Journal of the Physical Society of Japan 84, 034002 (2015)
http://dx.doi.org/10.7566/JPSJ.84.034002

Multi-Dark Soliton Solutions of the Two-Dimensional Multi-Component Yajima–Oikawa Systems

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(Received August 28, 2014; accepted January 5, 2015; published online February 6, 2015)

We present a general form of multi-dark soliton solutions of two-dimensional (2D) multi-component soliton systems. Multi-dark soliton solutions of the 2D and 1D multi-component Yajima–Oikawa (YO) systems, which are often called the 2D and 1D multi-component long wave-short wave resonance interaction systems, are studied in detail. Taking the 2D coupled YO system with two short wave and one long wave components as an example, we derive the general N-dark–dark soliton solution in both the Gram type and Wronski type determinant forms for the 2D coupled YO system via the KP hierarchy reduction method. By imposing certain constraint conditions, the general N-dark–dark soliton solution of the 1D coupled YO system is further obtained. The dynamics of one dark–dark and two dark–dark solitons are analyzed in detail. In contrast with bright–bright soliton collisions, it is shown that dark–dark soliton collisions are elastic and there is no energy exchange among solitons in different components. Moreover, the dark–dark soliton bound states including the stationary and moving ones are discussed. For the stationary case, the bound states exist up to arbitrary order, whereas, for the moving case, only the two-soliton bound state is possible under the condition that the coefficients of nonlinear terms have opposite signs.

1. Introduction

The two-dimensional (2D) coupled Yajima–Oikawa (YO) system, or the so-called 2D coupled long wave-short wave resonance interaction system:[1)]

$$i(S_t^{(1)} + S_y^{(1)}) - S_{xx}^{(1)} + LS^{(1)} = 0, \tag{1}$$

$$i(S_t^{(2)} + S_y^{(2)}) - S_{xx}^{(2)} + LS^{(2)} = 0, \tag{2}$$

$$L_t = 2(\sigma_1|S^{(1)}|^2 + \sigma_2|S^{(2)}|^2)_x, \tag{3}$$

where $\sigma_1 = \pm 1,\ \sigma_2 = \pm 1,\ S^{(1)}, S^{(2)} \in \mathbf{C},\ L, t, x, y \in \mathbf{R}$, was derived as a two-component generalization of the 2D YO system (the 2D long wave-short wave resonance interaction system) by virtue of the reductive perturbation method.[2,3)] The 2D coupled YO system can be written in the vector form:

$$i(\mathbf{S}_t + \mathbf{S}_y) - \mathbf{S}_{xx} + L\mathbf{S} = 0, \tag{4}$$

$$L_t = 2(\|\mathbf{S}\|^2)_x, \tag{5}$$

where $\mathbf{S} = (S^{(1)}, S^{(2)})^{\mathrm{T}}$ and $\|\cdots\|$ is defined as

$$\|\mathbf{S}\|^2 = \sigma_1 S^{(1)} S^{(1)*} + \sigma_2 S^{(2)} S^{(2)*}. \tag{6}$$

The above 2D coupled YO system can be generalized into a multi-component system, which is cast into the following vector form

$$i(\mathbf{S}_t + \mathbf{S}_y) - \mathbf{S}_{xx} + L\mathbf{S} = 0, \tag{7}$$

$$L_t = 2(\|\mathbf{S}\|^2)_x, \tag{8}$$

where $\mathbf{S} = (S^{(1)}, S^{(2)}, \ldots, S^{(M)})^{\mathrm{T}}$, $\sigma_k = \pm 1$ for $k = 1, 2, \ldots, M$, and $\|\cdots\|$ is defined as

$$\|\mathbf{S}\|^2 = \sum_{k=1}^{M} \sigma_k S^{(k)} S^{(k)*}. \tag{9}$$

The 1D YO system was proposed as a model equation for the interaction of a Langmuir wave with an ion-acoustic wave in a plasma by Yajima and Oikawa,[4)] which was also derived from several other physical contexts.[3,5–7)] The 1D YO system was solved exactly by the inverse scattering transform method[4,8)] and the (classical) Hirota's bilinear method (which uses the perturbation expansion).[9,10)] It admits both bright and dark soliton solutions. The 2D YO system for the resonant interaction between a long surface wave and a short internal wave in a two-layer fluid was presented and the bright and dark soltion solutions are provided by using the Hirota's bilinear method.[2,3)] The Painlevé property for the 2D YO system was investigated[11)] and some special solutions such as positons, dromions, instantons and periodic wave solutions were constructed.[11,12)] For the 2D coupled case, the multi-bright soliton solutions expressed by the Wronskian to Eqs. (1)–(3) were provided.[1)] Later, the bright N-soliton solutions in the Gram type determinant for the multi-component YO system were obtained.[13,14)] Similar to the single component case, the Painlevé property and dromion solutions to Eqs. (1)–(3) were discussed.[15)] In a recent paper by Kanna et al., one and two mixed soliton solutions for the multi-component YO system were constructed.[16)] Very recently, the rogue wave solutions for the single YO system in 1D case were derived.[17,18)]

However, to the best of our knowledge, general multi-dark soliton solutions for the multi-component 2D and 1D YO system have not been reported yet. Moreover, general multi-dark soliton solutions for the multi-component 2D soliton systems have never been previously reported in the literature. In this paper, by using the reduction method of the KP hierarchy, we derive and prove the general N-dark–dark soliton solutions to Eqs. (1)–(3) and their dynamics are discussed in detail. Based on the KP theory, the general N-dark–dark soliton solutions expressed by either the Gram type or Wronski type determinant are obtained directly from the τ-functions of the KP hierarchy by means of reductions. Similar to the 1D coupled nonlinear Schrödinger (NLS) equation case,[19)] it is very difficult to obtain multi-soliton solutions for the 1D coupled YO system since some nontrivial constraints for parameters need to be satisfied. In this paper, we show that the general N-dark–dark soliton solutions for the 1D coupled YO system can be obtained from the ones for the 2D coupled YO system by the reduction technique. Thus the constraint condition is naturally obtained.

Kanna et al. analyzed the bound states of the bright–dark solitons[16] and the bound states of the bright–bright solitons[20] for the 2D coupled YO system. In this paper, we investigate the bound states of dark–dark solitons for the 2D coupled YO system. The bound states of dark–dark solitons for the 1D coupled NLS equation with mixed focusing and defocusing nonlinearities was reported for the first time by Ohta et al.[19] The authors pointed out that the bound states of three or higher-dark–dark solitons do not exist. In the present paper, we show that the bound states of dark–dark solitons can be formed in the 2D coupled YO system including the stationary ones and moving ones. For the stationary case, the bound states of arbitrary order dark–dark solitons exist, whereas, for the moving case, only the bound states of two dark–dark solitons occur when the coefficients of nonlinear term take opposite signs.

The rest of the paper is organized as follows. In Sect. 2, we briefly present the bilinearization procedure for the 2D coupled YO system. The N-dark–dark soliton solutions with the implicit dispersion relation are derived through the classical Hirota's bilinear method which uses the perturbation expansion. In Sect. 3, the general N-dark–dark soliton solutions expressed by the Gram type and Wronski type determinants are obtained directly through the reduction method of the KP hierarchy. Moreover, the general N-dark–dark soliton solutions for the 1D coupled YO system are further obtained by imposing a constraint on parameters. Section 4 is devoted to the analysis of dynamics of one and two dark soltions, which suggests that the energy of solitons is completely transmitted through each component when two dark–dark solitons collide. In Sect. 5, the bound states including the stationary case and the moving case are discussed in detail. In Sect. 6, the general N-dark–dark soliton solutions for the 1D and 2D multi-component YO system are briefly presented. Appendix A and B present the proofs of Lemma 2.1 and Lemma 2.4, respectively.

2. Dark–Dark Soliton Solutions of the 2D Coupled YO System

Under the dependent variable transformation

$$S^{(1)} = \frac{G}{F}, \quad S^{(2)} = \frac{H}{F}, \quad L = -2(\log F)_{xx}, \tag{10}$$

Eqs. (1)–(3) can be converted into the following bilinear equations:

$$[i(D_t + D_y) - D_x^2]G \cdot F = 0, \tag{11}$$

$$[i(D_t + D_y) - D_x^2]H \cdot F = 0, \tag{12}$$

$$(D_tD_x - 2C)F \cdot F + 2\sigma_1 GG^* + 2\sigma_2 HH^* = 0, \tag{13}$$

where G and H are complex functions and F is a real function, C is an arbitrary constant and * denotes the complex conjugation hereafter. The Hirota's D-operators are defined as

$$D_x^n f(x) \cdot g(x) \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n f(x)g(x')\bigg|_{x=x'}.$$

2.1 Dark–dark soliton solutions by Hirota's direct method

In this subsection, we look for soliton solutions by the Hirota's bilinear method which uses the perturbation expansion.[10] To this end, we expand G, H, and F in terms of power series of a small parameter ε:

$$G = \rho_1 \exp(i\zeta_1)[1 + \varepsilon g_1 + \varepsilon^2 g_2 + \varepsilon^3 g_3 + \cdots], \tag{14}$$

$$H = \rho_2 \exp(i\zeta_2)[1 + \varepsilon h_1 + \varepsilon^2 h_2 + \varepsilon^3 h_3 + \cdots], \tag{15}$$

$$F = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \cdots, \tag{16}$$

where $\zeta_j = \alpha_j x + \beta_j y - \delta_j t + \zeta_{j0}$ and $\rho_j, \alpha_j, \beta_j, \delta_j, \zeta_{j0}$, $(j = 1, 2)$ are real parameters.

Substituting (14)–(16) into (11)–(13), we obtain the following constraint condition:

$$C = \sigma_1\rho_1^2 + \sigma_2\rho_2^2, \quad \delta_j = \beta_j - \alpha_j^2, \quad j = 1, 2. \tag{17}$$

Arranging each order of ε and solving the resultant set of linear partial differential equations recursively, we obtain the one-soliton solution:

$$G = \rho_1 \exp(i\zeta_1)[1 + A\exp(\eta)], \tag{18}$$

$$H = \rho_2 \exp(i\zeta_2)[1 + B\exp(\eta)], \tag{19}$$

$$F = 1 + \exp(\eta), \quad \eta = k_x x + k_y y + \omega t + \eta_0, \tag{20}$$

with

$$A = \frac{(2\alpha_1 k_x - k_y - \omega)i - k_x^2}{(2\alpha_1 k_x - k_y - \omega)i + k_x^2},$$

$$B = \frac{(2\alpha_2 k_x - k_y - \omega)i - k_x^2}{(2\alpha_2 k_x - k_y - \omega)i + k_x^2},$$

$$\omega k_x = \sigma_1\rho_1^2[2 - (A + A^*)] + \sigma_2\rho_2^2[2 - (B + B^*)],$$

where k_x, k_y, ω, and η_0 are arbitrary complex constants.

Furthermore, we obtain the 2-soliton solution

$$G = \rho_1 \exp(i\zeta_1)[1 + A_1\exp(\eta_1) + A_2\exp(\eta_2) + C_{12}A_1A_2\exp(\eta_1 + \eta_2)], \tag{21}$$

$$H = \rho_2 \exp(i\zeta_2)[1 + B_1\exp(\eta_1) + B_2\exp(\eta_2) + C_{12}B_1B_2\exp(\eta_1 + \eta_2)], \tag{22}$$

$$F = 1 + \exp(\eta_1) + \exp(\eta_2) + C_{12}\exp(\eta_1 + \eta_2), \tag{23}$$

with

$$A_j = \frac{(2\alpha_1 k_{x,j} - k_{y,j} - \omega_j)i - k_{x,j}^2}{(2\alpha_1 k_{x,j} - k_{y,j} - \omega_j)i + k_{x,j}^2},$$

$$B_j = \frac{(2\alpha_2 k_{x,j} - k_{y,j} - \omega_j)i - k_{x,j}^2}{(2\alpha_2 k_{x,j} - k_{y,j} - \omega_j)i + k_{x,j}^2},$$

$$\omega_j k_{x,j} = \rho_1^2[2 - (A_j + A_j^*)] + \rho_2^2[2 - (B_j + B_j^*)],$$

$$\eta_j = k_{x,j}x + k_{y,j}y + \omega_j t + \eta_{j0},$$

and

$$C_{12} = -\frac{C_{12}^-}{C_{12}^+},$$

$$C_{12}^- = (\theta_1 - \theta_2)(\omega_1 - \omega_2) + \sigma_1\rho_1^2(A_1A_2^* + A_2A_1^* - 2) + \sigma_2\rho_2^2(B_1B_2^* + B_2B_1^* - 2),$$

$$C_{12}^+ = (\theta_1 + \theta_2)(\omega_1 + \omega_2) + \sigma_1\rho_1^2(A_1A_2 + A_1^*A_2^* - 2) + \sigma_2\rho_2^2(B_1B_2 + B_1^*B_2^* - 2),$$

where $k_{x,j}, k_{y,j}, \omega_j$, and η_{j0} $(j = 1, 2)$ are arbitrary complex constants.

In general, one can get the N-dark–dark soliton solutions of the 2D coupled YO system (1)–(3):

$$G = \rho_1 e^{i\zeta_1} \times \sum_{\mu=0,1} \exp\left(\sum_{j=1}^{N} \mu_j(\eta_j + a_j) + \sum_{1\le j<l}^{N} \mu_j\mu_l c_{jl}\right), \tag{24}$$

$$H = \rho_2 e^{i\zeta_2} \times \sum_{\mu=0,1} \exp\left(\sum_{j=1}^{N} \mu_j(\eta_j + b_j) + \sum_{1\le j<l}^{N} \mu_j\mu_l c_{jl}\right), \tag{25}$$

$$F = \sum_{\mu=0,1} \exp\left(\sum_{j=1}^{N} \mu_j\eta_j + \sum_{1\le j<l}^{N} \mu_j\mu_l c_{jl}\right), \tag{26}$$

with

$$\sigma_1\rho_1^2 + \sigma_2\rho_2^2 = C,$$

$$\zeta_k = \alpha_k x + \beta_k y - (\beta_k - \alpha_k^2)t + \zeta_{k0}, \quad k = 1, 2$$

$$e^{a_j} \equiv A_j = \frac{(2\alpha_1 k_{x,j} - k_{y,j} - \omega_j)i - k_{x,j}^2}{(2\alpha_1 k_{x,j} - k_{y,j} - \omega_j)i + k_{x,j}^2},$$

$$e^{b_j} \equiv B_j = \frac{(2\alpha_2 k_{x,j} - k_{y,j} - \omega_j)i - k_{x,j}^2}{(2\alpha_2 k_{x,j} - k_{y,j} - \omega_j)i + k_{x,j}^2},$$

$$\omega_j k_{x,j} = \sigma_1\rho_1^2[2 - (A_j + A_j^*)] + \sigma_2\rho_2^2[2 - (B_j + B_j^*)],$$

and

$$C_{jl} \equiv e^{c_{jl}} = -\frac{C_{jl}^-}{C_{jl}^+},$$

$$C_{jl}^- = (k_{x,j} - \theta_l)(\omega_j - \omega_l) + \sigma_1\rho_1^2(A_jA_l^* + A_lA_j^* - 2) + \sigma_2\rho_2^2(B_jB_l^* + B_lB_j^* - 2),$$

$$C_{jl}^+ = (k_{x,j} + \theta_l)(\omega_j + \omega_l) + \sigma_1\rho_1^2(A_jA_l + A_j^*A_l^* - 2) + \sigma_2\rho_2^2(B_jB_l + B_j^*B_l^* - 2),$$

where the notation $\sum_{\mu=0,1}$ represents all possible combinations $\mu_j = 0, 1$ and $\eta_j = k_{x,j}x + k_{y,j}y + \omega_j t + \eta_{j0}$ for $j = 1, 2, 3 \ldots, N$.

2.2 *General N-dark–dark soliton solutions in the Gram determinant form*

In this subsection, we construct an alternative form of the N-soliton solution based on the KP hierarchy reduction method.

Lemma 2.1. *The following bilinear equations in the KP hierarchy:*[19]

$$(D_{x_2} - D_{x_1}^2 - 2aD_{x_1})\tau(k+1, l)\cdot\tau(k, l) = 0, \tag{27}$$

$$\left(\frac{1}{2}D_{x_1}D_{x_{-1}} - 1\right)\tau(k, l)\cdot\tau(k, l) = -\tau(k+1, l)\tau(k-1, l), \tag{28}$$

$$(D_{x_2} - D_{x_1}^2 - 2bD_{x_1})\tau(k, l+1)\cdot\tau(k, l) = 0, \tag{29}$$

$$\left(\frac{1}{2}D_{x_1}D_{y_{-1}} - 1\right)\tau(k, l)\cdot\tau(k, l) = -\tau(k, l+1)\tau(k, l-1), \tag{30}$$

where a and b are complex constants, and k and l are integers, have the Gram type determinant solutions

$$\tau(k, l) = |m_{ij}(k, l)|_{1\le i,j\le N}, \tag{31}$$

where the entries of the determinant are given by

$$m_{ij}(k, l) = c_{ij} + \int \phi_i(k, l)\psi_j(k, l)\, dx_1,$$

$$\phi_i(k, l) = (p_i - a)^k(p_i - b)^l \exp(\theta_i),$$

$$\psi_i(k, l) = \left(-\frac{1}{q_i + a}\right)^k\left(-\frac{1}{q_i + b}\right)^l \exp(\tilde\theta_i),$$

with

$$\theta_i = \frac{1}{p_i - a}x_{-1} + \frac{1}{p_i - b}y_{-1} + p_i x_1 + p_i^2 x_2 + \theta_{i0},$$

$$\tilde\theta_j = \frac{1}{q_i + a}x_{-1} + \frac{1}{q_i + b}y_{-1} + q_i x_1 - q_i^2 x_2 + \tilde\theta_{i0},$$

where $c_{ij}, p_i, q_i, \theta_{i0}$, and $\tilde\theta_{i0}$ $(i, j = 1, 2, \ldots, N)$ are arbitrary complex constants.

The proof is given in the Appendix.

Now we consider the reduction of the above bilinear equations in order to derive the general dark soliton solution. Assuming x_{-1}, y_{-1}, x_1 are real, x_2, a $(= i\alpha_1)$, b $(= i\alpha_2)$ are pure imaginary and $q_i = p_i^*$, $\tilde\theta_{j0} = \theta_{j0}^*$, $c_{ij} = c_{ij}^* = \delta_{ij}$, one can get

$$\tilde\theta_j = \theta_j^*, \quad m_{ji}(k, l) = m_{ij}^*(-k, -l),$$
$$\tau(k, l) = \tau^*(-k, -l). \tag{32}$$

Hence, Eqs. (27)–(30) can be recast into

$$(D_{x_2} - D_{x_1}^2 - 2i\alpha_1 D_{x_1})g\cdot f = 0, \tag{33}$$

$$\left(\frac{1}{2}D_{x_1}D_{x_{-1}} - 1\right)f\cdot f = -gg^*, \tag{34}$$

$$(D_{x_2} - D_{x_1}^2 - 2i\alpha_2 D_{x_1})h\cdot f = 0, \tag{35}$$

$$\left(\frac{1}{2}D_{x_1}D_{y_{-1}} - 1\right)f\cdot f = -hh^*, \tag{36}$$

by defining

$$f = \tau(0, 0), \quad g = \tau(1, 0), \quad h = \tau(0, 1),$$
$$g^* = \tau(-1, 0), \quad h^* = \tau(0, -1). \tag{37}$$

By introducing the independent variable transformation

$$x_1 = x, \quad x_2 = -iy,$$
$$x_{-1} = \sigma_1\rho_1^2(t - y), \quad y_{-1} = \sigma_2\rho_2^2(t - y), \tag{38}$$

i.e.,

$$\partial_x = \partial_{x_1}, \quad \partial_y = -i\partial_{x_2} - \sigma_1\rho_1^2\partial_{x_{-1}} - \sigma_2\rho_2^2\partial_{y_{-1}},$$
$$\partial_t = \sigma_1\rho_1^2\partial_{x_{-1}} + \sigma_2\rho_2^2\partial_{y_{-1}}, \tag{39}$$

Eqs. (33)–(36) become

$$[i(D_t + D_y - 2\alpha_1 D_x) - D_x^2]g\cdot f = 0, \tag{40}$$

$$[i(D_t + D_y - 2\alpha_1 D_x) + D_x^2]g^*\cdot f = 0, \tag{41}$$

$$[i(D_t + D_y - 2\alpha_2 D_x) - D_x^2]h\cdot f = 0, \tag{42}$$

$$[i(D_t + D_y - 2\alpha_2 D_x) + D_x^2]h^*\cdot f = 0, \tag{43}$$

$$[D_tD_x - 2(\sigma_1\rho_1^2 + \sigma_2\rho_2^2)]f\cdot f + 2\sigma_1\rho_1^2 gg^* + 2\sigma_2\rho_2^2 hh^* = 0. \tag{44}$$

By virtue of the following dependent variable transformation

$$S^{(1)} = \rho_1 e^{i\zeta_1}\frac{g}{f}, \quad S^{(2)} = \rho_2 e^{i\zeta_2}\frac{h}{f}, \quad L = -2(\log f)_{xx}, \tag{45}$$

with $\zeta_j = \alpha_j x + \beta_j y - (\beta_j - \alpha_j^2)t + \zeta_{j0}$ for $j = 1, 2$, the bilinear equations (40)–(44) are then transformed into the 2D YO system (1)–(3). Hence we immediately have the following theorem for the general N-dark–dark soliton solutions of Eqs. (1)–(3).

Theorem 2.2. *The N-dark–dark soliton solutions for the 2D coupled YO system (1)–(3) are*

$$S^{(1)} = \rho_1 e^{i[\alpha_1 x + \beta_1 y - (\beta_1 - \alpha_1^2)t + \zeta_{10}]} \frac{g}{f}, \tag{46}$$

$$S^{(2)} = \rho_2 e^{i[\alpha_2 x + \beta_2 y - (\beta_2 - \alpha_2^2)t + \zeta_{20}]} \frac{h}{f}, \tag{47}$$

$$L = -2(\log f)_{xx}, \tag{48}$$

where f, g, and h are Gram determinants given by

$$f = \left| \delta_{ij} + \frac{1}{p_i + p_j^*} e^{\xi_i + \xi_j^*} \right|_{N\times N},$$

$$g = \left| \delta_{ij} + \left(-\frac{p_i - i\alpha_1}{p_j^* + i\alpha_1} \right) \frac{1}{p_i + p_j^*} e^{\xi_i + \xi_j^*} \right|_{N\times N},$$

$$h = \left| \delta_{ij} + \left(-\frac{p_i - i\alpha_2}{p_j^* + i\alpha_2} \right) \frac{1}{p_i + p_j^*} e^{\xi_i + \xi_j^*} \right|_{N\times N},$$

with

$$\xi_j = p_j x - \left(\frac{\sigma_1 \rho_1^2}{p_j - i\alpha_1} + \frac{\sigma_2 \rho_2^2}{p_j - i\alpha_2} + i p_j^2 \right) y + \left(\frac{\sigma_1 \rho_1^2}{p_j - i\alpha_1} + \frac{\sigma_2 \rho_2^2}{p_j - i\alpha_2} \right) t + \xi_{j0},$$

where $\alpha_k, \beta_k, \rho_k, \zeta_{k0}$ $(k = 1, 2)$ are real constants, and p_j, ξ_{j0} are arbitrary complex constants.

Remark 2.3. By putting the parameters in (24)–(26) as

$$k_{x,j} = p_j + p_j^*, \quad k_{y,j} = -\omega_j - i(p_j^2 - p_j^{*2}),$$

$$\omega_j = \frac{\sigma_1 \rho_1^2 (p_j + p_j^*)}{|p_j - i\alpha_1|^2} + \frac{\sigma_2 \rho_2^2 (p_j + p_j^*)}{|p_j - i\alpha_2|^2},$$

$$\eta_{j0} = \xi_{j0} + \xi_{j0}^* + \frac{1}{p_j + p_j^*},$$

the N-dark–dark soliton solutions (24)–(26) are equivalent to the Gram type determinant solutions (46)–(48).

2.3 General N-dark–dark soliton solutions in the Wronskian form

In this subsection, we show that the general N-dark–dark soliton solutions for the 2D coupled YO system (1)–(3) can be expressed in the Wronskian from.

Lemma 2.4. *The following Wronskian satisfies the bilinear equations of the KP hierarchy (27)–(30):*

$$\tau(k,l) = \begin{vmatrix} \varphi_1(k,l) & \partial_{x_1}\varphi_1(k,l) & \cdots & \partial_{x_1}^{(N-1)}\varphi_1(k,l) \\ \varphi_2(k,l) & \partial_{x_1}\varphi_2(k,l) & \cdots & \partial_{x_1}^{(N-1)}\varphi_2(k,l) \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_N(k,l) & \partial_{x_1}\varphi_N(k,l) & \cdots & \partial_{x_1}^{(N-1)}\varphi_N(k,l) \end{vmatrix}, \tag{49}$$

with

$$\varphi_i(k,l) = (p_i - a)^k (p_i - b)^l \exp(\theta_i) + (q_i - a)^k (q_i - b)^l \exp(\tilde\theta_i),$$

$$\theta_i = \frac{1}{p_i - a} x_{-1} + \frac{1}{p_i - b} y_{-1} + p_i x_1 + p_i^2 x_2 + \theta_{i0},$$

$$\tilde\theta_i = \frac{1}{q_i - a} x_{-1} + \frac{1}{q_i - b} y_{-1} + q_i x_1 + q_i^2 x_2 + \tilde\theta_{i0},$$

where p_j, q_j, θ_{i0}, and $\tilde\theta_{i0}$ are arbitrary complex constants.

We provide the proof in the Appendix.

Next, we proceed to reductions. By applying the complex conjugate conditions

$$p_i^* = -q_i, \quad \theta_{i0}^* = -\tilde\theta_{i0}', \tag{50}$$

with $\exp(\tilde\theta_{i0}) = (\prod_{k=1,k\neq i}^N \frac{p_i - q_k}{q_i - q_k}) \exp(\tilde\theta_{i0}')$ and the determinant formula[21]

$$\det_{1\le i,j\le N}(p_i^{j-1}\mathcal{A}_i + q_i^{j-1}\mathcal{B}_i) = \left(\Delta(q_1, q_2, \ldots, q_N) \prod_{i=1}^N \mathcal{B}_i \right) \times \sum_{M=0}^N \sum_{1\le i_1 < \cdots < i_M \le N} \left(\prod_{1\le\mu<\nu\le M} \frac{(p_{i_\mu} - p_{i_\nu})(q_{i_\mu} - q_{i_\nu})}{(p_{i_\mu} - q_{i_\nu})(q_{i_\mu} - p_{i_\nu})} \right) \times \prod_{\nu=1}^M \frac{\mathcal{A}_{i_\nu}}{\mathcal{B}'_{i_\nu}}, \tag{51}$$

where $\mathcal{B}_i = (\prod_{k=1,k\neq i}^N \frac{p_i - q_k}{q_i - q_k}) \mathcal{B}'_i$ and Δ is the Vandermonde determinant, the following relation can be derived

$$\tau(0,0) = \mathcal{G} \sum_{M=0}^N \sum_{1\le i_1<\cdots<i_M\le N} |\mathcal{F}(i_\mu, i_\nu)|^2 \times \prod_{\nu=1}^M \exp(\theta_{i_\nu} + \theta_{i_\nu}^*), \tag{52}$$

$$\tau(1,0) = \mathcal{C}_a \cdot \mathcal{G} \sum_{M=0}^N \sum_{1\le i_1<\cdots<i_M\le N} |\mathcal{F}(i_\mu, i_\nu)|^2 \times \prod_{\nu=1}^M \left(-\frac{p_{i_\nu} - a}{p_{i_\nu}^* + a} \right) \exp(\theta_{i_\nu} + \theta_{i_\nu}^*), \tag{53}$$

$$\tau(-1,0) = \frac{1}{\mathcal{C}_a} \cdot \mathcal{G} \sum_{M=0}^N \sum_{1\le i_1<\cdots<i_M\le N} |\mathcal{F}(i_\mu, i_\nu)|^2 \times \prod_{\nu=1}^M \left(-\frac{p_{i_\nu}^* + a}{p_{i_\nu} - a} \right) \exp(\theta_{i_\nu} + \theta_{i_\nu}^*), \tag{54}$$

$$\tau(0,1) = \mathcal{C}_b \cdot \mathcal{G} \sum_{M=0}^N \sum_{1\le i_1<\cdots<i_M\le N} |\mathcal{F}(i_\mu, i_\nu)|^2 \times \prod_{\nu=1}^M \left(-\frac{p_{i_\nu} - b}{p_{i_\nu}^* + b} \right) \exp(\theta_{i_\nu} + \theta_{i_\nu}^*), \tag{55}$$

$$\tau(0,-1) = \frac{1}{\mathcal{C}_b} \cdot \mathcal{G} \sum_{M=0}^N \sum_{1\le i_1<\cdots<i_M\le N} |\mathcal{F}(i_\mu, i_\nu)|^2 \times \prod_{\nu=1}^M \left(-\frac{p_{i_\nu}^* + b}{p_{i_\nu} - b} \right) \exp(\theta_{i_\nu} + \theta_{i_\nu}^*), \tag{56}$$

where

$$\mathcal{G} = \Delta(-p_1^*, -p_2^*, \ldots, -p_N^*) \prod_{i=1}^{N} \left(\prod_{k=1,k\neq i}^{N} -\frac{p_i + p_k^*}{p_i^* - p_k^*} \right) \times \exp(-\theta_i^*),$$

$$\mathcal{C}_a = \prod_{i=1}^{N} (-1)^N (p_i^* + a), \quad \mathcal{C}_b = \prod_{i=1}^{N} (-1)^N (p_i^* + b),$$

$$\mathcal{F}(i_\mu, i_\nu) = \prod_{1\leq\mu<\nu\leq M} \frac{(p_{i_\mu} - p_{i_\nu})}{(p_{i_\mu} + p_{i_\nu}^*)}.$$

Next, setting

$$f = \frac{\tau(0,0)}{\mathcal{G}}, \quad g = \frac{\tau(1,0)}{\mathcal{C}_a \mathcal{G}}, \quad h = \frac{\tau(0,1)}{\mathcal{C}_b \mathcal{G}},$$
$$g^* = \frac{\mathcal{C}_a \tau(-1,0)}{\mathcal{G}}, \quad h^* = \frac{\mathcal{C}_b \tau(0,-1)}{\mathcal{G}}, \tag{57}$$

Eqs. (27)–(30) become

$$(D_{x_2} - D_{x_1}^2 - 2aD_{x_1}) g \cdot f = 0, \tag{58}$$

$$\left(\frac{1}{2} D_{x_1} D_{x_{-1}} - 1\right) f \cdot f = -gg^*, \tag{59}$$

$$(D_{x_2} - D_{x_1}^2 - 2bD_{x_1}) h \cdot f = 0, \tag{60}$$

$$\left(\frac{1}{2} D_{x_1} D_{y_{-1}} - 1\right) f \cdot f = -hh^*, \tag{61}$$

which are nothing but the bilinear equations (33)–(36) if $a = i\alpha_1$, $b = i\alpha_2$. Then, by applying the same transformations of independent variables (38) and dependent variables (45), the 2D coupled YO system (1)–(3) can be obtained.

In summary, we obtain an alternative form of the N-dark–dark soliton solution of Eqs. (1)–(3) in the following theorem:

Theorem 2.5. *The N-dark–dark soliton solutions for the 2D coupled YO system (1)–(3) are*

$$S^{(1)} = \rho_1 e^{i[\alpha_1 x + \beta_1 y - (\beta_1 - \alpha_1^2)t + \zeta_{10}]} \frac{g}{f}, \tag{62}$$

$$S^{(2)} = \rho_2 e^{i[\alpha_2 x + \beta_2 y - (\beta_2 - \alpha_2^2)t + \zeta_{20}]} \frac{h}{f}, \tag{63}$$

$$L = -2(\log f)_{xx}, \tag{64}$$

where f, g, and h are Wronskians expressed by the form

$$f = \frac{1}{\mathcal{G}} \begin{vmatrix} \varphi_1 & \partial_{x_1}\varphi_1 & \cdots & \partial_{x_1}^{(N-1)}\varphi_1 \\ \varphi_2 & \partial_{x_1}\varphi_2 & \cdots & \partial_{x_1}^{(N-1)}\varphi_2 \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_N & \partial_{x_1}\varphi_N & \cdots & \partial_{x_1}^{(N-1)}\varphi_N \end{vmatrix},$$

$$g = \frac{1}{\mathcal{C}_1\mathcal{G}} \begin{vmatrix} \bar\varphi_1 & \partial_{x_1}\bar\varphi_1 & \cdots & \partial_{x_1}^{(N-1)}\bar\varphi_1 \\ \bar\varphi_2 & \partial_{x_1}\bar\varphi_2 & \cdots & \partial_{x_1}^{(N-1)}\bar\varphi_2 \\ \vdots & \vdots & \cdots & \vdots \\ \bar\varphi_N & \partial_{x_1}\bar\varphi_N & \cdots & \partial_{x_1}^{(N-1)}\bar\varphi_N \end{vmatrix},$$

$$h = \frac{1}{\mathcal{C}_2\mathcal{G}} \begin{vmatrix} \tilde\varphi_1 & \partial_{x_1}\tilde\varphi_1 & \cdots & \partial_{x_1}^{(N-1)}\tilde\varphi_1 \\ \tilde\varphi_2 & \partial_{x_1}\tilde\varphi_2 & \cdots & \partial_{x_1}^{(N-1)}\tilde\varphi_2 \\ \vdots & \vdots & \cdots & \vdots \\ \tilde\varphi_N & \partial_{x_1}\tilde\varphi_N & \cdots & \partial_{x_1}^{(N-1)}\tilde\varphi_N \end{vmatrix},$$

with

$$\mathcal{G} = \Delta(-p_1^*, -p_2^*, \ldots, -p_N^*) \times \prod_{j=1}^{N} \left(\prod_{k=1,k\neq j}^{N} -\frac{p_j + p_k^*}{p_j^* - p_k^*} \right) \exp(-\xi_j^*),$$

$$\mathcal{C}_1 = \prod_{j=1}^{N} (-1)^N (p_j^* + i\alpha_1), \quad \mathcal{C}_2 = \prod_{j=1}^{N} (-1)^N (p_j^* + i\alpha_2),$$

and

$$\varphi_j = \exp(\xi_j) + \exp(-\xi_j^*),$$
$$\bar\varphi_j = (p_j - i\alpha_1)\exp(\xi_j) - (p_j^* + i\alpha_1)\exp(-\xi_j^*),$$
$$\tilde\varphi_j = (p_j - i\alpha_2)\exp(\xi_i) - (p_j^* + i\alpha_2)\exp(-\xi_j^*),$$
$$\xi_j = p_j x - \left(\frac{\sigma_2\rho_1^2}{p_j - i\alpha_1} + \frac{\sigma_2\rho_2^2}{p_j - i\alpha_2} + ip_j^2\right) y + \left(\frac{\sigma_1\rho_1^2}{p_j - i\alpha_1} + \frac{\sigma_2\rho_2^2}{p_j - i\alpha_2}\right) t + \xi_{j0},$$

where $\alpha_k, \beta_k, \rho_k, \zeta_{k0}$, $(k = 1, 2)$ are real constants, and p_j, ξ_{j0} are complex constants.

3. General N-Dark–Dark Soliton Solutions of the 1D Coupled YO System

The general N-dark–dark soliton solutions for the 1D coupled YO system can be derived from the one for 2D coupled YO system by further reductions. In what follows, we show the detailed process.

First, it is noted that the Gram determinant solution of the bilinear equations (27)–(30) in the KP hierarchy can be rewritten as

$$\tau(k,l) = \left| \delta_{ij} + \left(-\frac{p_i - i\alpha_1}{p_j^* + i\alpha_1}\right)^k \left(-\frac{p_i - i\alpha_2}{p_j^* + i\alpha_2}\right)^l \frac{1}{p_i + p_j^*} e^{\xi_i + \xi_j^*} \right|$$
$$= \exp\left(\sum_{k=1}^{N} (\xi_k + \xi_k^*)\right) \times \left| \delta_{ij} e^{-\xi_i - \xi_i^*} + \left(-\frac{p_i - i\alpha_1}{p_j^* + i\alpha_1}\right)^k \left(-\frac{p_i - i\alpha_2}{p_j^* + i\alpha_2}\right)^l \frac{1}{p_i + p_j^*} \right|, \tag{65}$$

with

$$\xi_i + \xi_i^* = \left(\frac{1}{p_i - i\alpha_1} + \frac{1}{p_i^* + i\alpha_1}\right) x_{-1} + \left(\frac{1}{p_i - i\alpha_2} + \frac{1}{p_i^* + i\alpha_2}\right) y_{-1} + (p_i + p_i^*) x_1 + (p_i^2 - p_i^{*2}) x_2 + \xi_{i0} + \xi_{i0}^*.$$

Thus, if p_i satisfies the constraint condition:

$$\sigma_1\rho_1^2 \left(\frac{1}{p_i - i\alpha_1} + \frac{1}{p_i^* + i\alpha_1}\right) + \sigma_2\rho_2^2 \left(\frac{1}{p_i - i\alpha_2} + \frac{1}{p_i^* + i\alpha_2}\right) = -i(p_i^2 - p_i^{*2}), \tag{66}$$

i.e.,

$$\frac{\sigma_1\rho_1^2}{|p_i-i\alpha_1|^2}+\frac{\sigma_2\rho_2^2}{|p_i-i\alpha_2|^2}=-i(p_i-p_i^*), \tag{67}$$

then we have

$$(\sigma_1\rho_1^2\partial_{x_{-1}}+\sigma_2\rho_2^2\partial_{y_{-1}})\tau(k,l)=-i\partial_{x_2}\tau(k,l), \tag{68}$$

which implies

$$(\sigma_1\rho_1^2\partial_{x_{-1}}+\sigma_2\rho_2^2\partial_{y_{-1}})f=-i\partial_{x_2}f, \tag{69}$$

by using $f=\tau(0,0)$. Moreover, differentiating with respect to x_1 once, we have

$$\sigma_1\rho_1^2 f_{x_1x_{-1}}+\sigma_2\rho_2^2 f_{x_1y_{-1}}=-if_{x_1x_2}. \tag{70}$$

Notice that Eqs. (34) and (36) can be rewritten as

$$f_{x_1x_{-1}}f-f_{x_1}f_{x_{-1}}-f^2=-gg^*, \tag{71}$$

$$f_{x_1y_{-1}}f-f_{x_1}f_{y_{-1}}-f^2=-hh^*. \tag{72}$$

From the above relations, we have

$$\begin{aligned}&-i(f_{x_1x_2}f-f_{x_2}f_{x_1})-(\sigma_1\rho_1^2+\sigma_2\rho_2^2)f^2\\&\quad+\sigma_1\rho_1^2gg^*+\sigma_2\rho_2^2hh^*=0,\end{aligned} \tag{73}$$

or the bilinear form

$$\begin{aligned}&-iD_{x_1}D_{x_2}f\cdot f-2(\sigma_1\rho_1^2+\sigma_2\rho_2^2)f^2\\&\quad+2\sigma_1\rho_1^2gg^*+2\sigma_2\rho_2^2hh^*=0.\end{aligned} \tag{74}$$

Finally, by using the transformations of independent variables

$$x_1=x,\quad x_2=-it, \tag{75}$$

i.e.,

$$\partial_{x_1}=\partial_x,\quad \partial_{x_2}=i\partial_t, \tag{76}$$

Eqs. (33)–(36) become

$$[i(D_t-2\alpha_1D_x)-D_x^2]g\cdot f=0, \tag{77}$$

$$[i(D_t-2\alpha_1D_x)+D_x^2]g^*\cdot f=0, \tag{78}$$

$$[i(D_t-2\alpha_2D_x)-D_x^2]h\cdot f=0, \tag{79}$$

$$[i(D_t-2\alpha_2D_x)+D_x^2]h^*\cdot f=0, \tag{80}$$

$$\begin{aligned}&[D_tD_x-2(\sigma_1\rho_1^2+\sigma_2\rho_2^2)]f\cdot f\\&\quad+2\sigma_1\rho_1^2gg^*+2\sigma_2\rho_2^2hh^*=0.\end{aligned} \tag{81}$$

By similar transformations of dependent variables (45), the above bilinear equations are converted into the 1D coupled YO system. Thus we have the following theorem about N-dark–dark soliton solutions for the 1D coupled YO system.

Theorem 3.6. *The two-component generalization of 1D YO system*

$$iS_t^{(1)}-S_{xx}^{(1)}+LS^{(1)}=0, \tag{82}$$

$$iS_t^{(2)}-S_{xx}^{(2)}+LS^{(2)}=0, \tag{83}$$

$$L_t=2(\sigma_1|S^{(1)}|^2+\sigma_2|S^{(2)}|^2)_x, \tag{84}$$

has N-dark–dark soliton solution:

$$S^{(1)}=\rho_1e^{i[\alpha_1x+\alpha_1^2t+\zeta_{10}]}\frac{g}{f}, \tag{85}$$

$$S^{(2)}=\rho_2e^{i[\alpha_2x+\alpha_2^2t+\zeta_{20}]}\frac{h}{f}, \tag{86}$$

$$L=-2(\log f)_{xx}, \tag{87}$$

where f, g, and h are Gram determinants given by

$$f=\left|\delta_{ij}+\frac{1}{p_i+p_j^*}e^{\xi_i+\xi_j^*}\right|_{N\times N},$$

$$g=\left|\delta_{ij}+\left(-\frac{p_i-i\alpha_1}{p_j^*+i\alpha_1}\right)\frac{1}{p_i+p_j^*}e^{\xi_i+\xi_j^*}\right|_{N\times N},$$

$$h=\left|\delta_{ij}+\left(-\frac{p_i-i\alpha_2}{p_j^*+i\alpha_2}\right)\frac{1}{p_i+p_j^*}e^{\xi_i+\xi_j^*}\right|_{N\times N},$$

where $\xi_j=p_jx-ip_j^2t+\xi_{j0}$, α_k, ρ_k, ζ_{k0} $(k=1,2)$ *are real constants,* p_j, ξ_{j0} *are complex constants, and these parameters satisfy the constraint conditions:*

$$\frac{\sigma_1\rho_1^2}{|p_j-i\alpha_1|^2}+\frac{\sigma_2\rho_2^2}{|p_j-i\alpha_2|^2}=-i(p_j-p_j^*). \tag{88}$$

Remark 3.7. Compared with the two dimensional case, the parameters in the N-dark–dark soliton solutions of the 1D coupled YO system need to satisfy some constraint conditions. In fact, by rewriting the solutions (46)–(48) in the two dimensional case into the similar forms as (65), one can get

$$\begin{aligned}\xi_i+\xi_i^*&=(p_i+p_i^*)\\&\times\left[x-\left(\frac{\sigma_1\rho_1^2}{|p_i-i\alpha_1|^2}+\frac{\sigma_2\rho_2^2}{|p_i-i\alpha_2|^2}+i(p_i-p_i^*)\right)y\right.\\&\left.+\left(\frac{\sigma_1\rho_1^2}{|p_i-i\alpha_1|^2}+\frac{\sigma_2\rho_2^2}{|p_i-i\alpha_2|^2}\right)t\right]+\xi_{i0}+\xi_{i0}^*.\end{aligned} \tag{89}$$

It is easy to find that the constraint conditions in the 1D case are nothing but the zero condition for the coefficients of y in the 2D case. It is interesting that the similar constraint conditions are obtained in finding the N-dark–dark soliton solutions for the coupled NLS equation.[19)]

4. Dynamics of Dark–Dark Solitons

4.1 Single dark–dark solitons

To obtain a single dark–dark soliton solution in Eqs. (1)–(3), we take $N=1$ in the formula (46)–(48). The Gram determinants read

$$f=1+\frac{1}{p_1+p_1^*}e^{\xi_1+\xi_1^*}, \tag{90}$$

$$g=1-\frac{1}{p_1+p_1^*}\frac{p_1-i\alpha_1}{p_1^*+i\alpha_1}e^{\xi_1+\xi_1^*}, \tag{91}$$

$$h=1-\frac{1}{p_1+p_1^*}\frac{p_1-i\alpha_2}{p_1^*+i\alpha_2}e^{\xi_1+\xi_1^*}, \tag{92}$$

and then the one-dark–dark soliton solution can be written as

$$\begin{aligned}S^{(1)}&=\frac{\rho_1}{2}e^{i[\alpha_1x+\beta_1y-(\beta_1-\alpha_1^2)t+\zeta_{10}]}\\&\times\left[1+K_1^{(1)}-(1-K_1^{(1)})\tanh\left(\frac{\xi_1+\xi_1^*+\Theta_1}{2}\right)\right],\end{aligned} \tag{93}$$

$$\begin{aligned}S^{(2)}&=\frac{\rho_2}{2}e^{i[\alpha_2x+\beta_2y-(\beta_2-\alpha_2^2)t+\zeta_{20}]}\\&\times\left[1+K_1^{(2)}-(1-K_1^{(2)})\tanh\left(\frac{\xi_1+\xi_1^*+\Theta_1}{2}\right)\right],\end{aligned} \tag{94}$$

$$L=-\frac{1}{2}(p_1+p_1^*)^2\,\mathrm{sech}^2\left(\frac{\xi_1+\xi_1^*+\Theta_1}{2}\right), \tag{95}$$

with

$$
\begin{aligned}
e^{\Theta_1} &= \frac{1}{p_1 + p_1^*} = \frac{1}{2a_1}, \\
K_1^{(1)} &= -\frac{p_1 - i\alpha_1}{p_1^* + i\alpha_1} = -\frac{a_1 + i(b_1 - \alpha_1)}{a_1 - i(b_1 - \alpha_1)}, \\
K_1^{(2)} &= -\frac{p_1 - i\alpha_2}{p_1^* + i\alpha_2} = -\frac{a_1 + i(b_1 - \alpha_2)}{a_1 - i(b_1 - \alpha_2)}, \\
\xi_1 + \xi_1^* &= 2a_1 x \\
&\quad - \left(\frac{2\sigma_1 a_1 \rho_1^2}{a_1^2 + (b_1 - \alpha_1)^2} + \frac{2\sigma_2 a_1 \rho_2^2}{a_1^2 + (b_1 - \alpha_2)^2} - 4a_1 b_1 \right) y \\
&\quad + \left(\frac{2\sigma_1 a_1 \rho_1^2}{a_1^2 + (b_1 - \alpha_1)^2} + \frac{2\sigma_2 a_1 \rho_2^2}{a_1^2 + (b_1 - \alpha_2)^2} \right) t + 2\xi_{10R},
\end{aligned}
$$

where $p_1 = a_1 + ib_1, a_1, b_1, \xi_{10R}, \alpha_i, \beta_i, \rho_i, \zeta_{i0}$, $(i = 1, 2)$ are real constants and ξ_{10} is a complex constant.

From (93)–(95), the intensity functions of the short wave components $|S^{(1)}|, |S^{(2)}|$ and the long-wave component L move at velocity $-\frac{\sigma_1\rho_1^2}{a_1^2+(b_1-\alpha_1)^2} - \frac{\sigma_2\rho_2^2}{a_1^2+(b_1-\alpha_2)^2}$ along the x-direction. As $x, y \to \pm\infty$, $|S^{(1)}| \to |\rho_1|, |S^{(2)}| \to |\rho_2|$, $-L \to 0$.

Denoting $K_1^{(1)} = \exp(2i\phi_1^{(1)})$ and $K_1^{(2)} = \exp(2i\phi_1^{(2)})$, the phases of the short wave components $S^{(1)}$ and $S^{(2)}$ acquire shifts in the amount of $2\phi_1^{(1)}$ and $2\phi_1^{(2)}$ but the long wave component $-L$ phase shifts is zero as x, y vary from $-\infty$ to $+\infty$ if $2\phi_1^{(1)}$ and $2\phi_1^{(2)}$ represent the phases of constants $K_1^{(1)}$ and $K_1^{(1)}$ respectively. Without loss of generality, we can assume $2\phi_1^{(1)}, 2\phi_1^{(2)} \in (-\pi, \pi]$, (or $\phi_1^{(1)}, \phi_1^{(2)} \in (-\frac{\pi}{2}, \frac{\pi}{2}]$). Then the intensities of the center of the solitons ($\xi_1 + \xi_1^* + \Theta_1 = 0$) are $|S^{(1)}|_{\text{center}} = |\rho_1| \cos\phi_1^{(1)}$, $|S^{(2)}|_{\text{center}} = |\rho_2| \cos\phi_1^{(2)}$ and $-L_{\text{center}} = 2a_1^2$. For the short wave components, the fact that the center intensities are lower than the background intensities implies these solitons are dark–dark solitons.

There are two different cases corresponding to values of α_1 and α_2:

(i) $\alpha_1 = \alpha_2$. In this case, $K_1^{(1)} = K_1^{(2)}$, i.e., $\phi_1^{(1)} = \phi_1^{(2)}$, this means the short wave components $S^{(1)}$ and $S^{(2)}$ are proportional to each other. In this situation, the dark–dark soliton solution for the coupled YO system is equivalent to the dark soliton solution in the single-component YO system, so it is viewed as degenerate case similar to the coupled NLS equation.[19] We illustrate these degenerate solitons in Fig. 1.

(ii) $\alpha_1 \neq \alpha_2$. The condition $K_1^{(1)} \neq K_1^{(2)}$, i.e., $\phi_1^{(1)} \neq \phi_1^{(2)}$ suggests that the components $S^{(1)}$ and $S^{(2)}$ have different degrees of darkness at the center. In this non-degenerate single dark–dark solitons of the coupled YO system (1)–(3), the components $S^{(1)}$ and $S^{(2)}$ are not proportional to each other. As is shown in Fig. 2, the intensity of the component $S^{(1)}$ is black, but the intensity of the component $S^{(2)}$ is gray at their centers.

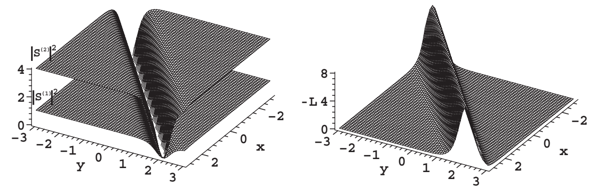
4.2 Two-dark–dark solitons

The two-dark–dark soliton solution can be obtained by taking $N = 2$ in the formula (46)–(48). In this case, we have

$$S^{(1)} = \rho_1 e^{i[\alpha_1 x + \beta_1 y - (\beta_1 - \alpha_1^2)t + \zeta_{10}]} \frac{g_2}{f_2}, \tag{96}$$

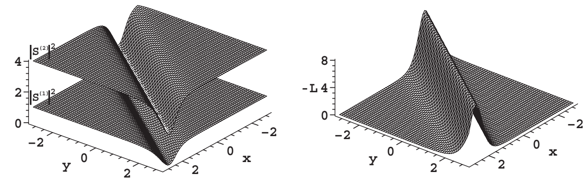
$$S^{(2)} = \rho_2 e^{i[\alpha_2 x + \beta_2 y - (\beta_2 - \alpha_2^2)t + \zeta_{20}]} \frac{h_2}{f_2}, \tag{97}$$

$$L = -2(\log f_2)_{xx}, \tag{98}$$



(a) (b)

Fig. 1. Single dark–dark solitons (degenerate) at the fixed time $t = 0$ with the parameters $\sigma_1 = \sigma_2 = 1$, $p_1 = 2$, $\rho_1 = 1$, $\rho_2 = 2$, $\alpha_1 = \alpha_2 = 0$, $\beta_1 = 1$, $\beta_2 = 2$.



(a) (b)

Fig. 2. Single dark–dark solitons (non-degenerate) at the fixed time $t = 0$ with the parameters $\sigma_1 = \sigma_2 = 1$, $p_1 = 2$, $\rho_1 = 1$, $\rho_2 = 2$, $\alpha_1 = 0$, $\alpha_2 = 2$, $\beta_1 = 1$, $\beta_2 = 2$.

with

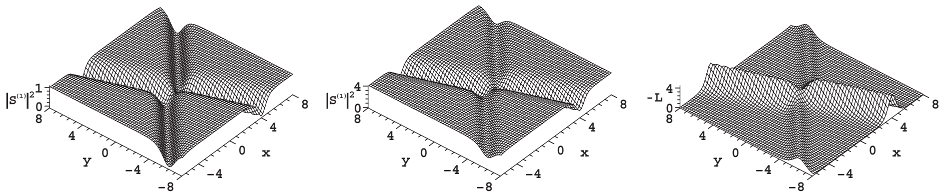
$$
\begin{aligned}
f_2 &= 1 + e^{\xi_1 + \xi_1^* + \Theta_1} + e^{\xi_2 + \xi_2^* + \Theta_2} && (99) \\
&\quad + \Omega_{12} e^{\xi_1 + \xi_1^* + \xi_2 + \xi_2^* + \Theta_1 + \Theta_2}, && (100) \\
g_2 &= 1 + K_1^{(1)} e^{\xi_1 + \xi_1^* + \Theta_1} + K_2^{(1)} e^{\xi_2 + \xi_2^* + \Theta_2} \\
&\quad + \Omega_{12} K_1^{(1)} K_2^{(1)} e^{\xi_1 + \xi_1^* + \xi_2 + \xi_2^* + \Theta_1 + \Theta_2}, && (101) \\
h_2 &= 1 + K_1^{(2)} e^{\xi_1 + \xi_1^* + \Theta_1} + K_2^{(2)} e^{\xi_2 + \xi_2^* + \Theta_2} \\
&\quad + \Omega_{12} K_1^{(2)} K_2^{(2)} e^{\xi_1 + \xi_1^* + \xi_2 + \xi_2^* + \Theta_1 + \Theta_2}, && (102)
\end{aligned}
$$

and

$$
\begin{aligned}
e^{\Theta_j} &= \frac{1}{p_j + p_j^*} = \frac{1}{2a_j}, \\
\Omega_{12} &= \left| \frac{p_1 - p_2}{p_1 + p_2^*} \right|^2 = \left| \frac{a_1 - a_2 + i(b_1 - b_2)}{a_1 + a_2 + i(b_1 - b_2)} \right|^2 \\
&= \frac{(a_1 - a_2)^2 + (b_1 - b_2)^2}{(a_1 + a_2)^2 + (b_1 - b_2)^2}, \\
K_j^{(1)} &= -\frac{p_j - i\alpha_1}{p_j^* + i\alpha_1} = -\frac{a_j + i(b_j - \alpha_1)}{a_j - i(b_j - \alpha_1)}, \\
K_j^{(2)} &= -\frac{p_j - i\alpha_2}{p_j^* + i\alpha_2} = -\frac{a_j + i(b_j - \alpha_2)}{a_j - i(b_j - \alpha_2)}, \\
\xi_j + \xi_j^* &= k_{x,j} x + k_{y,j} y + \omega_j t + 2\xi_{j0R}, \\
&= 2a_j x \\
&\quad - \left(\frac{2\sigma_1 a_j \rho_1^2}{a_j^2 + (b_j - \alpha_1)^2} + \frac{2\sigma_2 a_j \rho_2^2}{a_j^2 + (b_j - \alpha_2)^2} - 4a_j b_j \right) y \\
&\quad + \left(\frac{2\sigma_1 a_j \rho_1^2}{a_j^2 + (b_j - \alpha_1)^2} + \frac{2\sigma_2 a_j \rho_2^2}{a_j^2 + (b_j - \alpha_2)^2} \right) t + 2\xi_{j0R},
\end{aligned}
$$

where $p_j = a_j + ib_j, a_j, b_j, \alpha_j, \beta_j, \rho_j, \zeta_{j0}$, $(j = 1, 2)$ are real constants, and ξ_{10}, ξ_{20} are complex constants.

Remark 4.8. In the case of $a_2 = -a_1$ and $b_2 = b_1$ (i.e.,



(a) (b) (c)

Fig. 3. Two dark–dark solitons at the fixed time $t = 0$ the parameters $\sigma_1 = \sigma_2 = 1$, $p_1 = 1 + i$, $p_2 = 1.5 + i$, $\rho_1 = 1$, $\rho_2 = 2$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\beta_1 = 1$, $\beta_2 = 2$.

$p_2 = -p_1^*$), the denominator of Ω_{12} becomes zero. On this critical wave number, the soliton interaction shows Y-shape. This Y-shape type soliton solution is called the resonant soliton solution which was found in the KP equation. As the two-soliton solution of the KP equation, the above two-soliton solution are classified into two different types of soliton interactions:[22–25]

(1) If $a_1 a_2 < 0$, $1 < \Omega_{12}$. This case is called the O-type soliton interaction. In this case, two asymptotic soliton amplitudes $2a_1^2$ and $2a_2^2$ (in the variable $-L$) can be equivalent when $a_2 = -a_1$. The interaction peak (the maximum of L) is always greater than the sum of the asymptotic soliton amplitudes.
(2) If $a_1 a_2 > 0$, $0 < \Omega_{12} < 1$. This case is called the P-type soliton interaction. In this case, two asymptotic soliton amplitudes $2a_1^2$ and $2a_2^2$ (in the variable $-L$) cannot be equivalent. The interaction peak (the maximum of $-L$) is always less than the sum of the asymptotic soliton amplitudes.

Note that types of soliton interactions do not depend on the parameters b_1 and b_2 (i.e., the imaginary parts of p_1 and p_2). The resonant Y-shape soliton solution is obtained by taking the limit $b_2 \to b_1$ in the equal-amplitude O-type two-soliton ($a_2 = -a_1$). The interaction coefficient Ω_{12} for the two-soliton solution of the 2D coupled YO system is always non-negative although it can be negative for the KP two-soliton solution.

The collision of two dark–dark solitons is displayed in Fig. 3. It is easy to observe that the two solitons pass through each other without any change of shape, darkness and velocity in both components after collision. Hence there is no energy transfer between the two solitons or between the $S^{(1)}$ and $S^{(2)}$ components after collision. This complete transmission of energy of dark–dark soliton in both components occurs not only for $\sigma_1 = \sigma_2 = 1$ as in Fig. 3, but also for all other σ_1 and σ_2 values. For the coupled YO system, this kind of phenomenon is distinctly different from collisions of bright–bright solitons. As reported in the paper by Kanna et al.,[13] the bright–bright solitons in the short wave components $S^{(1)}$ and $S^{(2)}$ undergo shape changing (energy redistribution) collisions while the long wave component only have an elastic collision.

5. Dark–Dark Soliton Bound States

In this section, we investigate the soliton bound states. To obtain two dark–dark soliton bound states of the coupled YO system, the parameters need to satisfy $\frac{\omega_1}{k_{x,1}} = \frac{\omega_2}{k_{x,2}}$ and $\frac{\omega_1}{k_{y,1}} = \frac{\omega_2}{k_{y,2}}$, which results in two solitons with the same velocity in both short and long wave components.

5.1 *The stationary dark–dark soliton bound states*

The stationary dark–dark soliton bound states means that the common velocity equals zero. The stationary solitons for the 2D coupled YO system are possible when σ_1 and σ_2 take opposite signs. Requiring the coefficients of t in the solution (46)–(48) of the 2D coupled YO system ($\sigma_1 = 1, \sigma_2 = -1$) to be zero, i.e., $\frac{\rho_1^2}{|p_i - i\alpha_1|^2} - \frac{\rho_2^2}{|p_i - i\alpha_2|^2} = 0$. The degenerate case ($\alpha_1 = \alpha_2$) leads to $\rho_1^2 = \rho_2^2$, the dark soliton solutions for two short wave components are equivalent. The non-degenerate situation $\alpha_1 \neq \alpha_2$ can be further divided into two subcases ($p_j = a_j + ib_j$):

- Case (a) $\rho_1^2 = \rho_2^2$: We have $b_j = \frac{\alpha_1 + \alpha_2}{2}$ and $K_j^{(1)} = K_j^{(2)*}$. This case is trivial.
- Case (b) $\rho_1^2 \neq \rho_2^2$: $a_j = \sqrt{\frac{\rho_2^2 (b_j - \alpha_1)^2 - \rho_1^2 (b_j - \alpha_2)^2}{\rho_1^2 - \rho_2^2}}$.

In case (b), the soliton solution is independent of the time t and $|S^{(1)}|^2 - |S^{(2)}|^2 = \rho_1^2 - \rho_2^2$. Thus the original YO system reduces to two component linear Schrödinger equations with potential $L(x, y)$ if y is viewed as the time variable. That is to say, the linear Schrödinger equation

$$iS_y - S_{xx} + LS = 0, \tag{103}$$

possess two dark soliton solutions expressed by the form (46)–(48) with the constraints $\frac{\rho_1^2}{|p_i - i\alpha_1|^2} - \frac{\rho_2^2}{|p_i - i\alpha_2|^2} = 0$ and $\beta_k = \alpha_k^2$ for $k = 1, 2$.

Two examples of bound states are illustrated in Figs. 4 and 5, respectively. Figure 4 shows a case of $\frac{k_{y,1}}{k_{x,1}} \neq \frac{k_{y,2}}{k_{x,2}}$, which corresponds to an oblique bound state. Whereas, Fig. 5 displays a trivial case of $\frac{k_{y,1}}{k_{x,1}} = \frac{k_{y,2}}{k_{x,2}}$, which corresponds to a quasi-one-dimensional one. This kind of bound states can be viewed as two dark–dark soliton bound states of the linear Schrödinger equation (103) with potential L.

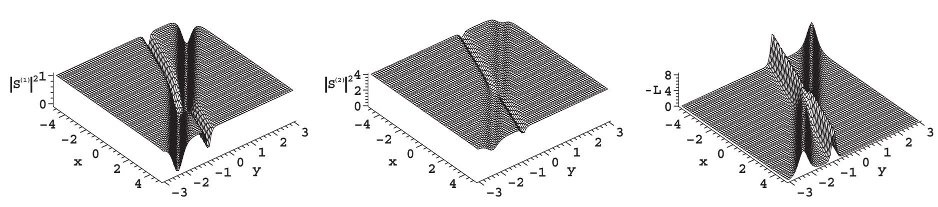
5.2 *The moving bound dark–dark soliton states*

The moving bound dark–dark soliton states require the common velocity being nonzero, i.e., $\omega_1 \neq 0$ and $\omega_2 \neq 0$. Then the parameters need to satisfy the following condition:

$$b_1 = b_2,$$
$$a_2 = \sqrt{-\frac{\sigma_1 \rho_1^2 \alpha_2'^2 (a_1^2 + \alpha_2'^2) + \sigma_2 \rho_2^2 \alpha_1'^2 (a_1^2 + \alpha_1'^2)}{\sigma_1 \rho_1^2 (a_1^2 + \alpha_2'^2) + \sigma_2 \rho_2^2 (a_1^2 + \alpha_1'^2)}}, \tag{104}$$

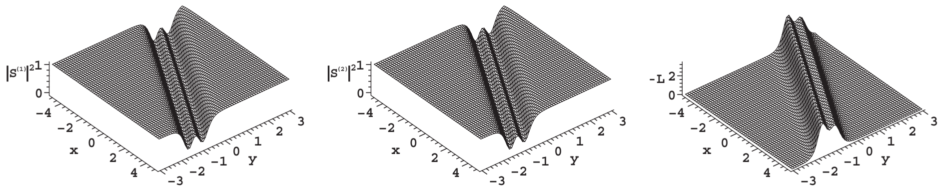
where $\alpha_1' = b_1 - \alpha_1$ and $\alpha_2' = b_2 - \alpha_2$. From the above expression, σ_1 and σ_2 must take different signs. To show these moving bound dark–dark soliton states, we choose the parameters as

$$\sigma_1 = -\sigma_2 = \alpha_1 = \beta_1 = \rho_1 = b_1 = b_2 = a_1 = 1,$$
$$\alpha_2 = \frac{1}{4}, \quad \beta_2 = 2, \quad a_2 = \frac{3}{2}, \quad \rho_2 = \frac{5\sqrt{5}}{8}, \tag{105}$$



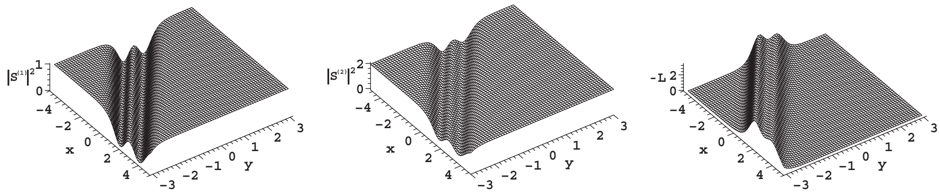
(a) (b) (c)

Fig. 4. The stationary dark–dark soliton bound states under the condition $\frac{k_{y,1}}{k_{x,1}} \neq \frac{k_{y,2}}{k_{x,2}}$ with the parameters $\sigma_1 = 1$, $\sigma_2 = -1$, $p_1 = \sqrt{3} + i$, $p_2 = \frac{9}{4} + \frac{3\sqrt{7}}{4} i$, $\rho_1 = 1$, $\rho_2 = 2$, $\alpha_1 = 1$, $\alpha_2 = -2$, $\beta_1 = 1$, $\beta_2 = 2$.



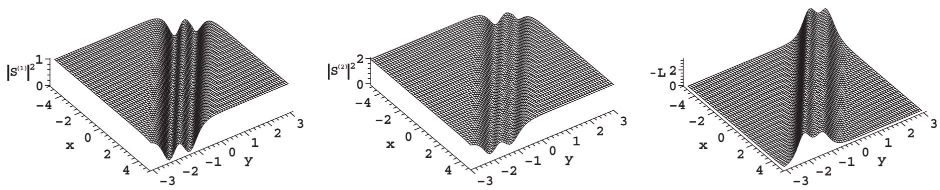
(a) (b) (c)

Fig. 5. The stationary dark–dark soliton bound states under the condition $\frac{k_{y,1}}{k_{x,1}} = \frac{k_{y,2}}{k_{x,2}}$ with the parameters $\sigma_1 = 1$, $\sigma_2 = -1$, $p_1 = 1 + \frac{3}{2} i$, $p_2 = \frac{3}{2} + \frac{3}{2} i$, $\rho_1 = 1$, $\rho_2 = 1$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\beta_1 = 1$, $\beta_2 = 2$.



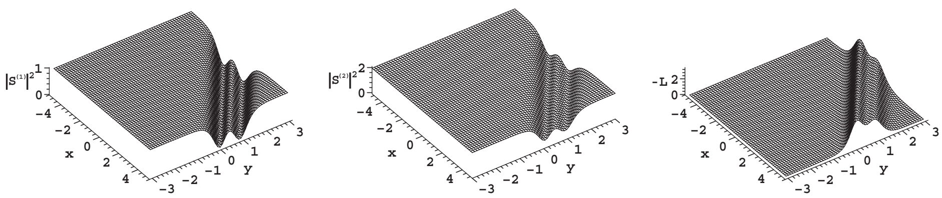
(a) (b) (c)

Fig. 6. The moving bound dark–dark soliton states with the parameters (105) at time $t = -20$.



(a) (b) (c)

Fig. 7. The moving bound dark–dark soliton states with the parameters (105) at time $t = 0$.



(a) (b) (c)

Fig. 8. The moving bound dark–dark soliton states with the parameters (105) at time $t = 20$.

and the corresponding profiles are displayed in Figs. 6–8 at different times.

We should point out that, in both stationary and moving bound states, the short wave components acquire non-zero phase shifts but the long wave component has no phase shift as x and y vary from $-\infty$ to $+\infty$. This feature is the same as the general two-dark–dark solitons. Indeed, if $2\phi_j^{(1)}$ and $2\phi_j^{(2)}$ represent the phases of complex constants $K_j^{(1)}$ and $K_j^{(2)}$ respectively, the phase shifts for all components are $S^{(1)}_{\text{phase shift}} = 2\phi_1^{(1)} + 2\phi_2^{(1)}$, $S^{(2)}_{\text{phase shift}} = 2\phi_1^{(2)} + 2\phi_2^{(2)}$ and $-L_{\text{phase shift}} = 0$. The total phase shifts of each short wave component are equal to the sum of the individual ones of the two dark solitons while the phase shifts of the long wave component are always zero.

Most recently, Sakkaravarthi and Kanna presented three bright–bright soltion bound states of the coupled YO system.[20] It is natural to see whether or not three- or higher-order dark–dark-soliton bound states exist in the coupled YO system. To ensure three- or higher-order dark–dark soliton bound states, at least three distinct values of p_j should exist. For the stationary bound states, as two subcases stated in Sect. 5.1, a_j can either take arbitrary positive value or is determined by b_j. So it is not difficult to construct the stationary dark–dark bound state up to arbitrary order. However, for the moving dark–dark soliton bound states, from (104), all b_j's values must be the same, which ends up at most two distinct values of a_j. This observation leads to a conclusion that there is no three- or higher-order moving bound states.

6. General *N* Dark Soliton Solutions of the 1D and 2D Multi-component YO Systems

As a matter of fact, we can extend our previous analysis to the 1D and 2D multi-component coupled YO systems. It is known that the multi-bright soliton solutions can be derived from the reduction of the multi-component KP hierarchy, whereas, the multi-dark soliton solutions are obtained from the reduction of the single KP hierarchy but with multiple copies of shifted singular points. Therefore, the general dark soliton solutions for the multi-component YO systems can be constructed in the same spirit as the two-component YO system. The details are omitted here, and we present only the results for both 1D and 2D multi-component YO systems.

To seek for N-dark soliton solutions, the 2D multi-component YO system consisting of M short wave components and one long wave component

$$i(S_t^{(k)} + S_y^{(k)}) - S_{xx}^{(k)} + LS^{(k)} = 0, \quad k = 1, 2, \ldots, M \tag{106}$$

$$L_t = 2\sum_{k=1}^{M} \sigma_k |S^{(k)}|_x^2, \tag{107}$$

is transformed to the following bilinear form

$$[i(D_t + D_y - 2\alpha_k D_x) - D_x^2] g_k \cdot f = 0, \quad k = 1, 2, \ldots, M \tag{108}$$

$$[D_t D_x - 2\sum_{k=1}^{M} \sigma_k \rho_k^2] f \cdot f + 2\sum_{k=1}^{M} \sigma_k \rho_k^2 g_k g_k^* = 0, \tag{109}$$

through the dependent variable transformations:

$$S^{(k)} = \rho_k e^{i[\alpha_k x + \beta_k y - (\beta_k - \alpha_k^2)t + \zeta_{k0}]} \frac{g_k}{f}, \quad k = 1, 2, \ldots, M$$

$$L = -2(\log f)_{xx}, \tag{110}$$

where $\alpha_k, \beta_k, \rho_k, \zeta_{k0}$ are real constants.

Similar to the procedure discussed in Sect. 2.2, taking into account the Gram type determinant solutions of the KP hierarchy, one can obtain N-dark soliton solutions as follows:

$$f = \left| \delta_{ij} + \frac{1}{p_i + p_j^*} e^{\xi_i + \xi_j^*} \right|_{N\times N}, \tag{111}$$

$$g_k = \left| \delta_{ij} + \left(-\frac{p_i - i\alpha_k}{p_j^* + i\alpha_k} \right) \frac{1}{p_i + p_j^*} e^{\xi_i + \xi_j^*} \right|_{N\times N}, \tag{112}$$

with

$$\xi_j = p_j x - \left(\sum_{k=1}^{M} \frac{\sigma_k \rho_k^2}{p_j - i\alpha_k} + i p_j^2 \right) y + \sum_{k=1}^{M} \frac{\sigma_k \rho_k^2}{p_j - i\alpha_k} t + \xi_{j0},$$

where p_j and ξ_{j0} are complex constants.

Starting from the Wronskian solution of the KP hierarchy, N-dark soliton solutions in the Wronskian form can be constructed in the same way in Sect. 2.3, which is of the following form

$$f = \frac{1}{\mathcal{G}} \begin{vmatrix} \varphi_1 & \partial_{x_1}\varphi_1 & \cdots & \partial_{x_1}^{(N-1)}\varphi_1 \\ \varphi_2 & \partial_{x_1}\varphi_2 & \cdots & \partial_{x_1}^{(N-1)}\varphi_2 \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_N & \partial_{x_1}\varphi_N & \cdots & \partial_{x_1}^{(N-1)}\varphi_N \end{vmatrix}, \tag{113}$$

$$g_k = \frac{1}{\mathcal{C}_k \mathcal{G}} \begin{vmatrix} \bar\varphi_1^{(k)} & \partial_{x_1}\bar\varphi_1^{(k)} & \cdots & \partial_{x_1}^{(N-1)}\bar\varphi_1^{(k)} \\ \bar\varphi_2^{(k)} & \partial_{x_1}\bar\varphi_2^{(k)} & \cdots & \partial_{x_1}^{(N-1)}\bar\varphi_2^{(k)} \\ \vdots & \vdots & \cdots & \vdots \\ \bar\varphi_N^{(k)} & \partial_{x_1}\bar\varphi_N^{(k)} & \cdots & \partial_{x_1}^{(N-1)}\bar\varphi_N^{(k)} \end{vmatrix}, \tag{114}$$

with

$$\mathcal{G} = \Delta(-p_1^*, -p_2^*, \ldots, -p_N^*) \times \prod_{j=1}^{N} \left(\prod_{k=1, k\neq j}^{N} -\frac{p_j + p_k^*}{p_j^* - p_k^*} \right) \exp(-\xi_j^*),$$

$$\mathcal{C}_k = \prod_{j=1}^{N} (-1)^N (p_j^* + i\alpha_k),$$

and

$$\varphi_j = \exp(\xi_j) + \exp(-\xi_j^*),$$

$$\bar\varphi_j^{(k)} = (p_j - i\alpha_k)\exp(\xi_j) - (p_j^* + i\alpha_k)\exp(-\xi_j^*),$$

$$\xi_j = p_j x - \left(\sum_{k=1}^{M} \frac{\sigma_k \rho_k^2}{p_j - i\alpha_k} + i p_j^2 \right) y + \sum_{k=1}^{M} \frac{\sigma_k \rho_k^2}{p_j - i\alpha_k} t + \xi_{j0},$$

where p_j and ξ_{j0} are complex constants.

By the similar procedure discussed in Sect. 3, N-dark soliton solutions for 1D multi-component YO system are provided with the same form as the one for 2D case. In other words, N-dark soliton solutions for 1D and 2D integrable systems can be deduced simultaneously without reformulating the problem. To be more specific, the following bilinear form

$$[i(D_t - 2\alpha_k D_x) - D_x^2] g_k \cdot f = 0, \quad k = 1, 2, \ldots, M \tag{115}$$

$$[D_t D_x - 2\sum_{k=1}^{M} \sigma_k \rho_k^2] f \cdot f + 2\sum_{k=1}^{M} \sigma_k \rho_k^2 g_k g_k^* = 0, \tag{116}$$

is converted from the 1D multi-component YO system

$$iS_t^{(k)} - S_{xx}^{(k)} + LS^{(k)} = 0, \quad k = 1, 2, \ldots, M \tag{117}$$

$$L_t = 2\sum_{k=1}^{M} \sigma_k |S^{(k)}|_x^2. \tag{118}$$

through dependent variable transformations

$$S^{(k)} = \rho_k e^{i[\alpha_k x + \alpha_k^2 t + \zeta_{k0}]} \frac{g_k}{f}, \quad k = 1, 2, \ldots, M \tag{119}$$

$$L = -2 \frac{\partial^2}{\partial x^2} \log f, \tag{120}$$

where $\alpha_k, \rho_k, \zeta_{k0}$ are real constants.

It is shown that, by imposing the constraint conditions

$$\sum_{k=1}^{M} \frac{\sigma_k \rho_k^2}{|p_j - i\alpha_k|^2} = -i(p_j - p_j^*), \quad j = 1, 2 \ldots, N, \tag{121}$$

the terms associated with D_y in (108)–(109) are dropped out, the bilinear equations for the 2D multi-component YO system are reduced to (115)–(116) for the 1D case. Therefore, the N-dark soliton solution for the 1D multi-component coupled YO system shares the same Gram determinant form (111)–(112) or Wronskian form (113)–(114) except the constraint conditions (121).

7. Conclusions

We have constructed the general multi-dark soliton solutions in both the 1D and the 2D multi-component coupled YO systems and analyzed their dynamical behaviors. General multi-dark soliton solutions for 2D multi-component soliton systems have never been reported in literature.

By using the classical Hirota bilinear method, we have presented the N-dark–dark soliton solutions with the implicit dispersion relation in the 2D coupled YO system containing two short wave component and one long wave component. By virtue of the reduction method of the KP hierarchy, N-dark–dark soliton solutions expressed by Gram type and Wronski type determinants are derived and proved. The process of obtaining N-dark–dark soliton solutions elucidates the connections of the YO system with other integrable systems in the KP hierarchy, which will be helpful for the further study of these systems. By further reduction, we also provide the general N-dark–dark soliton solutions for the 1D coupled YO system in the same form as the one for the 2D coupled YO system except some constraint conditions. The similar form of general N-dark soliton solutions in the 1D and 2D multi-component YO systems are constructed by simply inserting more copies of the shifts of singular points.

We have further investigated the dynamical behaviors of one and two dark–dark solitons in the 2D coupled YO system with two short wave components. In contrast with bright–bright soliton solutions, it is shown that dark–dark soliton collisions are elastic and there is no energy exchange in two components of each soliton.

Moreover, the dark–dark soliton bound states including the stationary and moving ones are discussed. For the stationary case, the bound states exist up to arbitrary order, whereas, for the moving case, only two-soliton bound state is possible under the condition that the coefficients of nonlinear terms have opposite signs.

Acknowledgments

J.C. appreciates the support by the China Scholarship Council. The project is supported by the Global Change Research Program of China (No. 2015CB953904), National Natural Science Foundation of China (Grant Nos. 11275072, 11435005, and 11428102), Research Fund for the Doctoral Program of Higher Education of China (No. 20120076110024), The Network Information Physics Calculation of basic research innovation research group of China (Grant No. 61321064), Shanghai Collaborative Innovation Center of Trustworthy Software for Internet of Things (Grant No. ZF1213), Shanghai Minhang District talents of high level scientific research project and CREST, JST.

Appendix A

In this appendix, we present the proof of Lemma 2.1 in Sect. 2.2.[19] Consider functions ϕ_i and ψ_i which satisfy the following differential and difference rules:

$$\begin{aligned}
&\partial_{x_2}\phi_i(k,l) = \partial_{x_1}^2\phi_i(k,l),\\
&\partial_{x_{-1}}\phi_i(k,l) = \phi_i(k-1,l),\\
&\phi_i(k+1,l) = (\partial_{x_1} - a)\phi_i(k,l),\\
&\partial_{x_2}\psi_i(k,l) = -\partial_{x_1}^2\psi_i(k,l),\\
&\partial_{x_{-1}}\psi_i(k,l) = -\psi_i(k+1,l),\\
&\psi_i(k-1,l) = -(\partial_{x_1} + a)\psi_i(k,l).
\end{aligned} \tag{A·1}$$

Define

$$m_{ij}(k,l) = c_{ij} + \int \phi_i(k,l)\psi_j(k,l)\, dx_1,$$

and the $N \times N$ matirx $\mathbf{m}(k,l) = (m_{ij}(k,l))_{1\le i,j\le N}$. Then one can easily verify that the matrix elements $m_{ij}(k,l)$ satisfy

$$\begin{aligned}
&\partial_{x_1} m_{ij}(k,l) = \phi_i(k,l)\psi_j(k,l),\\
&\partial_{x_2} m_{ij}(k,l)\\
&\quad = (\partial_{x_1}\phi_i(k,l))\psi_j(k,l) - \phi_i(k,l)(\partial_{x_1}\psi_j(k,l)),\\
&\partial_{x_{-1}} m_{ij}(k,l) = -\phi_i(k-1,l)\psi_j(k+1,l),\\
&m_{ij}(k+1,l) = m_{ij}(k,l) + \phi_i(k,l)\psi_j(k+1,l).
\end{aligned} \tag{A·2}$$

The functions ϕ_i, ψ_i and the matrix elements $m_{ij}(k,l)$ in Lemma 2.1 satisfy these relations.

Then with the help of (A·1) and (A·2), one can check that the derivatives and shifts of the τ function are expressed by the bordered determinants as follows:[10,19]

$$\partial_{x_1}\tau(k,l) = \begin{vmatrix} \mathbf{m}(k,l) & \Phi(k,l) \\ -\Psi(k,l) & 0 \end{vmatrix},$$

$$\partial_{x_1}^2\tau(k,l) = \begin{vmatrix} \mathbf{m}(k,l) & \partial_{x_1}\Phi(k,l) \\ -\Psi(k,l) & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{m}(k,l) & \Phi(k,l) \\ -\partial_{x_1}\Psi(k,l) & 0 \end{vmatrix},$$

$$\partial_{x_2}\tau(k,l) = \begin{vmatrix} \mathbf{m}(k,l) & \partial_{x_1}\Phi(k,l) \\ -\Psi(k,l) & 0 \end{vmatrix} - \begin{vmatrix} \mathbf{m}(k,l) & \Phi(k,l) \\ -\partial_{x_1}\Psi(k,l) & 0 \end{vmatrix},$$

$$\partial_{x_{-1}}\tau(k,l) = \begin{vmatrix} \mathbf{m}(k,l) & \Phi(k-1,l) \\ \Psi(k+1,l) & 0 \end{vmatrix},$$

$$(\partial_{x_1}\partial_{x_{-1}} - 1)\tau(k,l) = \begin{vmatrix} \mathbf{m}(k,l) & \Phi(k-1,l) & \Phi(k,l) \\ \Psi(k+1,l) & 0 & -1 \\ -\Psi(k,l) & -1 & 0 \end{vmatrix},$$

$$\tau(k+1,l) = \begin{vmatrix} \mathbf{m}(k,l) & \Phi(k,l) \\ -\Psi(k+1,l) & 1 \end{vmatrix},$$

$$\tau(k-1,l) = \begin{vmatrix} \mathbf{m}(k,l) & \Phi(k-1,l) \\ \Psi(k,l) & 1 \end{vmatrix},$$

$$(\partial_{x_1}+a)\tau(k+1,l) = \begin{vmatrix} \mathbf{m}(k,l) & \partial_{x_1}\Phi(k,l) \\ -\Psi(k+1,l) & a \end{vmatrix},$$

$$(\partial_{x_1}+a)^2\tau(k+1,l) = \begin{vmatrix} \mathbf{m}(k,l) & \partial_{x_1}^2\Phi(k,l) \\ -\Psi(k+1,l) & a^2 \end{vmatrix} + \begin{vmatrix} \mathbf{m}(k,l) & \partial_{x_1}\Phi_i(k,l) & \Phi_i(k,l) \\ -\Psi(k+1,l) & a & 1 \\ -\Psi(k,l) & 0 & 0 \end{vmatrix},$$

$$(\partial_{x_2}+a^2)\tau(k+1,l) = \begin{vmatrix} \mathbf{m}(k,l) & \partial_{x_1}^2\Phi(k,l) \\ -\Psi(k+1,l) & a^2 \end{vmatrix} - \begin{vmatrix} \mathbf{m}(k,l) & \partial_{x_1}\Phi(k,l) & \Phi(k,l) \\ -\Psi(k+1,l) & a & 1 \\ -\Psi(k,l) & 0 & 0 \end{vmatrix},$$

where the bordered determinants are defined as

$$\begin{vmatrix} \mathbf{m}(k,l) & \Phi(k,l) \\ -\Psi(k,l) & 0 \end{vmatrix} \equiv \begin{vmatrix} m_{11}(k,l) & m_{12}(k,l) & \cdots & m_{1N}(k,l) & \phi_1(k,l) \\ m_{21}(k,l) & m_{22}(k,l) & \cdots & m_{2N}(k,l) & \phi_2(k,l) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{N1}(k,l) & m_{N2}(k,l) & \cdots & m_{NN}(k,l) & \phi_N(k,l) \\ -\psi_1(k,l) & -\psi_2(k,l) & \cdots & -\psi_N(k,l) & 0 \end{vmatrix},$$

and $\Phi(k,l) = (\phi_1(k,l), \phi_2(k,l), \ldots, \phi_N(k,l))^{\mathrm{T}}$ and $\Psi(k,l) = (\psi_1(k,l), \psi_2(k,l), \ldots, \psi_N(k,l))$.

By using the above relations, one can verify

$$\begin{aligned}(D_{x_2} - D_{x_1}^2 - 2aD_{x_1})&\tau(k+1,l)\cdot\tau(k,l) \\ &= -2\begin{vmatrix} \mathbf{m}(k,l) & \partial_{x_1}\Phi(k,l) & \Phi(k,l) \\ -\Psi(k+1,l) & a & 1 \\ -\Psi(k,l) & 0 & 0 \end{vmatrix} \times |\mathbf{m}(k,l)| \\ &\quad + 2\begin{vmatrix} \mathbf{m}(k,l) & \partial_{x_1}\Phi(k,l) \\ -\Psi(k+1,l) & 0 \end{vmatrix} \times \begin{vmatrix} \mathbf{m}(k,l) & \Phi(k,l) \\ -\Psi(k,l) & 0 \end{vmatrix} \\ &\quad - 2\begin{vmatrix} \mathbf{m}(k,l) & \Phi(k,l) \\ -\Psi(k+1,l) & 0 \end{vmatrix} \times \begin{vmatrix} \mathbf{m}(k,l) & \partial_{x_1}\Phi(k,l) \\ -\Psi(k,l) & 0 \end{vmatrix},\end{aligned} \tag{A·3}$$

$$\begin{aligned}\left(\frac{1}{2}D_{x_1}D_{x_{-1}} - 1\right)&\tau(k,l)\cdot\tau(k,l) + \tau(k+1,l)\tau(k-1,l) \\ &= \begin{vmatrix} \mathbf{m}(k,l) & \Phi(k-1,l) & \Phi(k,l) \\ \Psi(k+1,l) & 0 & -1 \\ -\Psi(k,l) & -1 & 0 \end{vmatrix} \times |\mathbf{m}(k,l)| \\ &\quad - \begin{vmatrix} \mathbf{m}(k,l) & \Phi(k,l) \\ -\Psi(k,l) & 0 \end{vmatrix} \times \begin{vmatrix} \mathbf{m}(k,l) & \Phi(k-1,l) \\ \Psi(k+1,l) & 0 \end{vmatrix} \\ &\quad + \begin{vmatrix} \mathbf{m}(k,l) & \Phi(k,l) \\ -\Psi(k+1,l) & 1 \end{vmatrix} \times \begin{vmatrix} \mathbf{m}(k,l) & \Phi(k-1,l) \\ \Psi(k,l) & 1 \end{vmatrix}.\end{aligned} \tag{A·4}$$

Both (A·3) and (A·4) are identically zero because of the Jacobi identities[10] and hence $\tau(k,l)$, $\tau(k+1,l)$ and $\tau(k-1,l)$ satisfy the bilinear equations (27) and (28). In a similar way, one can prove the other two bilinear identities (29) and (30).

Appendix B

Here we present the proof of Lemma 2.4 in Sect. 2.3. Consider the τ-function

$$\tau(k,l) = \begin{vmatrix} \varphi_1(k,l) & \partial_{x_1}\varphi_1(k,l) & \cdots & \partial_{x_1}^{(N-1)}\varphi_1(k,l) \\ \varphi_2(k,l) & \partial_{x_1}\varphi_2(k,l) & \cdots & \partial_{x_1}^{(N-1)}\varphi_2(k,l) \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_N(k,l) & \partial_{x_1}\varphi_N(k,l) & \cdots & \partial_{x_1}^{(N-1)}\varphi_N(k,l) \end{vmatrix}, \tag{B·1}$$

where functions $\varphi_i(k,l)$ satisfy the following linear dispersion relations:

$$\partial_{x_1}\varphi_i^{(n)}(k,l) = \varphi_i^{(n)}(k+1,l) + a\varphi_i^{(n)}(k,l), \tag{B·2}$$

$$\begin{aligned}\partial_{x_2}\varphi_i^{(n)}(k,l) &= \partial_{x_1}^2\varphi_i^{(n)}(k,l) \\ &= \varphi_i^{(n)}(k+2,l) + 2a\varphi_i^{(n)}(k+1,l) + a^2\varphi_i^{(n)}(k,l),\end{aligned} \tag{B·3}$$

$$\partial_{x_{-1}}\varphi_i^{(n)}(k,l) = \varphi_i^{(n)}(k-1,l), \tag{B·4}$$

$$\partial_{x_1}\varphi_i^{(n)}(k,l) = \varphi_i^{(n)}(k,l+1) + b\varphi_i^{(n)}(k,l), \tag{B·5}$$

$$\begin{aligned}\partial_{x_2}\varphi_i^{(n)}(k,l) &= \partial_{x_1}^2\varphi_i^{(n)}(k,l) \\ &= \varphi_i^{(n)}(k,l+2) + 2b\varphi_i^{(n)}(k,l+1) + b^2\varphi_i^{(n)}(k,l),\end{aligned} \tag{B·6}$$

$$\partial_{y_{-1}}\varphi_i^{(n)}(k,l) = \varphi_i^{(n)}(k,l-1). \tag{B·7}$$

The functions $\varphi_i(k,l)$ in Lemma 2.4 satisfy these relations.

Let us introduce a simplified notation,

$$|n_{k,l}, n+1_{k,l}, \ldots, n+N-1_{k,l}| \equiv \begin{vmatrix} \partial_{x_1}^n\varphi_1(k,l) & \partial_{x_1}^{n+1}\varphi_1(k,l) & \cdots & \partial_{x_1}^{n+N-1}\varphi_1(k,l) \\ \partial_{x_1}^n\varphi_2(k,l) & \partial_{x_1}^{n+1}\varphi_2(k,l) & \cdots & \partial_{x_1}^{n+N-1}\varphi_2(k,l) \\ \vdots & \vdots & \cdots & \vdots \\ \partial_{x_1}^n\varphi_N(k,l) & \partial_{x_1}^{n+1}\varphi_N(k,l) & \cdots & \partial_{x_1}^{n+N-1}\varphi_N(k,l) \end{vmatrix}.$$

One can rewrite the above τ-function as

$$\begin{aligned}\tau(k,l) &= |0_{k,l}, 1_{k,l}, \ldots, N-2_{k,l}, N-1_{k,l}| \\ &= |0_{k,l}, 1_{k,l}, \ldots, N-2_{k,l}, (N-2_{k+1,l}) + a(N-2_{k,l})| \\ &= |0_{k,l}, 1_{k,l}, \ldots, N-2_{k,l}, N-2_{k+1,l}| \\ &\quad \ldots \\ &= |0_{k,l}, 0_{k+1,l}, 1_{k+1,l}, \ldots, N-3_{k+1,l}, N-2_{k+1,l}| \\ &\quad \ldots \\ &= |0_{k,l}, 0_{k+1,l}, 0_{k+2,l}, \ldots, 0_{k+N-2,l}, 0_{k+N-1,l}|.\end{aligned} \tag{B·8}$$

For simplicity, we omit subscripts ${}_{k,l}$. Thus the above τ-function is written as

$$\begin{aligned}\tau(k,l) &= |0_{k,l}, 1_{k,l}, \ldots, N-2_{k,l}, N-1_{k,l}| \\ &= |0_{k,l}, 0_{k+1,l}, 0_{k+2,l}, \ldots, 0_{k+N-2,l}, 0_{k+N-1,l}| \\ &= |0, 1, \ldots, N-2, N-1|.\end{aligned}$$

The differential formulas for $\tau(k,l)$ are derived as follows:

$$\begin{aligned}\partial_{x_1}\tau(k,l) &= \partial_{x_1}|0,1,\ldots,N-2,N-1| \\ &= |0,1,\ldots,N-2,N| + Na\tau(k,l), \qquad \text{(B·9)}\end{aligned}$$

$$\begin{aligned}\partial_{x_1}^2\tau(k,l) &= \partial_{x_1}|0,1,\ldots,N-2,N| + Na\partial_{x_1}\tau(k,l), \\ &= |0,1,\ldots,N-3,N-1,N| \\ &\quad + |0,1,\ldots,N-3,N-2,N+1| \\ &\quad + Na(\partial_{x_1} - Na)\tau(k,l) + Na\partial_{x_1}\tau(k,l) \\ &= |0,1,\ldots,N-3,N-1,N| \\ &\quad + |0,1,\ldots,N-3,N-2,N+1| \\ &\quad + 2Na\partial_{x_1}\tau(k,l) - N^2a^2\tau(k,l), \qquad \text{(B·10)}\end{aligned}$$

$$\begin{aligned}\partial_{x_2}\tau(k,l) &= \partial_{x_2}|0,1,\ldots,N-2,N-1| \\ &= -|0,1,\ldots,N-3,N-1,N| \\ &\quad + |0,1,\ldots,N-3,N-2,N+1| \\ &\quad + 2a\partial_{x_1}\tau(k,l) - Na^2\tau(k,l), \qquad \text{(B·11)}\end{aligned}$$

$$\begin{aligned}\partial_{x_{-1}}\tau(k,l) &= \partial_{x_{-1}}|0,1,\ldots,N-2,N-1| \\ &= |-1,1,\ldots,N-2,N-1|, \qquad \text{(B·12)}\end{aligned}$$

$$\begin{aligned}&\partial_{x_{-1}}\partial_{x_1}\tau(k,l) \\ &\quad = \partial_{x_{-1}}|0,1,\ldots,N-2,N| + Na\partial_{x_{-1}}\tau(k,l) \\ &\quad = |-1,1,\ldots,N-2,N| \\ &\qquad + |0,1,\ldots,N-2,N-1| \\ &\qquad + Na\partial_{x_{-1}}\tau(k,l) \\ &\quad = |-1,1,\ldots,N-2,N| + \tau(k,l) \\ &\qquad + Na\partial_{x_{-1}}\tau(k,l). \qquad \text{(B·13)}\end{aligned}$$

Consider the following determinant identities:

$$\left|\begin{array}{ccc:c:c:c:cccc} 1 & \cdots & N-2 & N-1 & N & N+1 & & & \emptyset & \\ \hdashline & \emptyset & & N-1 & N & N+1 & 0 & 1 & \cdots & N-2 \end{array}\right| = 0,$$

and

$$\left|\begin{array}{c:ccc:c:cccc:c} -1 & 1 & \cdots & N-2 & N & & & \emptyset & & N-1 \\ \hdashline -1 & & \emptyset & & N & 0 & 1 & \cdots & N-2 & N-1 \end{array}\right| = 0.$$

Applying the Laplace expansion to the left-hand side of these identities, we obtain the Plücker relations

$$\begin{aligned}&|1,\ldots,N-2,N,N+1| \times |0,1,\ldots,N-2,N-1| \\ &\quad - |1,\ldots,N-2,N-1,N+1| \times |0,1,\ldots,N-2,N| \\ &\quad + |1,\ldots,N-2,N-1,N| \times |0,1,\ldots,N-2,N+1| = 0, \\ &|-1,1,\ldots,N-2,N| \times |0,1,\ldots,N-2,N-1| \\ &\quad - |-1,1,\ldots,N-2,N-1| \times |0,1,\ldots,N-2,N| \\ &\quad + |1,2,\ldots,N-2,N-1,N| \\ &\quad \times |-1,0,1,\ldots,N-3,N-2| = 0.\end{aligned}$$

By using the τ-functions, these determinant identities are rewritten as

$$\begin{aligned}&\frac{1}{2}(-\partial_{x_2}\tau(k+1,l) + \partial_{x_1}^2\tau(k+1,l) \\ &\quad - 2(N-1)a\partial_{x_1}\tau(k+1,l) + N(N-1)a^2\tau(k+1,l)) \\ &\quad \times \tau(k,l) \\ &\quad - (\partial_{x_1}\tau(k+1,l) - Na\tau(k+1,l)) \\ &\quad \times (\partial_{x_1}\tau(k,l) - Na\tau(k,l)) \\ &\quad + \tau(k+1,l) \\ &\quad \times \frac{1}{2}(\partial_{x_2}\tau(k,l) + \partial_{x_1}^2\tau(k,l) - 2(N+1)a\partial_{x_1}\tau(k,l) \\ &\quad + N(N+1)a^2\tau(k,l)) = 0, \\ &(\partial_{x_{-1}}\partial_{x_1}\tau(k,l) - \tau(k,l) - Na\partial_{x_{-1}}\tau(k,l)) \times \tau(k,l) \\ &\quad - \partial_{x_{-1}}\tau(k,l) \times (\partial_{x_1}\tau(k,l) - Na\tau(k,l)) \\ &\quad + \tau(k+1,l) \times \tau(k-1,l) = 0,\end{aligned}$$

which are nothing but bilinear equations (27) and (28), respectively. Equations (29) and (30) can be proved in a similar way.

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