorname{cn}\xi \to \operatorname{sech}\xi. \tag{2.16}$$

When $m \to 0$, the Jacobi functions degenerate to the triangular functions, i.e.

$$\operatorname{sn}\xi \to \sin\xi, \quad \operatorname{cn}\xi \to \cos\xi. \tag{2.17}$$

The more detailed notations for the Weierstrass and Jacobi elliptic functions can be found in [25–27].

Remark 1. Compared with the method proposed by Fan [18], our ansates is more general than the ansates in [18]. When $b_{ij} = f_{ij} = k_{ij} = 0$ in Eq. (2.4), Eq. (2.4) becomes the ansates proposed by Fan.

Remark 2. The method can be extend to find soliton-like solutions and more types double periodic solutions of PDE Eq. (2.1). Only will the restriction on $\xi(x, y, t)$ as merely a linear function x, y, t and the restrictions on the coefficients a_{i0}, a_{ij}, b_{ij}, f_{ij}, k_{ij} and c_i as constants be remove.

3. Exact solutions of WBK equation

According to the above method, to seek the solutions of WBK,

$$\begin{cases} u_t + uu_x + v_x + \beta u_{xx} = 0, \\ v_t + u_x v + uv_x + \alpha u_{xxx} - \beta v_{xx} = 0, \end{cases} \tag{3.1}$$

we make the following transformation:

$$u(x,t) = \sigma(\xi), \quad v(x,t) = \tau(\xi), \quad \xi = x - \lambda t, \tag{3.2}$$

where λ is constant to be determined later, and thus Eq. (3.1) becomes

$$\begin{cases} -\lambda\sigma' + \sigma\sigma' + \tau' + \beta\sigma'' = 0, \\ -\lambda\tau' + \sigma\tau' + \sigma'\tau + \alpha\sigma''' - \beta\tau'' = 0. \end{cases} \tag{3.3}$$

According to *Step 1* in Section 2, if $a \neq 0$ and $\beta \neq 0$, by balancing $\sigma''(\xi)$ and $\sigma(\xi)\sigma'(\xi)$ in Eq. (3.3) and balancing $\tau''(\xi)$ and $\sigma'(\xi)\tau(\xi)$ in Eq. (3.3), we suppose that Eq. (3.3) has the following formal solutions:

$$\begin{cases} \sigma = a_0 + a_1\phi + \frac{b_1}{\phi} + f_1\sqrt{\sum_{l=0}^4 c_i\phi^i} + k_1\frac{\sqrt{\sum_{l=0}^4 c_i\phi^i}}{\phi}, \\ \tau = A_0 + A_1\phi + \frac{B_1}{\phi} + F_1\sqrt{\sum_{l=0}^4 c_i\phi^i} + K_1\frac{\sqrt{\sum_{l=0}^4 c_i\phi^i}}{\phi} + A_2\phi^2 + \frac{B_2}{\phi^2} + F_2\phi\sqrt{\sum_{l=0}^4 c_i\phi^i} + K_2\frac{\sqrt{\sum_{l=0}^4 c_i\phi^i}}{\phi^2}, \end{cases} \tag{3.4}$$

where $\phi(\xi)$ satisfies (2.5), where $a_0, a_1, b_1, f_1, k_1, A_0, A_1, B_1, F_1, K_1, A_2, B_2, F_2$ and K_2 are constants to be determined later.

With the aid of *Maple*, substituting (3.4) along with (2.5) into (3.3), yields a set of algebraic equations for $\phi^i(\xi)\left(\sqrt{\sum_{l=0}^4 c_l\phi^l}\right)^j$ $(i = 0, 1, \ldots;\ j = 0, 1)$. Setting the coefficients of these terms $\phi^i(\xi)\left(\sqrt{\sum_{l=0}^4 c_l\phi^l}\right)^j$ to zero yields a set of over-determined algebraic equations with respect to a_0, a_1, b_1, f_1, k_1, A_0, A_1, B_1, F_1, K_1, A_2, B_2, F_2, K_2 and λ.

By use of the *Maple*, solving the over-determined algebraic equations, then we get the following results:

Case 1

$$A_2 = c_4\sqrt{\beta^2+\alpha}\left(-2\sqrt{\beta^2+\alpha} \pm 2\beta\right), \quad k_1 = \pm 2\sqrt{\beta^2+\alpha}, \quad \lambda = a_0,$$
$$c_0 = c_1 = c_3 = a_1 = b_1 = f_1 = A_0 = A_1 = B_1 = B_2 = F_1 = F_2 = K_1 = K_2 = 0. \tag{3.5}$$

Case 2

$$B_1 = \frac{1}{2}\sqrt{\beta^2+\alpha}c_1\left(-\sqrt{\beta^2+\alpha} \pm \beta\right), \quad \lambda = a_0, \quad A_1 = \frac{1}{2}\sqrt{\beta^2+\alpha}c_3\left(-\sqrt{\beta^2+\alpha} \pm \beta\right),$$
$$c_4 = \frac{a_1^2}{\beta^2+\alpha}, \quad F_1 = -a_1\left(\pm\sqrt{\beta^2+\alpha} + \beta\right), \quad A_2 = -\frac{a_1^2\left(\beta^2+\alpha \pm \beta\sqrt{\beta^2+\alpha}\right)}{\beta^2+\alpha},$$
$$k_1 = \pm\sqrt{\beta^2+\alpha}, \quad c_0 = b_1 = f_1 = A_0 = B_2 = F_2 = K_1 = K_2 = 0. \tag{3.6}$$

Case 3

$$K_2 = -b_1\left(\pm\sqrt{\beta^2+\alpha} - \beta\right), \quad \lambda = a_0, \quad k_1 = \pm\sqrt{\beta^2+\alpha}, \quad c_0 = \frac{b_1^2}{\beta^2+\alpha},$$
$$B_1 = \frac{1}{2}\sqrt{\beta^2+\alpha}c_1\left(-\sqrt{\beta^2+\alpha} \pm \beta\right), \quad A_1 = \frac{1}{2}\sqrt{\beta^2+\alpha}c_3\left(-\sqrt{\beta^2+\alpha} \pm \beta\right),$$
$$B_2 = -\frac{b_1^2\left(\beta^2+\alpha \pm \beta\sqrt{\beta^2+\alpha}\right)}{\beta^2+\alpha}, c_4 = a_1 = f_1 = A_0 = A_2 = F_1 = F_2 = K_1 = 0. \tag{3.7}$$

Case 4

$$A_2 = c_4\sqrt{\beta^2+\alpha}\left(-2\sqrt{\beta^2+\alpha} \pm 2\beta\right), \quad k_1 = \pm 2\sqrt{\beta^2+\alpha},$$
$$B_2 = \sqrt{\beta^2+\alpha}c_0\left(-2\sqrt{\beta^2+\alpha} \pm 2\beta\right), \quad \lambda = a_0,$$
$$c_1 = c_3 = a_1 = b_1 = f_1 = A_0 = A_1 = B_1 = F_1 = F_2 = K_1 = K_2 = 0. \tag{3.8}$$

Case 5

$$\lambda = -\frac{-a_0 b_1 + \beta^2 c_1 + \alpha c_1}{b_1}, \quad A_0 = \frac{-\beta^2 c_2 b_1^2 + \beta^4 c_1^2 + 2\beta^2 c_1^2 \alpha + \alpha^2 c_1^2 - \alpha c_2 b_1^2}{b_1^2},$$
$$K_2 = \beta b_1, \quad B_1 = -\beta^2 c_1 - \alpha c_1, \quad c_0 = \frac{1}{4}\frac{b_1^2}{\beta^2+\alpha}, \quad B_2 = -\frac{1}{2}b_1^2,$$
$$a_1 = f_1 = k_1 = A_1 = A_2 = F_1 = F_2 = K_1 = 0. \tag{3.9}$$

From (3.2), (3.5) and Cases 1–5, we obtain the following solutions for Eqs. (3.1):

Family 1. From Eqs. (3.5), we obtain the following solutions for the WBK equations, as follows:

$$u_1 = a_0 \pm 2\sqrt{\beta^2+\alpha}\sqrt{c_2 - c_2\,\mathrm{sech}^2(\sqrt{c_2}\xi)}, \tag{3.10}$$
$$v_1 = -\sqrt{\beta^2+\alpha}\left(-2\sqrt{\beta^2+\alpha} \pm 2\beta\right)c_2\,\mathrm{sech}^2(\sqrt{c_2}\xi), \tag{3.11}$$

where $\xi = x - \lambda t$, $\lambda = a_0$, $c_2 > 0$ and a_0 are arbitrary constants.

Family 2. From Eqs. (3.6), when $c_1 = 0$, then we obtain the following solutions for the WBK equations, as follows:

$$\begin{aligned} u_2 = {} & a_0 + a_1\frac{c_2\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2c_4}\,\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3} \\ & \pm k_1\sqrt{c_2 + c_3\frac{c_2\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2c_4}\,\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3} + c_4\left(\frac{c_2\,\mathrm{sech}^2(\frac{1}{2}\sqrt{c_2}\xi)}{2\sqrt{c_2c_4}\,\tanh(\frac{1}{2}\sqrt{c_2}\xi) - c_3}\right)^2}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} v_2 = {} & \frac{A_1c_2\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3} + A_2\left(\frac{c_2\,\mathrm{sech}^2(\frac{1}{2}\sqrt{c_2}\xi)}{2\sqrt{c_2c_4}\,\tanh(\frac{1}{2}\sqrt{c_2}\xi) - c_3}\right)^2 \\ & \pm\left(\frac{F_1c_2\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2c_4}\,\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3}\right)\sqrt{c_2 + \frac{c_3c_2\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\sqrt{c_2c_4}\,\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3} + c_4\left(\frac{c_2\,\mathrm{sech}^2(\frac{1}{2}\sqrt{c_2}\xi)}{2\sqrt{c_2c_4}\,\tanh(\frac{1}{2}\sqrt{c_2}\xi) - c_3}\right)^2}, \end{aligned} \tag{3.13}$$

where $\xi = x - \lambda t$, $\lambda = a_0$, $A_1 = \frac{1}{2}\sqrt{\beta^2+\alpha}c_3\left(-\sqrt{\beta^2+\alpha} \pm \beta\right)$, $F_1 = -a_1\left(\pm\sqrt{\beta^2+\alpha} + \beta\right)$, $A_2 = -\frac{a_1^2\left(\beta^2+\alpha\pm\beta\sqrt{\beta^2+\alpha}\right)}{\beta^2+\alpha}$, $k_1 = \pm\sqrt{\beta^2+\alpha}$, $c_4 = \frac{a_1^2}{\beta^2+\alpha}$, $c_2 > 0$, a_0, a_1 and c_3 are arbitrary constants.

Family 3. From Eqs. (3.7), when $c_2 = 0$, then we can obtain the following solutions for the WBK equations, as follows:

$$u_3 = a_0 + \frac{b_1}{\wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)} + k_1\frac{\sqrt{c_0 + c_1\wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right) + c_3\wp^3\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)}}{\wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)}, \tag{3.14}$$

$$v_3 = A_1\wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right) + \frac{B_1}{\wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)} + \frac{B_2}{\wp^2\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)} + k_2\frac{\sqrt{c_0 + c_1\wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right) + c_3\wp^3\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)}}{\wp^2\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right)}, \tag{3.15}$$

where $\xi = x - \lambda t$, $\lambda = a_0$, $g_2 = -4\frac{c_1}{c_3}$, $g_3 = -4\frac{c_0}{c_3}$, $K_2 = -b_1\left(\pm\sqrt{\beta^2+\alpha} - \beta\right)$, $k_1 = \pm\sqrt{\beta^2+\alpha}$, $B_1 = \frac{1}{2}\sqrt{\beta^2+\alpha}c_1 \times \left(-\sqrt{\beta^2+\alpha} \pm \beta\right)$, $A_1 = \frac{1}{2}\sqrt{\beta^2+\alpha}c_3\left(-\sqrt{\beta^2+\alpha} \pm \beta\right)$, $B_2 = -\frac{b_1^2\left(\beta^2+\alpha\pm\beta\sqrt{\beta^2+\alpha}\right)}{\beta^2+\alpha}$, $c_0 = \frac{b_1^2}{\beta^2+\alpha}$, $c_3 > 0$, b_1, c_1 and a_0 are arbitrary constants.

Family 4. From Eqs. (3.8), we can obtain the following solutions for the WBK equations, as follows:

$$u_4 = a_0 + k_1\frac{\sqrt{c_0 - \frac{c_2^2m^2\,\mathrm{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)}{c_4(2m^2-1)} + \frac{c_2^2m^4\,\mathrm{cn}^4\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)}{c_4(2m^2-1)^2}}}{\sqrt{\frac{-c_2m^2}{c_4(2m^2-1)}}\mathrm{cn}\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)}, \tag{3.16}$$

$$v_4 = -\frac{A_2c_2m^2}{c_4(2m^2-1)}\mathrm{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right) - B_2\frac{c_4(2m^2-1)}{c_2m^2}nc^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right), \tag{3.17}$$

where $\xi = x - \lambda t$, $\lambda = a_0$, $k_1 = \pm 2\sqrt{\beta^2+\alpha}$, $A_2 = c_4\sqrt{\beta^2+\alpha}\left(-2\sqrt{\beta^2+\alpha} \pm 2\beta\right)$, $B_2 = \sqrt{\beta^2+\alpha}c_0\left(-2\sqrt{\beta^2+\alpha} \pm 2\beta\right)$, $c_0 = \frac{c_2^2m^2(1-m^2)}{c_4(2m^2-1)^2}$, $c_2 > 0$, c_4 and a_0 are arbitrary constants.

Family 5. From Eqs. (3.8), we can obtain the following solutions for the WBK equations, as follows:

$$u_5 = a_0 + k_1\frac{\sqrt{c_0 - \frac{c_2m^2\,\mathrm{dn}^2\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}{c_4(2-m^2)} + \frac{m^4\,\mathrm{dn}^4\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}{c_4(2-m^2)^2}}}{\sqrt{\frac{-m^2}{c_4(2-m^2)}}\mathrm{dn}\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}, \tag{3.18}$$

$$v_5 = -\frac{A_2m^2}{c_4(2-m^2)}\mathrm{dn}^2\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right) - B_2\frac{c_4(2-m^2)}{m^2}n\mathrm{d}^2\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right), \tag{3.19}$$

where $\xi = x - \lambda t$, $\lambda = a_0$, $k_1 = \pm 2\sqrt{\beta^2+\alpha}$, $A_2 = c_4\sqrt{\beta^2+\alpha}\left(-2\sqrt{\beta^2+\alpha} \pm 2\beta\right)$, $B_2 = \sqrt{\beta^2+\alpha}c_0\left(-2\sqrt{\beta^2+\alpha} \pm 2\beta\right)$, $c_0 = \frac{c_2^2(1-m^2)}{c_4(2-m^2)^2}$, $c_2 > 0$, c_4 and a_0 are arbitrary constants.

Family 6. From Eqs. (3.8), we can obtain the following solutions for the WBK equations, as follows:

$$u_6 = a_0 + k_1\frac{\sqrt{c_0 - \frac{c_2^2m^2\,\mathrm{sn}^2\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right)}{c_4(m^2+1)} + \frac{c_2^2m^4\,\mathrm{sn}^4\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right)}{c_4(m^2+1)^2}}}{\sqrt{\frac{-c_2m^2}{c_4(m^2+1)}}\mathrm{sn}\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right)}, \tag{3.20}$$

$$v_6 = -A_2\frac{c_2m^2}{c_4(m^2+1)}\mathrm{sn}^2\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right) - B_2\frac{c_4(m^2+1)}{c_2m^2}ns^2\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right), \tag{3.21}$$

where $\xi = x - \lambda t$, $\lambda = a_0$, $k_1 = \pm 2\sqrt{\beta^2+\alpha}$, $A_2 = c_4\sqrt{\beta^2+\alpha}\left(-2\sqrt{\beta^2+\alpha} \pm 2\beta\right)$, $B_2 = \sqrt{\beta^2+\alpha}c_0\left(-2\sqrt{\beta^2+\alpha} \pm 2\beta\right)$, $c_0 = \frac{c_2^2m^2}{c_4(m^2+1)^2}$, $c_2 < 0$, c_4 and a_0 are arbitrary constants.

Family 7. From Eqs. (3.9), we can obtain the following solutions for the WBK equations, as follows:

$$u_7 = a_0 + b_1\sqrt{\frac{c_4(2m^2-1)}{-c_2m^2}}nc\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right), \tag{3.22}$$

$$v_7 = A_0 - B_2\frac{c_4(2m^2-1)}{c_2m^2}nc^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right) - K_2\frac{\sqrt{c_0 - \frac{c_2^2m^2\,\mathrm{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)}{c_4(2m^2-1)} + \frac{c_2^2m^4\,\mathrm{cn}^4\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)}{c_4(2m^2-1)^2}}}{\frac{c_2m^2}{c_4(2m^2-1)}\,\mathrm{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)}, \tag{3.23}$$

where $\xi = x - \lambda t$, $\lambda = a_0$, $A_0 = -(\beta+\alpha)c_2$, $K_2 = \beta b_1$, $B_2 = -\frac{1}{2}b_1^2$, $c_0 = \frac{1}{4}\frac{b_1^2}{\beta^2+\alpha}$, $c_4 = \frac{c_2^2m^2(1-m^2)}{c_0(2m^2-1)^2}$, $c_2 > 0$, b_1 and a_0 are arbitrary constants.

Family 8. From Eqs. (3.9), we can obtain the following solutions for the WBK equations, as follows:

$$u_8 = a_0 + b_1\sqrt{\frac{c_4(2-m^2)}{-m^2}}nd\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right), \tag{3.24}$$

$$v_8 = A_0 - B_2\frac{c_4(2-m^2)}{m^2}nd^2\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right) - K_2\frac{\sqrt{c_0 - \frac{c_2m^2\,\mathrm{dn}^2\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}{c_4(2-m^2)} + \frac{m^4\,\mathrm{dn}^4\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}{c_4(2-m^2)^2}}}{\frac{m^2}{c_4(2-m^2)}\,\mathrm{dn}^2\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}, \tag{3.25}$$

where $\xi = x - \lambda t$, $\lambda = a_0$, $A_0 = -(\beta+\alpha)c_2$, $K_2 = \beta b_1$, $B_2 = -\frac{1}{2}b_1^2$, $c_0 = \frac{1}{4}\frac{b_1^2}{\beta^2+\alpha}$, $c_4 = \frac{c_2^2(1-m^2)}{c_0(2-m^2)^2}$, $c_2 > 0$, b_1 and a_0 are arbitrary constants.

Family 9. From Eqs. (3.9), we can obtain the following solutions for the WBK equations, as follows:

$$u_9 = a_0 + b_1\sqrt{\frac{c_4(m^2+1)}{-c_2m^2}}ns\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right), \tag{3.26}$$

$$v_8 = A_0 - B_2\frac{c_2m^2}{c_4(m^2+1)}ns^2\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right) - K_2\frac{\sqrt{c_0 - \frac{c_2^2m^2\,\mathrm{sn}^2\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right)}{c_4(m^2+1)} + \frac{c_2^2m^4\,\mathrm{sn}^4\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right)}{c_4(m^2+1)^2}}}{\frac{c_2m^2}{c_4(m^2+1)}\,\mathrm{sn}^2\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right)}, \tag{3.27}$$

where $\xi = x - \lambda t$, $\lambda = a_0$, $A_0 = -(\beta+\alpha)c_2$, $K_2 = \beta b_1$, $B_2 = -\frac{1}{2}b_1^2$, $c_0 = \frac{1}{4}\frac{b_1^2}{\beta^2+\alpha}$, $c_4 = \frac{c_2^2m^2}{c_0(m^2+1)^2}$, $c_2 < 0$, b_1 and a_0 are arbitrary constants.

Remark 1. Up to now we have obtained some new forms solutions which cannot be obtained by Fan's method, such as the solutions of Families 2–9. When $b_1 = f_1 = k_1 = B_1 = B_2 = F_1 = F_2 = K_1 = K_2 = 0$, the form of the solutions of Families 2–9 become the form which can be obtained by Fan's method. This further shows our method is more general than Fan's method.

Remark 2. The some kinds of solutions derived by the generalized transformation are singular soliton solution and Jacobi elliptic doubly periodic wave solution. There is much current interest in the formation of so-called "hot spots" or "blow up" of solutions. It appears that these singular solutions will model this physical phenomena.

4. Summary and conclusions

In summary, based on a more general ansatz than one presented by Fan [18], we have presented the generalized method used to find more formal solutions of nonlinear evolution equations. To illustrate the method, the WBK equation are studied by the method and more types solutions are obtained, which include polynomial, exponential, solitary wave, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions of the WBK equation. The method can easily be extended to other NLEEs and is sufficient to seek more new formal solution of NLEEs.

Acknowledgements

We would express our sincere thanks to Prof. Wadati for his enthusiastic help and valuable suggestions. The work is supported by the Outstanding Youth Foundation (No. 19925522).

References

[1] Ablowitz MJ, Clarkson PA. Soliton, nonlinear evolution equations and inverse scatting. New York: Cambridge University Press; 1991.
[2] Wadati M. J Phys Soc Jpn 1983;53:2642–8;
Wadati M, Konno K, Ichikawa YH. J Phys Soc Jpn 1979;46:1965–6.
[3] Wadati M. J Phys Soc Jpn 1975;38:673–80;
Wadati M. J Phys Soc Jpn 1975;38:681–6;
Wadati M, Sanuki H, Konno K. Prog Theor Phys 1975;53:419;
Konno K, Wadati M. Prog Theor Phys 1975;53:1652.
[4] Chen Y, Li B, Zhang HQ. Chaos, Solitons & Fractals 2003;17:693–8;
Chen Y, Yan ZY, Zhang HQ. Theor Math Phys 2002;132(1):970–5;
Chen Y, Li B, Zhang HQ. Commun Theor Phys (Beijing) 2003;39:135–40;
Li B, Chen Y, Zhang HQ. Phys Lett A 2002;305:377–82.
[5] Matveev VB, Salle MA. Darbooux transformation and soliton. Berlin: Springer; 1991.
[6] Dubrousky VG, Konopelchenko BG. J Phys A 1994;27:4619.
[7] Hirta R, Satsuma J. Phys Lett A 1981;85:407.
[8] Tam HW, Ma WX, Hu XB, Wang DL. J Phys Soc Jpn 2000;69:45.
[9] Lou SY, Lu JZ. J Phys A 1996;29:4029;
Lou SY. Phys Lett A 2000;277:94;
Lou SY, Ruan HY. J Phys A 2001;34:305;
Tang XY, Lou SY, Zhang Y. Phys Rev E 2002;66:046601.
[10] Wang ML. Phys Lett A 1996;216:67;
Wang ML, Wang YM, Zhou YB. Phys Lett A 2002;303:45.
[11] Lou SY, Tang XY, Lin J. J Math Phys 2000;41:8286.
[12] Parkes EJ, Duffy BR. Comput Phys Commun 1996;98:288;
Parkes EJ, Duffy BR. Phys Lett A 1997;229:217.
[13] Fan E. Phys Lett A 2000;277:212–8;
Fan E. Phys Lett A 2002;294:26–30;
Fan E. Phys Lett A 2001;285:373–6.
[14] Elwakil SA, El-labany SK, Zahran MA, Sabry R. Phys Lett A 2002;299:179–88.
[15] Yan ZY. Phys Lett A 2001;292:100;
Yan ZY, Zhang HQ. Appl Math Mech 2000;21:382;
Chen Y, Yan ZY. Phys Lett A 2003;307:107.
[16] Chen Y, Yan ZY, Li B, Zhang HQ. Common Theor Phys 2002;38:261;
Chen Y, Li B, Zhang HQ. J Mod Phys C 2003;14:99;
Chen Y, Li B, Zhang HQ. Chaos, Soliton & Fractals 2003;307:107;
Chen Y, Li B. Chaos, Soliton & Fractals 2004;19:977.
[17] Fan E. Comput Phys Commun 2003;153:17–30;
Fan E. Chaos, Solitons & Fractals 2003;15:559C566;
Fan E. Chaos, Solitons & Fractals 2003;16:819C839;
Fan E. Chaos, Solitons & Fractals 2004;19:1141C1146;
Fan E. Phys Lett A 2002;300:243C249.
[18] Whitham GB. Proc R Soc A 1967;299:6.
[19] Broer LJ. Appl Sci Res 1975;31:377.
[20] Kaup DJ. Prog Theor Phys 1975;54:369.
[21] Kupershmidt BA. Commun Math Phys 1985;99:51.
[22] Yan ZY, Zhang HQ. Phys Lett A 2001;285:355C362.
[23] Xie FD, Yan ZY, Zhang HQ. Phys Lett A 2001;285:76C80.
[24] Chen Y, Zheng Y. Int J Mod Phys C 2003;14:601–12.
[25] Chandrasekharan K. Elliptic function. Berlin: Springer; 1978.
[26] Patrick Du Val. Elliptic function and elliptic curves. Cambridge: Cambridge University Press; 1973.
[27] Wang ZX, Xia XJ. Special functions. Singapore: World Scientific; 1989.