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PDEBellII: A Maple package for finding bilinear forms, bilinear Bäcklund transformations, Lax pairs and conservation laws of the KdV-type equations☆

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A B S T R A C T

Based on the Bell polynomials scheme, this paper presents a Maple computer algebra program *PDEBellII* which can automatically construct the bilinear forms, bilinear Bäcklund transformations, Lax pairs and conservation laws of the KdV-type soliton equations. Some examples are given to verify the validity of our program.

Program summary

Program title: PDEBellII

Catalogue identifier: AEQP_v1_0

Program summary URL: http://cpc.cs.qub.ac.uk/summaries/AEQP_v1_0.html

Program obtainable from: CPC Program Library, Queen's University, Belfast, N. Ireland

Licensing provisions: Standard CPC licence, http://cpc.cs.qub.ac.uk/licence/licence.html

No. of lines in distributed program, including test data, etc.: 2170

No. of bytes in distributed program, including test data, etc.: 43 827

Distribution format: tar.gz

Programming language: Maple internal language.

Computer: PCs, Dell OptiPlex 390.

Operating system: Windows XP and Windows 7.

RAM: Depends on the complexity of the problem (MB)

Classification: 4.3, 5.

Nature of problem:
Determination of integrability of the nonlinear evolution equations, including bilinear forms, bilinear Bäcklund transformations, Lax pairs and conservation laws.

Solution method:
The package PDEBellII is developed by using the Bell polynomials which is linked with Hirota operators.

Restrictions:
The program can only handle single nonlinear evolution equations.

☆ This paper and its associated computer program are available via the Computer Physics Communication homepage on ScienceDirect (http://www.sciencedirect.com/science/journal/00104655).

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Unusual features:
The program PDEBellII is the first Maple automatic program to construct bilinear Bäcklund transformations, Lax pairs and the infinite conservation laws for nonlinear evolution equations.

Running time:
Depends strongly on the complexity of the equation. Different tasks take from 0.4 to 20 s.

1. Introduction

Solitons are among the most exciting features of nonlinear dynamics: they correspond to nonlinear wave solutions with particle-like interaction properties [1]. Investigation of integrability for a soliton equation can be regarded as a pretest and the first step of its exact solvability. Among the direct algebraic methods employed to study the integrability of soliton equations, the Hirota method has been proved particularly powerful [2–7]. Once a given soliton equation is written in bilinear form, on one hand, such results as multi-soliton solutions, quasi periodic wave solutions and other exact solutions are usually obtained, and on the other hand, the integrable properties of the soliton equation, such as the bilinear Bäcklund transformation (BT) and Lax pair can also be investigated. However, the construction of bilinear form and bilinear BT of the original soliton equation is not as one would wish. It relies on a particular skill in using appropriate dependent variable transformation, exchange formulas and bilinear identities.

Recently, F. Lambert et al. have proposed an alternative procedure based on the use of Bell polynomials to obtain bilinear forms, bilinear BTs, Lax pairs and Darboux covariant Lax pairs for soliton equations in a lucid and systematic way [1,8–12]. Fan developed this method to find infinite conservation laws of soliton equations [13–15] and proposed the super Bell polynomials [16,17]. Ma systematically analyzed the connection between Bell polynomials and new bilinear equations [18].

In addition, the characteristic of direct algebraic methods, e.g., the Hirota method [2–7] and various function expansion methods [19–25], enables us to implement corresponding algorithms with any symbolic manipulation programs, such as Maple, Reduce and Macsyma. For instance, different packages in computer algebra systems exist implementing the Hirota method: J. Hietarinta designed a program for searching for integrable bilinear equations which include the KdV-type, mKdV-type, SG-type and NLS-type equations [26–30]. Li et al. presented two Maple programs *Bilinearization* and *Multisoliton* to automatically calculate bilinear equations for soliton equations and to compute their multi-soliton solutions for $N = 1, 2, 3$, respectively [31]. Yang et al. presented some Maple programs to construct the bilinear forms for soliton equations by using the logarithmic transformation [32–34].

To the best knowledge of the authors, however, there have been no programs to derive the bilinear BTs, Lax pairs and infinite conservation laws of soliton equations. One of our two authors, Chen and Yang [35], developed a Maple program *PDEBell* to construct bilinear forms of soliton equations based on the use of Bell polynomials. For the Bell polynomials approach, on the one hand, the transformation between Hirota representation and soliton equation can be directly derived through the derivatives of dimensionless variables, and on the other hand, the bilinear forms can be directly obtained by the Bell polynomials ($\mathscr{Y}$-polynomials or P-polynomials) expression which is linked with the Hirota D-operators. Thus, the program *PDEBell* is very efficient to find bilinear forms of KdV-type soliton equations. However, the program *PDEBell* is only appropriate for the soliton equations which are invariant under the scalar transformations. The aim of this paper is to overcome this disadvantage, and find the bilinear transformations by applying the homogeneous balance method [36]. Furthermore, we design a systematic algorithm to construct the bilinear BTs, Lax pairs and infinite conservation laws based on the use of binary Bell polynomials.

The structure of this paper is as follows. In Section 2, we briefly present necessary notations on Bell polynomials that will be used in this paper. In Section 3, taking the typical Korteweg–de Vries equation as an example to introduce the procedure of the Bell polynomials approach. In Sections 4 and 5, a systematic computational algorithm is presented to construct the bilinear forms, bilinear BTs, Lax pairs and conservation laws of soliton solutions based on the use of Bell polynomials. Based on the algorithm, a Maple program *PDEBellII* is outlined, several different types of examples are investigated to illustrate and verify the effectiveness of our program *PDEBEllII*. Section 6 will be our conclusions. Finally, some introductions of global parameters for program *PDEBellII* are given in the Appendix.

2. Bell polynomials

With the assumption that $f = f(x_1, x_2, \ldots, x_l)$ is a $\mathbb{C}^\infty$ function with multi-variables, the following polynomials

$$Y_{n_1x_1,\ldots,n_lx_l}(f) \equiv Y_{n_1,\ldots,n_l}(\{f_{r_1x_1,\ldots,r_lx_l}(1 \le r_i \le n_i, 0 \le i \le l)\}) = e^{-f}\partial_{x_1}^{n_1}\cdots\partial_{x_l}^{n_l}e^f, \tag{2.1}$$

are the multi-dimensional Bell polynomials, in which we denote that

$$f_{r_1x_1,\ldots,r_lx_l} = \partial_{x_1}^{r_1}\cdots\partial_{x_l}^{r_l}f, \qquad f_{0x_i} \equiv f. \tag{2.2}$$

The multi-dimensional binary Bell polynomials take the following forms

$$\mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(v,w) \equiv Y_{n_1x_1,\ldots,n_lx_l}(f) \equiv Y_{n_1,\ldots,n_l}(\{f_{r_1x_1,\ldots,r_lx_l}\})\Bigg|_{f_{r_1x_1,\ldots,r_lx_l}=\begin{cases} v_{r_1x_1,\ldots,r_lx_l}, & \sum_{i=1}^l r_i \text{ is odd},\\ w_{r_1x_1,\ldots,r_lx_l}, & \sum_{i=1}^l r_i \text{ is even},\end{cases}} \tag{2.3}$$

where the vertical line means that the elements on the left-hand side are chosen according to the rule on the right-hand side, v and w are both the $\mathbb{C}^\infty$ functions of $(x_1, x_2, \ldots, x_l)$.

Proposition 2.1. *Under the mixing variables*

$$v = \ln F/G, \qquad w = \ln FG, \tag{2.4}$$

the relations between the binary Bell polynomials and the Hirota D-operators can be given by the identity

$$\mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(v = \ln F/G, w = \ln FG) = (FG)^{-1}D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}F\cdot G, \quad \sum_{i=1}^{l} n_i \ge 1, \tag{2.5}$$

where the Hirota D-operators are defined by

$$D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}F\cdot G=(\partial_{x_1}-\partial_{x_1'})^{n_1}\cdots(\partial_{x_l}-\partial_{x_l'})^{n_l}\times F(x_1,\ldots,x_l)G(x_1',\ldots,x_l')\Big|_{x_1'=x_1,\ldots,x_l'=x_l}. \quad (2.6)$$

In particular, if $F=G$*, the formula* (2.5) *can be rewritten as*

$$F^{-2}D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}F\cdot F=\mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(0,q=2\ln F)=\begin{cases}0, & \sum_{i=1}^{l}n_i \text{ is odd},\\ P_{n_1x_1,\ldots,n_lx_l}(q), & \sum_{i=1}^{l}n_i \text{ is even},\end{cases} \quad (2.7)$$

where

$$P_{n_1x_1,\ldots,n_lx_l}(q)=\mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(0,q=2\ln F), \quad (2.8)$$

is called P*-polynomials.*

If a given soliton equation can be written as a P-polynomials expression

$$E(q)=\sum_i c_iP_{n_1x_1,\ldots,n_lx_l}=0,\quad q=2\ln F, \quad (2.9)$$

on account of which, suppose that $q'=2\ln G$ is another solution of (2.9), then

$$q'=w+v,\qquad q=w-v. \quad (2.10)$$

Proposition 2.2. *Given a soliton equation for a primary field* q *of form* (2.9)*, one can find a pair of constraint conditions*

$$\sum_j c_{1j}\mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(v,w)=0, \quad (2.11a)$$

$$\sum_j c_{2j}\mathscr{Y}_{m_1x_1,\ldots,m_lx_l}(v,w)=0, \quad (2.11b)$$

which satisfy

$$C(q',q)=E(q')-E(q)\equiv E(w+v)-E(w-v)=0, \quad (2.12)$$

then system (2.11) *is called* $\mathscr{Y}$*-polynomials Bäcklund transformation.*

Moreover, the $\mathscr{Y}$-polynomials expression (2.3) is related to the Lax pair by use of the Hopf–Cole transformation through the following identity

$$\begin{aligned}&\mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(v,w)\Big|_{v=\ln F/G,w=\ln FG}\\ &=(FG)^{-1}D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}F\cdot G\Big|_{G=\exp(q/2),F/G=\psi}\\ &=\psi^{-1}\sum_{p_1=0}^{n_1}\cdots\sum_{p_l=0}^{n_l}\binom{n_1}{p_1}\cdots\binom{n_l}{p_l}\\ &\quad\times P_{n_1x_1,\ldots,n_lx_l}(q)\psi_{(n_1-r_1)x_1,\ldots,(n_l-r_l)x_l}(v).\end{aligned} \quad (2.13)$$

3. Bell polynomials approach for Korteweg–de Vries equation

In this section, we take the Korteweg–de Vries (KdV) equation as an example to introduce the procedure of finding bilinear form, bilinear BT, Lax pairs and infinite conservation laws by using the Bell polynomials approach [12,13].

The KdV equation, given here in canonical form [12],

$$u_t+6uu_x+u_{3x}=0, \quad (3.1)$$

is widely recognized as a paradigm for the description of weakly nonlinear long waves in many branches of physics and engineering.

3.1. Bilinear representation

KdV equation (3.1) is invariant under the following scale transformation

$$x\longrightarrow\lambda x,\qquad t\longrightarrow\lambda^3t,\qquad u\longrightarrow\lambda^{-2}u, \quad (3.2)$$

in terms of which a potential field q can be introduced by setting

$$u=cq_{2x}, \quad (3.3)$$

with c being free constant to be the appropriate choice such that KdV equation (3.1) connects with P-polynomials (2.8).

In terms of transformation (3.3), KdV equation (3.1) is rewritten in the form

$$E(q)\equiv q_{x,t}+(q_{4x}+3cq_{2x}^2)=0, \quad (3.4)$$

which can be cast into the following P-polynomials expression

$$E(q)\equiv P_{x,t}(q)+P_{4x}(q)=0, \quad (3.5)$$

by setting $c=1$.

Under the change of dependent variable

$$q=2\ln F\Longleftrightarrow u=q_{2x}=2(\ln F)_{2x}, \quad (3.6)$$

P-polynomials expression (3.5) directly produces the bilinear representation of KdV equation (3.1)

$$(D_xD_t+D_x^4)F\cdot F=0. \quad (3.7)$$

3.2. Bilinear BT and Lax pair

Suppose that $q=2\ln G$ and $q'=2\ln F$ are two different solutions of Eq. (3.4), respectively. On introducing two new variables

$$v=(q'-q)/2=\ln(F/G),\qquad w=(q'+q)/2=\ln(FG), \quad (3.8)$$

we associate the two-field condition

$$\begin{aligned}E(q')-E(q)&=E(w+v)-E(w-v)\\&=2(v_{x,t}+v_{4x}+6v_{2x}w_{2x})\\&=2\partial_x[\mathscr{Y}_t(v)+\mathscr{Y}_{3x}(v,w)]+\mathscr{R}(v,w)=0,\end{aligned} \quad (3.9)$$

with $\mathscr{R}(v,w)=6(v_{2x}w_{2x}-v_xw_{3x}-v_x^2v_{2x})$.

In order to find the bilinear BT of the KdV equation (3.1), the next step is to decompose the two-field condition (3.9) into a pair of equations in the form of linear combinations of $\mathscr{Y}$-polynomials. It suffices to impose a constraint (2.11a) on v and w of lowest possible order (or weight).

The simplest possible choice of such a constraint may be

$$\mathscr{Y}_{2x}(v,w)\equiv w_{2x}+v_x^2=\lambda, \quad (3.10)$$

on account of which, $\mathscr{R}(v,w)$ can be rewritten as the x-derivative of a $\mathscr{Y}$-polynomial

$$\mathscr{R}(v,w)=6\lambda v_{2x}=2\partial_x[3\lambda\mathscr{Y}_x(v)]. \quad (3.11)$$

Then, combining the relations (3.9)–(3.11), we deduce a $\mathscr{Y}$-polynomials BT

$$\mathscr{Y}_{2x}(v,w)-\lambda=0, \quad (3.12a)$$

$$\partial_t\mathscr{Y}_x(v)+\partial_x[\mathscr{Y}_{3x}(v,w)+3\lambda\mathscr{Y}_x(v)]=0, \quad (3.12b)$$

where the second Eq. (3.12b) is useful to construct conservation laws later. By application of the identity (2.5), the system (3.12) immediately leads to the bilinear BT

$$(D_x^2-\lambda)F\cdot G=0, \quad (3.13a)$$

$$(D_t+D_x^3+3\lambda D_x-\mu)F\cdot G=0, \quad (3.13b)$$

with μ an arbitrary constant.

Moreover, it follows from the formulae (2.13) that

$$\mathscr{Y}_t(v)=\frac{\psi_t}{\psi},\qquad \mathscr{Y}_{2x}(v,w)=q_{2x}+\frac{\psi_{2x}}{\psi},$$
$$\mathscr{Y}_{3x}(v,w)=3q_{2x}\frac{\psi_x}{\psi}+\frac{\psi_{3x}}{\psi}, \tag{3.14}$$

in terms of which, the $\mathscr{Y}$-polynomials system (3.12) is then linearized into a system with double parameters λ and μ

$$\psi_{2x}+(u-\lambda)\psi=0, \tag{3.15a}$$
$$\psi_t+\psi_{3x}+3(u+\lambda)\psi_x-\mu\psi=0, \tag{3.15b}$$

with the q_{2x} replaced by u.

It can be proved that the compatibility condition $\psi_{2x,t}=\psi_{t,2x}$ just gives rise to KdV equation (3.1). Thus, system (3.15) can be regarded as the Lax pair of KdV equation (3.1).

3.3. Infinite conservation laws

The conservation laws actually have been hinted in the $\mathscr{Y}$-polynomials system (3.12). By introducing a new potential function

$$\eta=\frac{q'_x-q_x}{2}, \tag{3.16}$$

it follows from the relation (3.8) that

$$v_x=\eta,\qquad w_x=q_x+\eta. \tag{3.17}$$

Substituting (3.17) into (3.12), we get a Riccati-type equation

$$\eta_x+\eta^2+u-\lambda=0, \tag{3.18}$$

and a divergence-type equation

$$\eta_t+\partial_x[\eta_{2x}-2\eta^3+6\epsilon^2\eta]=0,\qquad \lambda=\epsilon^2. \tag{3.19}$$

To proceed, inserting the expansion

$$\eta=\epsilon+\sum_{n=1}^{\infty}\mathscr{I}_n(u,u_x,\ldots)\epsilon^{-n}, \tag{3.20}$$

into (3.18) and equating the coefficients for the power of ϵ, we then obtain the recursion relations for the conserved densities

$$\mathscr{I}_1=-\frac{1}{2}u, \tag{3.21a}$$
$$\mathscr{I}_2=-\frac{1}{2}\mathscr{I}_{1,x}=\frac{1}{4}u_x, \tag{3.21b}$$
$$\mathscr{I}_3=-\frac{\mathscr{I}_1^2}{2}-\frac{1}{2}\mathscr{I}_{2,x}=-\frac{1}{8}(u_{2x}+u^2), \tag{3.21c}$$
$$\ldots, \tag{3.21d}$$
$$\mathscr{I}_{n+1}=-\frac{1}{2}\left(\mathscr{I}_{n,x}+\sum_{k=1}^{n}\mathscr{I}_k\mathscr{I}_{n-k}\right),\quad n=3,4,\ldots. \tag{3.21e}$$

Again substituting (3.20) into (3.19) yields

$$\sum_{n=1}^{\infty}\mathscr{I}_{n,t}\epsilon^{-n}+\partial_x\left[\sum_{n=1}^{\infty}\mathscr{I}_{n,2x}\epsilon^{-n}-2\left(\epsilon+\sum_{n=1}^{\infty}\mathscr{I}_n\epsilon^{-n}\right)^3\right.$$
$$\left.+6\epsilon^2\left(\epsilon+\sum_{n=1}^{\infty}\mathscr{I}_n\epsilon^{-n}\right)\right]=0, \tag{3.22}$$

which provides us with an infinite consequence of conservation laws

$$\mathscr{I}_{n,t}+\mathscr{F}_{n,x}=0,\quad n=1,2,\ldots. \tag{3.23}$$

In Eq. (3.23), the conversed densities $\mathscr{I}'_n s$ are given by formula (3.21), while the fluxes $\mathscr{F}'_n s$ are given by recursion formulas explicitly

$$\mathscr{F}_1=-\frac{1}{2}u_{2x}-\frac{3}{2}u^2, \tag{3.24a}$$
$$\mathscr{F}_2=\frac{1}{4}u_{3x}+\frac{3}{2}uu_x, \tag{3.24b}$$
$$\mathscr{F}_3=-\frac{5}{8}u_x^2-uu_{2x}-\frac{1}{8}u_{4x}-\frac{1}{2}u^3, \tag{3.24c}$$
$$\ldots, \tag{3.24d}$$
$$\mathscr{F}_n=\mathscr{I}_{n,2x}-6\sum_{k=1}^{n}\mathscr{I}_k\mathscr{I}_{n+1-k}-2\sum_{i+j+k=n}\mathscr{I}_i\mathscr{I}_j\mathscr{I}_k. \tag{3.24e}$$

The first equation of conservation law Eq. (3.23) is exactly the KdV equation (3.1).

Compared with the Hirota method to derive the bilinear forms, bilinear BTs, Lax pairs, and infinite conservation laws of the soliton equations, the Bell polynomials approach employed here is shown to be more direct and systematic, which establishes a deep relation between the integrabilities of soliton equations and Bell polynomials. Based on the Bell polynomials approach, in the following sections, we will develop a Maple program for finding bilinear forms, bilinear BTs, Lax pairs and conservation laws of the KdV-type equation.

4. Bilinearization: algorithm and Maple program

In this section, we will develop a Maple program *PDEBellII* to construct bilinear forms of soliton equations based on the use of Bell polynomials.

4.1. Algorithm

Consider a $(N+1)$-dimensional soliton equation

$$\Delta(u,u_{x_0},u_{x_1},u_{x_2},\ldots,u_{x_N},u_{x_ix_j},\ldots)=0,\quad 0\le i,j\le N, \tag{4.1}$$

in which $u=u(x_0,x_1,\ldots,x_N)$, x_0 usually denotes time variable t, u_{x_N} denotes the partial derivatives with respect to independent variable x_N.

1. Identify the set of independent variables $\{x_0,x_1,\ldots,x_N\}$ and dependent variables $\{u\}$ as well as its differential orders to each independent variable in the given soliton equation. If the coefficients of the given soliton equation are not constants, the set of these coefficients $\{\alpha_i, i=1,\ldots,\iota\}$ is also given. The relations satisfied by these coefficients will be determined in step 5.
2. Detect whether the given equation satisfies the homogeneous balance principle. Assuming that
$$u=f^{p_1+\cdots+p_N}(\phi)\phi_{x_1}^{p_1}\cdots\phi_{x_N}^{p_N}+\phi,\quad \phi\equiv\phi(x_1,\ldots,x_N), \tag{4.2}$$
satisfy Eq. (4.1), on account of which, substituting (4.2) into Eq. (4.1), then nonnegative integers $p_1,\ldots,p_N$ can be determined by balancing the highest-order linear derivative term and the highest-order nonlinear terms in the resulting equation. For the cases where any $p_i(1\le i\le N)$ is negative or a fraction or $p_1,\ldots,p_N$ cannot be worked out, the algorithm terminates.
3. Determine the appropriate variable transformation. On introducing a new potential field q by setting
$$u=cq_{p_ix_i},\quad 1\le i\le N, \tag{4.3}$$
with c being a free constant to be the appropriate choice such that the resulting equation connects with P-polynomials (2.8). After substituting variable transformation (4.3) into Eq. (4.1), the resulting equation is customarily called the dimensionless field equation.

4. Determine the order of the highest-order linear derivative term, and generate a set of polynomials which include P-polynomials or $\mathscr{Y}$-polynomials whose order is less than or equal to maximal order. For instance, if the highest-order linear derivative term is $q_{n_i x_i}$, then the set of $\mathscr{Y}$-polynomials is $\{\mathscr{Y}_{m_1x_1,m_2x_2,\ldots,m_Nx_N}(q) | \sum_{i=1}^{N} m_i \leq n_i\}$.
5. This step contains two kinds of polynomials matching the process: P-polynomials and $\mathscr{Y}$-polynomials.
   (a) Check whether the dimensionless field equation can be cast into a combination form of P-polynomials. If it succeeds, turn to 5c. Otherwise, turn to 5b.
   (b) Check whether the dimensionless field equation can be cast into a combination form of $\mathscr{Y}$-polynomials. If it succeeds, turn to 5c. Otherwise, turn to 5d.
   (c) Give out the P-polynomials expression $E(q)$ or $\mathscr{Y}$-polynomials expression along with associated bilinear representation. Meanwhile, determine the value of c in Eq. (4.3) and return the current equation. If the coefficients of the given soliton equation are not constants, the relations satisfied by these coefficients along with c will also be determined in this step.
   (d) Integrate the current equation with respect to x. If the obtained equation has no integral term, go back to 5a. If any integral term exists, the algorithm terminates.

Remark 4.1. In step 5, some additional constraint conditions might be imposed by using P-polynomials or $\mathscr{Y}$-polynomials. The rule of choosing constraint conditions is that the linear combination form of P-polynomials or $\mathscr{Y}$-polynomials is as simple as possible and the order of any term is not more than 3 as a rule of thumb.

4.2. Maple program

We have developed an automated Maple program *PDEBellII* to implement the algorithm described above. For global parameters for program *PDEBellII* refer to Table A.1.

The program is initialized by the command

```
> with(PDEBellII):
```

The main procedure is *Get_Bell(eq, num)*, in which the parameter *eq* represents the soliton equation to be handled, *num* denotes the number of $\mathscr{I}_n$. All sub-procedures called in this procedure are described in the following:

- ***Homogen_Balance(eq)***: Introduces a new variable *new_devar* to the equation *eq* by homogeneous balance method and gives the new equation *Eq*.
- ***NewEq(eq, a)***: Integrates the equation *eq* whose dependent variable is a with respect to x.
- ***H_Order(eq)***: For the specified equation *eq*, obtains the differential order sequence of the highest-order derivative item with respect to all the independent variables.
- ***P_Match(eq, indev)***: Checks whether the current equation *eq* can be written as a linear combination of related P-polynomials which takes *indev* as the independent variables. If it succeeds, it outputs the matched results with the corresponding bilinear representation of *eq* and returns value 0. Otherwise it returns value 1.
- ***Y_Match(eq, indev)***: Checks whether the current equation *eq* can be written as a linear combination of related $\mathscr{Y}$-polynomials which takes *indev* as the independent variables. If it succeeds, it outputs the matched results with the corresponding bilinear representation of *eq* and returns value 0. Otherwise it returns value 1.

Other program commands and corresponding parameters are given in the following listing:

- ***BellPoly(dev, indev::sequential, order::sequential)***: With *dev* as dependent variable, *indev* as independent variables, generates Bell polynomials of specified order sequence *order*.

```
> BellPoly(Y,[t,x],[1,1]);
```

the corresponding output is

$$B_{x,t}(Y) = \left(\frac{\partial^2}{\partial x \partial t} Y\right) + \left(\frac{\partial}{\partial x} Y\right)\left(\frac{\partial}{\partial t} Y\right).$$

- ***YPoly(devs::sequential, indev::sequential, order::sequential)***: With *devs* as dependent variables, *indev* as independent variables, generates a $\mathscr{Y}$-polynomial of specified order sequence *order*.

```
> YPoly([v,w],[t,x],[1,2]);
```

the corresponding output is

$$\mathscr{Y}_{t,2x}(v,w) = \left(\frac{\partial^3}{\partial x^2 \partial t} v\right) + 2\left(\frac{\partial^2}{\partial x \partial t} w\right)\left(\frac{\partial}{\partial x} v\right) + \left(\frac{\partial}{\partial t} v\right)\left(\frac{\partial^2}{\partial x^2} w\right) + \left(\frac{\partial}{\partial t} v\right)\left(\frac{\partial}{\partial x} v\right)^2.$$

- ***PPoly(dev, indev::sequential, order::sequential)***: With *dev* as dependent variable, *indev* as independent variables, generates a P-polynomial of specified order sequence *order*.

```
> PPoly(q,[t,x],[3,1]);
```

the corresponding output is

$$P_{3t,x}(q) = \left(\frac{\partial^4}{\partial x \partial t^3} q\right) + 3\left(\frac{\partial^2}{\partial t^2} q\right)\left(\frac{\partial^2}{\partial x \partial t} q\right).$$

- ***YPolyset(devs::sequential, indev::sequential, order::sequential)***: With *devs* as dependent variables, *indev* as independent variables, generates a set of $\mathscr{Y}$-polynomials of specified order sequence *order*. The sum of the differential order sequence of any polynomial in the set with respect to all the independent variables is less than or equal to that of sequence *order*.

```
> YPolyset([v,w],[t,x],[1,1]);
```

the corresponding output is

$$\Big\{\mathscr{Y}_{0,x}(v,w) = \left(\frac{\partial}{\partial x} v\right), \mathscr{Y}_{0,2x}(v,w) = \left(\frac{\partial^2}{\partial x^2} w\right) + \left(\frac{\partial}{\partial x} v\right)^2, \mathscr{Y}_{t,0}(v,w) = \left(\frac{\partial}{\partial t} v\right), \mathscr{Y}_{2t,0}(v,w) = \left(\frac{\partial}{\partial t^2} w\right) + \left(\frac{\partial}{\partial t} v\right)^2, \mathscr{Y}_{t,x}(v,w) = \left(\frac{\partial^2}{\partial x \partial t} w\right) + \left(\frac{\partial}{\partial t} v\right)\left(\frac{\partial}{\partial x} v\right)\Big\}.$$

- ***PPolyset(dev, indev::sequential, order::sequential)***: With *dev* as dependent variable, *indev* as independent variables, generates a set of P-polynomials of specified order sequence *order*. The sum of the differential order sequence of any polynomial in the set with respect to all the independent variables is less than or equal to that of sequence *order*.

```
>PPolyset(q,[t,x],[1,1]);
```

the corresponding output is

$$\left\{P_{0,2x}(q) = \left(\frac{\partial^2}{\partial x^2} q\right), P_{t,x}(q) = \left(\frac{\partial^2}{\partial x \partial t} q\right), P_{2t,0}(q) = \left(\frac{\partial^2}{\partial t^2} q\right)\right\}.$$

- ***P_Operator(dev, indev::sequential, order::sequential)***: Generates a set of bilinear operators related to the set of P-polynomials ***PPolyset(dev, indev, order)***.

```
>P_Operator(q,[t,x],[1,1]);
```

the corresponding output is

$$\left\{P_{0,2x}(q) = D_x^2 F \cdot F, P_{x,t}(q) = D_x D_t F \cdot F, P_{2t,0}(q) = D_t^2 F \cdot F\right\}.$$

- ***Y_Operator(devs::sequential, indev::sequential,order:: sequential)***: Generates a set of bilinear operators related to the set of $\mathscr{Y}$-polynomials ***YPolyset(devs, indev, order)***.

```
>Y_Operator([v,w],[t,x],[1,1]);
```

the corresponding output is

$$\{\mathscr{Y}_{0,x}(v,w) = D_x F \cdot G, \mathscr{Y}_{0,2x}(v,w) = D_x^2 F \cdot G,$$
$$\mathscr{Y}_{t,0}(v,w) = D_t F \cdot G, \mathscr{Y}_{t,x}(v,w) = D_t D_x F \cdot G,$$
$$\mathscr{Y}_{2t,0}(v,w) = D_t^2 F \cdot G\}.$$

- ***orders(s::sequential)***: Generates a set of number sequences in which the sum of all the numbers is less than or equal to the sum of all the numbers in sequence s.

```
>orders([1,2]);
```

the corresponding output is

[[0, 0], [0, 1], [0, 2], [0, 3], [1, 0],
[1, 1], [1, 2], [2, 0], [2, 1], [3, 0]].

- ***listOrders(s::sequential)***: Generates a set of number sequences in which each number is less than or equal to the corresponding number in sequence s.

```
>listOrders([1,2]);
```

the corresponding output is

[[0, 0], [0, 1], [0, 2], [1, 0], [1, 1], [1, 2]].

4.3. Illustrative examples

The program *PDEBellII* can be used to handle several different kinds of soliton equations, which includes (1 + 1)-dimensional, (1 + 2)-dimensional, (1 + 3)-dimensional soliton equations, variable coefficient soliton equations, and soliton equations with integration terms.

Example 4.3.1. KdV equation [5]

$$u_t + 6uu_x + u_{3x} = 0, \tag{4.4}$$

the input and corresponding output are:

```
> with(PDEBellII):
> alias(u=u(x,t)):
> KdV:=diff(u,t)+6*u*diff(u,x)+diff(u,x$3):
> Get_Bell(KdV,3);
```

This equation can be related to a new equation by setting

$$u = c\frac{\partial^2}{\partial x^2}q$$

and the new equation is

$$\left(\frac{\partial^2}{\partial x \partial t}q\right) + 3c\left(\frac{\partial^2}{\partial x^2}q\right)^2 + \left(\frac{\partial^4}{\partial x^4}q\right) = 0.$$

Under the constraint condition:

$$c = 1.$$

The new equation can be written in the linear combination of P-polynomials

$$P_{0,4x}(q) + P_{t,x}(q) = 0.$$

The bilinear form of this equation is

$$(D_x^4 + D_t D_x)F \cdot F = 0.$$

Example 4.3.2. Boussinesq equation [5]

$$u_{2t} + (u^2)_{2x} - u_{4x} = 0. \tag{4.5}$$

The input is similar to the KdV case, therefore, we only give the output:

This equation can be related to a new equation by setting

$$u = c\frac{\partial^2}{\partial x^2}q$$

and the new equation is

$$\left(\frac{\partial^2}{\partial t^2}q\right) - \left(\frac{\partial^4}{\partial x^4}q\right) + c\left(\frac{\partial^2}{\partial x^2}q\right)^2 = 0.$$

Under the constraint condition:

$$c = -3.$$

The new equation can be written in the linear combination of P-polynomials

$$P_{2t,0}(q) - P_{0,4x}(q) = 0.$$

The bilinear form of this equation is

$$(D_t^2 - D_x^4)F \cdot F = 0.$$

Example 4.3.3. The program *PDEBellII* can be applied into the variable coefficient soliton equations, such as variable coefficient fifth-order KdV equation [37]

$$u_t + u_{5x} + \gamma uu_{3x} + \beta u_x u_{2x} + \alpha u^2 u_x = 0, \tag{4.6}$$

the corresponding output is:

This equation can be related to a new equation by setting

$$u = c\frac{\partial^2}{\partial x^2}q$$

and the new equation is

$$\left(\frac{\partial^2}{\partial x \partial t}q\right) + \left(\frac{\partial^6}{\partial x^6}q\right) + \gamma c\left(\frac{\partial^4}{\partial x^4}q\right)\left(\frac{\partial^2}{\partial x^2}q\right)$$
$$-\frac{\gamma c}{2}\left(\frac{\partial^3}{\partial x^3}q\right)^2 + \frac{\beta c}{2}\left(\frac{\partial^3}{\partial x^3}q\right)^2 + \frac{\alpha c^2}{3}\left(\frac{\partial^2}{\partial x^2}q\right)^3 = 0.$$

Under the constraint condition:

$$c = \frac{15}{\gamma}.$$

The new equation can be written in the linear combination of P-polynomials

$$P_{0,6x}(q) + P_{t,x}(q) = 0.$$

The bilinear form of this equation is

$$(D_x^6 + D_t D_x)F \cdot F = 0.$$

Example 4.3.4. The extended KdV equation [38]

$$u_t + u_x + \alpha(6uu_x + u_{3x})$$
$$+ \alpha^2(u_{5x} + 15uu_{3x} + 15u_x u_{2x} + 45u^2 u_x) = 0, \tag{4.7}$$

the corresponding output is:

This equation can be related to a new equation by setting

$$u = c\frac{\partial^2}{\partial x^2}q$$

and the new equation is

$$\left(\frac{\partial^2}{\partial x\partial t}q\right)+\left(\frac{\partial^2}{\partial x^2}q\right)+3\alpha c\left(\frac{\partial^2}{\partial x^2}q\right)^2+\alpha\left(\frac{\partial^2}{\partial x^4}q\right)$$
$$+15\alpha^2c^2\left(\frac{\partial^2}{\partial x^2}q\right)^3+15\alpha^2c\left(\frac{\partial^2}{\partial x^4}q\right)\left(\frac{\partial^2}{\partial x^2}q\right)$$
$$+\alpha^2\left(\frac{\partial^2}{\partial x^6}q\right)=0.$$

Under the constraint condition:

$c = 1.$

The new equation can be written in the linear combination of P-polynomials

$P_{0,2x}(q)+\alpha P_{0,4x}(q)+\alpha^2P_{0,6x}(q)+P_{t,x}(q)=0.$

The bilinear form of this equation is

$(D_x^2+\alpha D_x^4+\alpha^2D_x^6+D_tD_x)F\cdot F=0.$

Example 4.3.5. The program *PDEBellII* can also be applied into soliton equations with integration terms, such as the shallow water waves equation [39]

$$u_t-u_{2x,t}-3uu_t-3u_x\int u_tdx+u_x=0, \tag{4.8}$$

the corresponding outputs are:

This equation can be related to a new equation by setting

$$u=c\frac{\partial^2}{\partial x^2}q$$

and the new equation is

$$\left(\frac{\partial^2}{\partial x\partial t}q\right)-\left(\frac{\partial^4}{\partial x^3\partial t}q\right)-3c\left(\frac{\partial^2}{\partial x\partial t}q\right)\left(\frac{\partial^2}{\partial x^2}q\right)+\left(\frac{\partial^2}{\partial x^2}q\right)=0.$$

Under the constraint condition:

$c = 1.$

The new equation can be written in the linear combination of p-polynomial

$P_{0,2x}(q)+P_{t,x}(q)-P_{t,3x}(q)=0.$

The bilinear form of this equation is

$(D_x^2+D_tD_x-D_tD_x^3)F\cdot F=0.$

The program *PDEBellII* can also handle the higher dimensional soliton equations, such as

Example 4.3.6. Kadomtsev–Petviashvili (KP) equation [5]

$$u_{x,t}+6u_x^2+6uu_{2x}+u_{4x}+3u_{2y}=0, \tag{4.9}$$

the corresponding outputs are:

This equation can be related to a new equation by setting

$$u=c\frac{\partial^2}{\partial x^2}q$$

and the new equation is

$$\left(\frac{\partial^2}{\partial x\partial t}q\right)+3c\left(\frac{\partial^2}{\partial x^2}q\right)^2+\left(\frac{\partial^4}{\partial x^4}q\right)+3\left(\frac{\partial^2}{\partial y^2}q\right)=0.$$

Under the constraint condition:

$c = 1.$

The new equation can be written in the linear combination of p-polynomial

$P_{t,x,0}(q)+P_{0,4x,0}(q)+3P_{0,0,2y}(q)=0.$

The bilinear form of this equation is

$(D_tD_x+D_x^4+3D_y^2)F\cdot F=0.$

Example 4.3.7. (2 + 1)-dimensional Sawada–Kotera (SK) equation [40]

$$u_t-u_{5x}-5u_xu_{2x}-5uu_{3x}-5u^2u_x-5u_{2x,y}-5uu_y$$
$$+5\int u_{2y}dx-5u_x\int u_ydx=0, \tag{4.10}$$

the corresponding outputs are:

This equation can be related to a new equation by setting

$$u=c\frac{\partial^2}{\partial x^2}q$$

and the new equation is

$$\left(\frac{\partial^2}{\partial x\partial t}q\right)-\left(\frac{\partial^6}{\partial x^6}q\right)-5c\left(\frac{\partial^4}{\partial x^4}q\right)\left(\frac{\partial^2}{\partial x^2}q\right)-\frac{5c^2}{3}\left(\frac{\partial^2}{\partial x^2}q\right)^3$$
$$-5\left(\frac{\partial^4}{\partial x^3\partial y}q\right)+5\left(\frac{\partial^2}{\partial y^2}q\right)-5c\left(\frac{\partial^2}{\partial x^2}q\right)\left(\frac{\partial^2}{\partial x\partial y}q\right)=0.$$

Under the constraint condition:

$c = 3.$

The new equation can be written in the linear combination of p-polynomial

$5P_{0,0,2y}(q)-5P_{0,3x,y}(q)-P_{0,6x,0}(q)+P_{t,x,0}(q)=0.$

The bilinear form of this equation is

$(5D_y^2-5D_x^3D_y-D_x^6+D_tD_x)F\cdot F=0.$

Example 4.3.8. (3 + 1)-dimensional KP equation [41]

$$u_{x,t}+6u_x^2+6uu_{2x}-u_{4x}-u_{2y}-u_{2z}=0, \tag{4.11}$$

the corresponding outputs are:

This equation can be related to a new equation by setting

$$u=c\frac{\partial^2}{\partial x^2}q$$

and the new equation is

$$\left(\frac{\partial^2}{\partial x\partial t}q\right)+3c\left(\frac{\partial^2}{\partial x^2}q\right)^2-\left(\frac{\partial^4}{\partial x^4}q\right)$$
$$-\left(\frac{\partial^2}{\partial y^2}q\right)-\left(\frac{\partial^2}{\partial z^2}q\right)=0.$$

Under the constraint condition:

$c = -1.$

The new equation can be written in the linear combination of p-polynomial

$P_{t,x,0,0}(q)-P_{0,4x,0,0}(q)-P_{0,0,2y,0}(q)-P_{0,0,0,2z}(q)=0.$

The bilinear form of this equation is

$(D_tD_x-D_x^4-D_y^2-D_z^2)F\cdot F=0.$

Example 4.3.9. (3 + 1)-dimensional Jimbo–Miwa (JM) equation [41]

$$u_{3x,y} + 3u_x u_{x,y} + 3u_y u_{2x} + 2u_{y,t} - 3u_{x,z} = 0, \tag{4.12}$$

the corresponding outputs are:

This equation can be related to a new equation by setting

$$u = c\frac{\partial}{\partial x}q$$

and the new equation is

$$3c\left(\frac{\partial^2}{\partial x^2}q\right)\left(\frac{\partial^2}{\partial x\partial y}q\right) + \left(\frac{\partial^4}{\partial x^3\partial y}q\right) + 2\left(\frac{\partial^2}{\partial y\partial t}q\right) - 3\left(\frac{\partial^2}{\partial x\partial z}q\right) = 0.$$

Under the constraint condition:

$$c = 1.$$

The new equation can be written in the linear combination of p-polynomial

$$P_{0,3x,y,0}(q) + 2P_{t,0,y,0}(q) - 3P_{0,x,0,z}(q) = 0.$$

The bilinear form of this equation is

$$(D_x^3 D_y + 2D_t D_y - 3D_x D_z)F \cdot F = 0.$$

Example 4.3.10. The program *PDEBellII* can also be applied to mKdV-type equations, such as mKdV equation [5]

$$u_t + 6u^2 u_x + u_{3x} = 0, \tag{4.13}$$

the corresponding outputs are:

This equation can be related to a new equation by setting

$$u = c\frac{\partial}{\partial x}v$$

and the new equation is

$$\left(\frac{\partial}{\partial t}v\right) + 2c^2\left(\frac{\partial}{\partial x}v\right)^3 + \left(\frac{\partial^3}{\partial x^3}v\right).$$

We introduce the new constraint:

$$\left(\frac{\partial^2}{\partial x^2}w\right) + \left(\frac{\partial}{\partial x}v\right)^2 = 0.$$

Under the constraint condition:

$$c^2 = -1.$$

The new equation can be written in the linear combination of $\mathscr{Y}$-polynomial

$$\mathscr{Y}_{0,2x}(v, w) = 0$$

$$\mathscr{Y}_{0,3x}(v, w) + \mathscr{Y}_{t,0}(v, w) = 0.$$

The bilinear form of this equation is

$$D_x^2 F \cdot G = 0$$

$$(D_x^3 + D_t)F \cdot G = 0.$$

5. Bilinear BTs, Lax pairs and conservation laws: algorithm and Maple program

It is well known that bilinear BTs, Lax pairs and infinite conservation laws can characterize the integrability of soliton equations. Compared with the program that was used to find bilinear forms, to the best knowledge of the authors, there have been no programs to derive the bilinear BTs, Lax pairs and infinite conservation laws of soliton equations. The difficulty is that the procedure of finding bilinear BTs needed rich practical experience and complicated mathematical skills. Fortunately, compared with the Hirota method, the Bell polynomials approach is a more direct procedure in finding bilinear BTs of KdV-type equations, and it is very easy to follow. In this section, based on the Bell polynomials approach, we further improve our program *PDEBellII* to find the bilinear BTs, Lax pairs and conservation laws of the KdV-type equations.

5.1. Algorithm

Our algorithm focuses on the soliton equations which can be written as P-polynomials expression $E(q)$. From the Section 3.2, the two-field condition can be written in the form

$$E(q') - E(q) = E(w + v) - E(w - v) = 0, \tag{5.1}$$

on account of which, the bilinear BTs can be obtained by decomposing (5.1) into a pair of $\mathscr{Y}$-polynomials expression.

1. Figure out the order n_i of the linear highest-order derivative term $q_{n_i x_i}$ in Eq. (5.1), and generate a set of $\mathscr{Y}$-polynomials which the differential order sequence of any polynomial $\mathscr{Y}_{m_1 x_1,\ldots,m_N x_N}$ in the sets satisfies that $\sum_{i=1}^{N} m_i \le n_i - 1$.
2. Split Eq. (5.1) into two parts: $\mathscr{L}(v, w)$ matched with the first derivative of the linear combination of $\mathscr{Y}$-polynomials appearing in step 1 with respect to t or x, and the rest denoted by $\mathscr{R}(v, w)$.
3. Introduce appropriate constraint (linear combination of $\mathscr{Y}$-polynomials) to ensure that $\mathscr{R}(v, w)$ can be expressed as a linear combination form of $\mathscr{Y}$-polynomials with respect to t or x. The rule of choosing constraints is that the linear combination of $\mathscr{Y}$-polynomials is as simple as possible. As a rule of thumb, the order of the $\mathscr{Y}$-polynomial is less than or equal to 3. If it succeeds, turn to 4. Otherwise, the algorithm terminates.
4. Give the Bäcklund transformation in the form of $\mathscr{Y}$-polynomials and bilinear representations.
5. Linearize the Bäcklund transformation into Lax pairs.
6. By introducing appropriate transformation, rewrite the $\mathscr{Y}$-polynomials Bäcklund transformation as a Riccati-type equation and a divergence-type equation. In terms of the series expansion method, obtain the infinite conservation laws.

5.2. Maple program

We further improve program *PDEBellII* to implement the algorithm described above.

The sub-procedure called in this procedure is described in the following:

- ***Bäcklund_Lax_Conservation(eq, indev)***: For the specified equation *eq* whose independent variables are *indev*, gives the bilinear Bäcklund transformation, Lax pair and infinite conservation laws. Some appropriate constraint conditions may need to be introduced during the computation.

Other program commands and corresponding parameters are given in the following listing:

- **YExp(devs::sequential, indev::sequential, order::sequential)**: With *devs* as dependent variables, *indev* as independent variables, generates the linear representation of an $\mathscr{Y}$-polynomial of specified order sequence *order*.

```
> YExp([v,w],[t,x],[1,1]);
```

the corresponding output is

$$\left(\frac{\frac{\partial^2}{\partial x\partial_t}\phi}{\phi}\right) + \left(\frac{\partial^2}{\partial x\partial_t}q\right)$$

- **YExpset(devs::sequential, indev::sequential, order:: sequential)**: With *devs* as dependent variable, *indev* as independent variables, generates a set of linear representations of $\mathscr{Y}$-polynomials of specified order sequence *order*. The sum of the differential order sequence of any $\mathscr{Y}$-polynomial with respect to all the independent variables is less than or equal to that of sequence *order*.

```
> YExpset([v,w],[t,x],[1,1]);
```

the corresponding output is

$$\left\{\mathscr{Y}_{0,x}(v,w)=\left(\frac{\frac{\partial}{\partial x}\phi}{\phi}\right),\mathscr{Y}_{0,2x}(v,w)=\left(\frac{\frac{\partial^2}{\partial x^2}\phi}{\phi}\right)+\left(\frac{\partial^2}{\partial x^2}q\right),\right.$$
$$\mathscr{Y}_{t,0}(v,w)=\left(\frac{\frac{\partial}{\partial t}\phi}{\phi}\right),\mathscr{Y}_{2t,0}(v,w)=\left(\frac{\frac{\partial^2}{\partial t^2}\phi}{\phi}\right)+\left(\frac{\partial^2}{\partial t^2}q\right),$$
$$\left.\mathscr{Y}_{t,x}(v,w)=\left(\frac{\frac{\partial^2}{\partial x\partial t}\phi}{\phi}\right)+\left(\frac{\partial^2}{\partial x\partial t}q\right)\right\}$$

Conservation_Law(eq1, eq2, n): Constructs the infinite conservation laws of input equation according to *eq*1 and *eq*2 with respect to specified number *n*. All conserved densities and fluxes are given with explicit recursion formulas and accurate expressions.

5.3. Illustrative examples

The program *PDEBellII* can be used to derive the bilinear BTs and Lax pairs of the KdV-type equation. In particular, the conservation laws can also be given if the $\mathscr{Y}$-polynomials BTs can be transformed into a Riccati-type equation and a divergence-type equation, such as the KdV equation and KP equation.

Example 5.3.1. Boussinesq equation 4.3.2, the corresponding outputs are:

****Bäcklund transformation****

After pretreatment, a new equation is:

$$-2\left(\frac{\partial^4}{\partial x^4}v\right)-12\left(\frac{\partial^2}{\partial x^2}w\right)\left(\frac{\partial^2}{\partial x^2}v\right)+2\left(\frac{\partial^2}{\partial t^2}v\right). \tag{5.2}$$

With

$$C=(\sqrt{3}I,-\sqrt{3}I). \tag{5.3}$$

We introduce the new constraint:

$$\left(\frac{\partial}{\partial t}v\right)+C\left(\left(\frac{\partial^2}{\partial x^2}w\right)+\left(\frac{\partial}{\partial x}v\right)^2\right)=0. \tag{5.4}$$

It can be written in the linear combination of $\mathscr{Y}$-polynomial:

$C\mathscr{Y}_{0,2x}(v,w)+\mathscr{Y}_{t,0}(v,w)=0$
$\mathscr{Y}_{0,3x}(v,w)+C\mathscr{Y}_{t,x}(v,w)=0.$

The Bäcklund transformation is:

$(CD_x^2+D_t)F\cdot G=0$
$(D_x^3+CD_tD_x)F\cdot G=0.$

The Lax pair is:

$$C\left(\frac{\partial^2}{\partial x^2}\phi\right)-\frac{C\phi u}{3}+\left(\frac{\partial}{\partial t}\phi\right)-\lambda\phi=0,$$
$$\left(\frac{\partial^3}{\partial x^3}\phi\right)-u\left(\frac{\partial}{\partial x}\phi\right)+C\left(\frac{\partial^2}{\partial x\partial t}\phi\right)+C\left(\frac{\partial^2}{\partial x\partial t}q\right)\phi=0.$$

Example 5.3.2. Variable coefficient fifth-order KdV equation 4.3.3, the corresponding outputs are:

****Bäcklund transformation****

After pretreatment, a new equation is:

$$2\left(\frac{\partial^6}{\partial x^6}v\right)+30\left(\frac{\partial^4}{\partial x^4}w\right)\left(\frac{\partial^2}{\partial x^2}v\right)+30\left(\frac{\partial^4}{\partial x^4}v\right)\left(\frac{\partial^2}{\partial x^2}w\right)$$
$$+90\left(\frac{\partial^2}{\partial x^2}w\right)^2\left(\frac{\partial^2}{\partial x^2}v\right)+30\left(\frac{\partial^2}{\partial x^2}v\right)^3+2\left(\frac{\partial^2}{\partial x\partial t}v\right). \tag{5.5}$$

We introduce the new constraint:

$$\left(\frac{\partial^3}{\partial x^3}v\right)+3\left(\frac{\partial^2}{\partial x^2}w\right)\left(\frac{\partial}{\partial x}v\right)+\left(\frac{\partial}{\partial x}v\right)^3-\lambda=0. \tag{5.6}$$

It can be written in the linear combination of $\mathscr{Y}$-polynomial:

$\mathscr{Y}_{0,3x}(v,w)-\lambda=0$
$2\mathscr{Y}_{t,0}(v,w)-3\mathscr{Y}_{0,5x}(v,w)-15\lambda\mathscr{Y}_{0,2x}(v,w)=0.$

The Bäcklund transformation is:

$(D_x^3-\lambda)F\cdot G=0$
$(2D_t-3D_x^5-15\lambda D_x^2)F\cdot G=0.$

The Lax pair is:

$$\left(\frac{\partial^3}{\partial x^3}\phi\right)+\frac{\lambda u}{5}\left(\frac{\partial}{\partial x}\phi\right)-\lambda\phi=0,$$
$$2\left(\frac{\partial}{\partial t}\phi\right)-3\left(\frac{\partial^5}{\partial x^5}\phi\right)-15\lambda\left(\frac{\partial^2}{\partial x^2}\phi\right)-\gamma\lambda u\phi$$
$$-2\gamma u\left(\frac{\partial^3}{\partial x^3}\phi\right)-\gamma\left(\frac{\partial}{\partial x}\phi\right)\left(\frac{\partial^2}{\partial x^2}u\right)-\frac{\gamma^2u^2}{5}\left(\frac{\partial}{\partial x}\phi\right)=0.$$

Example 5.3.3. Shallow water waves equation 4.3.5, the corresponding outputs are:

****Bäcklund transformation****

After pretreatment, a new equation is:

$$2\left(\frac{\partial^2}{\partial x\partial t}v\right)-2\left(\frac{\partial^4}{\partial x^3\partial t}v\right)-6\left(\frac{\partial^2}{\partial x\partial t}w\right)\left(\frac{\partial^2}{\partial x^2}v\right)$$
$$-6\left(\frac{\partial^2}{\partial x\partial t}v\right)\left(\frac{\partial^2}{\partial x^2}w\right)+2\left(\frac{\partial^2}{\partial x^2}v\right). \tag{5.7}$$

We introduce the new constraint:

$$\left(\frac{\partial^2}{\partial x\partial t}w\right)+\left(\frac{\partial}{\partial t}v\right)\left(\frac{\partial}{\partial x}v\right)-\gamma\left(\frac{\partial}{\partial x}v\right)-\frac{1}{3}=0. \tag{5.8}$$

It can be written in the linear combination of $\mathscr{Y}$-polynomial:

$-\gamma\mathscr{Y}_{0,x}(v,w)+\mathscr{Y}_{t,x}(v,w)-\frac{1}{3}=0$
$\mathscr{Y}_{0,x}(v,w)-\mathscr{Y}_{0,3x}(v,w)=0.$

The Bäcklund transformation is:

$\left(-\gamma D_x+D_tD_x-\frac{1}{3}\right)F\cdot G=0$
$(D_x-D_x^3)F\cdot G=0.$

The Lax pair is:

$$-3\gamma\left(\frac{\partial}{\partial x}\phi\right)+3\left(\frac{\partial^2}{\partial x\partial t}\phi\right)+3\left(\frac{\partial^2}{\partial x\partial t}q\right)\phi-\phi-3\lambda\phi=0$$
$$\left(\frac{\partial}{\partial x}\phi\right)-\left(\frac{\partial^3}{\partial x^3}\phi\right)-3u\left(\frac{\partial}{\partial x}\phi\right)=0.$$

Example 5.3.4. (2 + 1)-dimensional SK equation 4.3.7, the corresponding outputs are:

****Bäcklund transformation****

After pretreatment, a new equation is:

$$10\left(\frac{\partial^2}{\partial y^2}v\right)-30\left(\frac{\partial^2}{\partial x^2}w\right)\left(\frac{\partial^2}{\partial y\partial x}v\right)-30\left(\frac{\partial^2}{\partial x^2}v\right)\left(\frac{\partial^2}{\partial y\partial x}w\right)$$
$$-30\left(\frac{\partial^4}{\partial x^4}w\right)\left(\frac{\partial^2}{\partial x^2}v\right)-30\left(\frac{\partial^4}{\partial x^4}v\right)\left(\frac{\partial^2}{\partial x^2}w\right)$$
$$-90\left(\frac{\partial^2}{\partial x^2}w\right)^2\left(\frac{\partial^2}{\partial x^2}v\right)-30\left(\frac{\partial^2}{\partial x^2}v\right)^3-10\left(\frac{\partial^4}{\partial y\partial x^3}v\right)$$
$$-2\left(\frac{\partial^6}{\partial x^6}v\right)+2\left(\frac{\partial^2}{\partial x\partial t}v\right). \quad (5.9)$$

We introduce the new constraint:

$$\left(\frac{\partial^3}{\partial x^3}v\right)+3\left(\frac{\partial^2}{\partial x^2}w\right)\left(\frac{\partial}{\partial x}v\right)+\left(\frac{\partial}{\partial x}v\right)^3$$
$$+\left(\frac{\partial}{\partial y}v\right)-\lambda=0. \quad (5.10)$$

It can be written in the linear combination of $\mathscr{Y}$-polynomial:

$$\mathscr{Y}_{0,0,y}(v,w)+\mathscr{Y}_{0,3x,0}(v,w)-\lambda=0$$
$$3\mathscr{Y}_{0,5x,0}(v,w)+2\mathscr{Y}_{t,0,0}(v,w)$$
$$+15\lambda\mathscr{Y}_{0,2x,0}(v,w)-15\mathscr{Y}_{0,2x,y}(v,w)=0.$$

The Bäcklund transformation is:

$$(D_x^3+D_y-\lambda)F\cdot G=0$$
$$(2D_t-15D_x^2D_y+15\lambda D_x^2+3D_x^5)F\cdot G=0.$$

The Lax pair is:

$$u\left(\frac{\partial}{\partial x}\phi\right)+\left(\frac{\partial^3}{\partial x^3}\phi\right)+\left(\frac{\partial}{\partial y}\phi\right)-\lambda\phi=0$$
$$2\left(\frac{\partial}{\partial t}\phi\right)-30\left(\frac{\partial^2}{\partial y\partial x}q\right)\left(\frac{\partial}{\partial x}\phi\right)-5u\left(\frac{\partial}{\partial y}\phi\right)+3\left(\frac{\partial^5}{\partial x^5}\phi\right)$$
$$+15\lambda\left(\frac{\partial^2}{\partial x^2}\phi\right)+5\lambda u\phi+10u\left(\frac{\partial^3}{\partial x^3}\phi\right)$$
$$+5\left(\frac{\partial}{\partial x}\phi\right)\left(\frac{\partial^2}{\partial x^2}u\right)+5u^2\left(\frac{\partial}{\partial x}\phi\right)-15\left(\frac{\partial^3}{\partial y\partial x^2}\phi\right)=0.$$

Example 5.3.5. KdV equation 4.3.1, the corresponding outputs are:

****Bäcklund transformation****

After pretreatment, a new equation is:

$$2\left(\frac{\partial^2}{\partial x\partial t}v\right)+12\left(\frac{\partial^2}{\partial x^2}w\right)\left(\frac{\partial^2}{\partial x^2}v\right)+2\left(\frac{\partial^4}{\partial x^4}v\right).$$

We introduce the new constraint:

$$\left(\frac{\partial^2}{\partial x^2}w\right)+\left(\frac{\partial}{\partial x}v\right)^2=\lambda.$$

It can be written in the linear combination of y-polynomial:

$$\mathscr{Y}_{0,2x}(v,w)-\lambda=0$$
$$\mathscr{Y}_{0,3x}(v,w)+\mathscr{Y}_{t,0}(v,w)+3\lambda\mathscr{Y}_{0,x}(v,w)=0.$$

The Bäcklund transformation is:

$$(D_x^2-\lambda)F\cdot G=0$$
$$(D_x^3+D_t+3\lambda D_x)F\cdot G=0.$$

The Lax pair is:

$$\left(\frac{\partial^2}{\partial x^2}\phi\right)+u\phi-\lambda\phi=0$$
$$\left(\frac{\partial^3}{\partial x^3}\phi\right)+3u\left(\frac{\partial}{\partial x}\phi\right)+\left(\frac{\partial}{\partial t}\phi\right)+3\lambda\left(\frac{\partial}{\partial x}\phi\right)=0.$$

****Conservation Law****

The expressions of $\mathscr{I}_n$ are:

$$\mathscr{I}_1=-\frac{u}{2},\qquad \mathscr{I}_2=\frac{1}{4}\left(\frac{\partial}{\partial x}u\right),$$
$$\mathscr{I}_3=-\frac{1}{8}\left(\frac{\partial^2}{\partial x^2}u\right)-\frac{u^2}{8},\qquad \mathscr{I}_4=\frac{1}{4}u\frac{\partial}{\partial x}u+\frac{1}{16}\left(\frac{\partial^3}{\partial x^3}u\right).$$

The expressions of $\mathscr{F}_n$ are:

$$\mathscr{F}_1=-\frac{3u^2}{2}-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}u\right),\qquad \mathscr{F}_2=\frac{1}{4}\left(\frac{\partial^3}{\partial x^3}u\right)+\frac{3}{2}u\left(\frac{\partial}{\partial x}u\right),$$
$$\mathscr{F}_3=-\frac{u^3}{2}-u\left(\frac{\partial^2}{\partial x^2}u\right)-\frac{1}{8}\left(\frac{\partial^4}{\partial x^4}u\right)-\frac{5}{8}\left(\frac{\partial}{\partial x}u\right)^2.$$

Example 5.3.6. KP equation 4.3.6, the corresponding outputs are:

****Bäcklund transformation****

After pretreatment, a new equation is:

$$2\left(\frac{\partial^2}{\partial x\partial t}v\right)+12\left(\frac{\partial^2}{\partial x^2}w\right)\left(\frac{\partial^2}{\partial x^2}v\right)+2\left(\frac{\partial^4}{\partial x^4}v\right)+6\left(\frac{\partial^2}{\partial y^2}v\right).$$

We introduce the new constraint:

$$\left(\frac{\partial^2}{\partial x^2}w\right)+\left(\frac{\partial}{\partial x}v\right)^2+\left(\frac{\partial}{\partial y}v\right)=\lambda.$$

It can be written in the linear combination of $\mathscr{Y}$-polynomial:

$$\mathscr{Y}_{0,0,y}(v,w)+\mathscr{Y}_{0,2x,0}(v,w)-\lambda=0$$
$$\mathscr{Y}_{t,0,0}(v,w)+\mathscr{Y}_{0,3x,0}(v,w)$$
$$-3\mathscr{Y}_{0,x,y}(v,w)+3\lambda\mathscr{Y}_{0,x,0}(v,w)=0.$$

The Bäcklund transformation is:

$$(D_x^2+D_y-\lambda)F\cdot G=0$$
$$(D_t+D_x^3-3D_xD_y+3\lambda D_x)F\cdot G=0.$$

The Lax pair is:

$$\left(\frac{\partial^2}{\partial x^2}\phi\right)+\left(\frac{\partial}{\partial y}\phi\right)+u\phi-\lambda\phi=0$$
$$\left(\frac{\partial^3}{\partial x^3}\phi\right)+3u\left(\frac{\partial}{\partial x}\phi\right)+\left(\frac{\partial}{\partial t}\phi\right)-3\left(\frac{\partial^2}{\partial x\partial y}q\right)\phi$$
$$-3\left(\frac{\partial^2}{\partial x\partial y}\phi\right)+3\lambda\left(\frac{\partial}{\partial x}\phi\right)=0.$$

****Conservation Law****

The expressions of $\mathscr{I}_n$ are:

$$\mathscr{I}_1=-\frac{u}{2},\qquad \mathscr{I}_2=\frac{1}{4}\left(\frac{\partial}{\partial x}u\right)+\frac{1}{4}\left(\int\frac{\partial}{\partial y}udx\right),$$
$$\mathscr{I}_3=-\frac{1}{8}u^2-\frac{1}{8}\left(\frac{\partial^2}{\partial x^2}u\right)-\frac{1}{4}\left(\frac{\partial}{\partial y}u\right)-\frac{1}{8}\int\int\frac{\partial^2}{\partial y^2}udxdx.$$

The expressions of $\mathscr{F}_n$ are:

$$\mathscr{F}_1=-\frac{3}{2}u^2-\frac{3}{2}\left(\frac{\partial}{\partial y}u\right)-\frac{3}{2}\int\int\left(\frac{\partial^2}{\partial y^2}u\right)dxdx-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}u\right),$$

$$\mathscr{F}_2 = \frac{3}{2}\left(u\frac{\partial}{\partial x}u\right) + \frac{3}{2}\int\int\left(\frac{\partial^2}{\partial y^2}u\right)dxdx + \left(\frac{\partial^2}{\partial x\partial y}u\right) + \frac{1}{4}\left(\frac{\partial^3}{\partial x^3}u\right) + \frac{3}{4}\int\int\int\left(\frac{\partial^3}{\partial y^3}u\right)dxdxdx + \frac{3}{2}\int u\left(\frac{\partial}{\partial y}u\right)dx.$$

The expressions of $\mathscr{G}_n$ are:

$$\mathscr{G}_1 = \frac{3}{2}\left(\frac{\partial}{\partial x}u\right), \qquad \mathscr{G}_2 = -\frac{3}{4}\frac{\partial^2}{\partial x^2}u - \frac{3}{4}\left(\frac{\partial}{\partial y}u\right).$$

6. Conclusions

In this paper, with the help of the Bell polynomials, a Maple program *PDEBellII* is developed to construct the bilinear forms, bilinear BTs, Lax pairs and conservation laws of the KdV-type equations. The main points of program *PDEBellII* should be as below:

- Based on the relation between the Hirota operators and Bell polynomials, the Bell polynomials expression of a given soliton equation can be directly mapped into its corresponding bilinear equation(s). Thus, the key to bilinearization is to find the appropriate variable transformation which can be used to transform the original soliton equation into the corresponding Bell polynomials expression. For program *PDEBellII*, the homogeneous balance principle is used for finding this kind of variable transformation. Package *PDEBellII* can be applied to several different kinds of soliton equations, which include (1 + 1)-dimensional, (1 + 2)-dimensional, (1 + 3)-dimensional soliton equations, variable coefficient soliton equations, and soliton equations with integration terms. Compared with program *PDEBell* [35], program *PDEBellII* can also handle the mKdV-type equation and the soliton equations which dissatisfy the dimensionless scheme.
- Bilinear BTs, Lax pairs and infinite conservation laws are important characteristics of integrable equations. Based on the two-field condition, the $\mathscr{Y}$-polynomials BTs and bilinear BTs of some KdV-type equations can be obtained in a quick and natural manner. Moreover, with the help of identity (2.13), their corresponding Lax pairs can be directly derived from the $\mathscr{Y}$-polynomials BTs. Furthermore, the infinite conservation laws can also be obtained by transforming the $\mathscr{Y}$-polynomials BTs into a Riccati-type equation and divergence-type equation, respectively. In terms of this algorithm, we further develop program *PDEBellII* to derive bilinear BTs, Lax pairs and infinite conservation laws of KdV-type equations. To the best knowledge of the authors, this is the first Maple program on this work.
- For the mKdV-type equations, the construction of bilinear BTs needs more complex combinations and classification, and this is our following work.

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Appendix. Global parameters

Here we briefly describe some parameters and program commands available in *PDEBellII*. In Table A.1, the abbreviations are used for the global parameters.

Table A.1
Global parameters for program *PDEBellII*.

Parameter name	Parameter description
devar	Dependent variable in the input equation.
devars	Function expressions with respect to ***devar***.
coevar	The set of coefficients in the input equation.
indevar	Independent variables.
indevar_num	The number of ***indevar***.
new_devar	New dependent variable introduced in terms of homogeneous balance principle.
Eq	New equation after introducing ***new_devar***.
P_Set	The required set of P-polynomials.
Y_Set	The required set of $\mathscr{Y}$-polynomials.
PO_Set	The bilinear representations of polynomials in ***P_Set***.
YO_Set	The bilinear representations of polynomials in ***Y_Set***.
YE_Set	The set of $\mathscr{Y}$-polynomials in ***Y_Set***.
P_Eq	The equation written in P-polynomials form for ***Eq***.
PO_Eq	The bilinear representations of ***P_Eq***.
Y_Eq	The equation written in $\mathscr{Y}$-polynomials form for ***Eq***.
YO_Eq	The bilinear representations of ***Y_Eq***.
c_value	The value of c in ***new_devar***.
cflag	The real–imaginary symbol for c. It will be set to be 0 if c is a real number while it will be set to be 1 if c is an imaginary number.
Lax_x	The first equation of Lax pair.
Lax_t	The second equation of Lax pair.
In_exp	Expressions of conserved densities $\mathscr{I}_n$.
In_term	Recursion relations for conserved densities $\mathscr{I}_n$.
Fn_exp	The expressions of the first fluxes $\mathscr{F}_n$.
Fn_term	Recursion relations for the first fluxes $\mathscr{F}_n$.
Gn_exp	The expressions of the second fluxes $\mathscr{G}_n$.
Gn_term	Recursion relations for the second fluxes $\mathscr{G}_n$.
n	The number of $\mathscr{I}_n$ which should be calculated. The value of n is indicated by the user and it is default value is 3.

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