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Rational Form Solitary Wave Solutions and Doubly Periodic Wave Solutions to (1+1)-Dimensional Dispersive Long Wave Equation*

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Abstract *In this work we devise an algebraic method to uniformly construct rational form solitary wave solutions and Jacobi and Weierstrass doubly periodic wave solutions of physical interest for nonlinear evolution equations. With the aid of symbolic computation, we apply the proposed method to solving the (1+1)-dimensional dispersive long wave equation and explicitly construct a series of exact solutions which include the rational form solitary wave solutions and elliptic doubly periodic wave solutions as special cases.*

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1 Introduction

Recently, in Ref. [1], Fan proposed an algebraic method which further exceeds the applicability of the tanh method[2−8] in obtaining a series of travelling wave solutions including the soliton, triangular periodic, Jacobi, and Weierstrass doubly periodic solutions. More recently, making use of a new more general ansatz, we[9] presented the generalized algebraic method to uniformly construct a series of new and general travelling wave solution for nonlinear evolution equations (NLEEs). On the other hand, in order to extend the algebraic method[1] by Fan and the generalized method[9] by us, we[10] presented the generalized method to uniformly construct a series of soliton-like solutions and double-like periodic solutions for NLEEs. In this paper, we propose a new algebraic method, named elliptic equation rational expansion (EERE) method, which further exceeds the applicability of the above methods[1,9] in obtaining a series of travelling wave solutions including rational form solitary wave solutions, triangular periodic wave solutions and rational wave solutions.

For illustration, we apply the proposed method to solve the (1+1)-dimensional dispersive long wave equation (DLWE) and successfully construct various exact solutions especially including rational form kink-shaped and bell-shaped soliton solutions, and rational form Jacobi and Weierstrass doubly periodic solutions.

This paper is arranged as follows. In Sec. 2, a detailed derivation of the proposed method will be given. The application of the proposed method to DLWE is illustrated in Sec. 3. Conclusions will be presented finally.

2 Summary of Elliptic Equation Rational Expansion Method

In the following we would like to outline the main steps of our method:

Step 1 For a given NLEE system with some physical fields $u_i(x,y,t)$ in three variables x, y, t,

$$F_i(u_i, u_{it}, u_{ix}, u_{iy}, u_{itt}, u_{ixt}, u_{iyt}, u_{ixx}, u_{iyy}, u_{ixy}, \ldots) = 0\,, \tag{1}$$

by using the wave transformation

$$u_i(x,y,t) = U_i(\xi)\,, \qquad \xi = k(x + ly + \lambda\, t)\,, \tag{2}$$

where k, l, and λ are constants to be determined later, the nonlinear partial differential equation (1) is reduced to a nonlinear ordinary differential equation (ODE):

$$G_i(U_i, U_i', U_i'', \ldots) = 0\,. \tag{3}$$

Step 2 We introduce a new ansätz in terms of finite rational formal expansion in the following forms:

$$U_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \frac{a_{ij}\phi^j(\xi)}{(\mu\phi(\xi)+1)^j}\,, \tag{4}$$

where the new variable $\phi = \phi(\xi)$ satisfies

$$\phi'^2 = \Big(\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\Big)^2 = h_0 + h_1\phi + h_2\phi^2 + h_3\phi^3 + h_4\phi^4\,, \tag{5}$$

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and h_ρ, a_{i0}, and a_{ij} ($\rho = 0, 1, \ldots, 4$; $i = 1, 2, \ldots$; $j = 1, 2, \ldots, m_i$) are constants to be determined later.

Step 3 The underlying mechanism for a series of fundamental solutions such as polynomial, exponential, solitary wave, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions to occur is that different effects act to change wave forms in many nonlinear equations, i.e. dispersion, dissipation and nonlinearity, either separately or various combination are able to balance out. We define the degree of $U_i(\xi)$ as $D[U_i(\xi)] = n_i$, which gives rise to the degrees of other expressions as

$$D[U_i^{(\alpha)}] = n_i + \alpha\,, \quad D[U_i^{\beta}(U_j^{(\alpha)})^s] = n_i\beta + (\alpha + n_j)s\,. \quad (6)$$

Therefore we can get the value of m_i in Eq. (4). If n_i is a nonnegative integer, then we first make the transformation $U_i = V_i^{n_i}$.

Step 4 Substitute Eq. (4) into Eq. (3) along with Eq. (5) and then set all coefficients of

$$\phi^p(\xi)\left(\sqrt{\sum_{\rho=0}^{4} h_\rho \phi^\rho}\right)^q \qquad p = 1, 2, \ldots; q = 0, 1\,,$$

of the resulting system's numerator to be zero to get an over-determined system of nonlinear algebraic equations with respect to k, μ, a_{i0}, and a_{ij} ($i = 1, 2, \ldots$; $j = 1, 2, \ldots, m_i$).

Step 5 Solving the over-determined system of nonlinear algebraic equations by use of *Maple*, we will end up with the explicit expressions for k, μ, a_{i0}, and a_{ij} ($i = 1, 2, \ldots$; $j = 1, 2, \ldots, m_i$).

Step 6 It is well known that the general solutions of Eq. (5) are

Case A If $h_3 = h_4 = 0$, equation (5) possesses the following solutions:

$$\phi = \sqrt{h_0}\,\xi\,, \quad h_1 = h_2 = 0\,, \quad h_0 > 0\,; \quad (7)$$

$$\phi = -\frac{h_0}{h_1} + \frac{1}{4}h_1\xi^2\,, \quad h_2 = 0\,, \quad h_1 \neq 0\,; \quad (8)$$

$$\phi = -\frac{h_1}{2h_2} + \exp(\sqrt{h_2}\,\xi)\,, \quad h_0 = \frac{{h_1}^2}{4h_2}\,, \quad h_2 > 0\,; \quad (9)$$

$$\phi = -\frac{h_1}{2h_2} + \frac{h_1}{2h_2}\sin(\sqrt{-h_2}\,\xi)\,, \quad h_0 = 0\,, \quad h_2 < 0\,; \quad (10)$$

$$\phi = -\frac{h_1}{2h_2} + \frac{h_1}{2h_2}\sinh(\sqrt{h_2}\,\xi)\,, \quad h_0 = 0\,, \quad h_2 > 0\,. \quad (11)$$

Case B If $h_1 = h_3 = 0$, equation (5) possesses the following solutions:

$$\phi = \sqrt{-\frac{h_2}{h_4}}\,\mathrm{sech}\,(\sqrt{h_2}\,\xi)\,, \quad h_0 = 0\,, \quad h_2 > 0\,, \quad h_4 < 0\,; \quad (12)$$

$$\phi = \sqrt{-\frac{h_2}{2h_4}}\tanh\left(\sqrt{-\frac{h_2}{2}}\,\xi\right)\,, \quad h_0 = \frac{{h_2}^2}{4h_4}\,, \quad h_2 < 0\,, \quad h_4 > 0\,; \quad (13)$$

$$\phi = \sqrt{-\frac{h_2}{h_4}}\sec(\sqrt{-h_2}\,\xi)\,, \quad h_0 = 0\,, \quad h_2 < 0\,, \quad h_4 > 0\,; \quad (14)$$

$$\phi = \sqrt{\frac{h_2}{2h_4}}\tan\left(\sqrt{\frac{h_2}{2}}\,\xi\right)\,, \quad h_0 = \frac{{h_2}^2}{4h_4}\,, \quad h_2 > 0\,, \quad h_4 > 0\,; \quad (15)$$

$$\phi = -\frac{1}{\sqrt{h_4}\xi}\,, \quad h_0 = h_2 = 0\,, \quad h_4 > 0\,; \quad (16)$$

$$\phi = \mathrm{sn}\,(\xi)\,, \quad h_0 = 1\,, \quad h_2 = -(m^2+1)\,, \quad h_4 = m^2\,; \quad (17)$$

$$\phi = \mathrm{cd}\,(\xi)\,, \quad h_0 = 1\,, \quad h_2 = -(m^2+1)\,, \quad h_4 = m^2\,; \quad (18)$$

$$\phi = \mathrm{cn}\,(\xi)\,, \quad h_0 = 1 - m^2\,, \quad h_2 = 2m^2 - 1\,, \quad h_4 = -m^2\,; \quad (19)$$

$$\phi = \mathrm{dn}\,(\xi)\,, \quad h_0 = m^2 - 1\,, \quad h_2 = 2 - m^2\,, \quad h_4 = -1\,; \quad (20)$$

$$\phi = \mathrm{ns}\,(\xi)\,, \quad h_0 = m^2\,, \quad h_2 = -(m^2+1)\,, \quad h_4 = 1\,; \quad (21)$$

$$\phi = \mathrm{dc}\,(\xi)\,, \quad h_0 = m^2\,, \quad h_2 = -(m^2+1)\,, \quad h_4 = 1\,; \quad (22)$$

$$\phi = \mathrm{nc}\,(\xi)\,, \quad h_0 = -m^2\,, \quad h_2 = 2m^2 - 1\,, \quad h_4 = 1 - m^2\,; \quad (23)$$

$$\phi = \mathrm{nd}\,(\xi)\,, \quad h_0 = -1\,, \quad h_2 = 2 - m^2\,, \quad h_4 = 1 - m^2\,; \quad (24)$$

$$\phi = \mathrm{cs}\,(\xi)\,, \quad h_0 = 1 - m^2\,, \quad h_2 = 2 - m^2\,, \quad h_4 = 1\,; \quad (25)$$

$$\phi = \mathrm{sc}\,(\xi)\,, \quad h_0 = 1\,, \quad h_2 = 2 - m^2\,, \quad h_4 = 1 - m^2\,; \quad (26)$$

$$\phi = \mathrm{sd}\,(\xi)\,, \qquad h_0 = 1\,, \qquad h_2 = 2m^2 - 1\,, \qquad h_4 = m^2(m^2-1)\,; \tag{27}$$

$$\phi = \mathrm{ds}\,(\xi)\,, \qquad h_0 = m^2(m^2-1)\,, \qquad h_2 = 2m^2 - 1\,, \qquad h_4 = 1\,; \tag{28}$$

$$\phi = \mathrm{ns}\,(\xi) \pm \mathrm{cs}\,(\xi)\,, \qquad h_0 = \frac{1}{4}\,, \qquad h_2 = \frac{1-2m^2}{2}\,, \qquad h_4 = \frac{1}{4}\,; \tag{29}$$

$$\phi = \mathrm{nc}\,(\xi) \pm \mathrm{sc}\,(\xi)\,, \qquad h_0 = \frac{1-m^2}{4}\,, \qquad h_2 = \frac{1+m^2}{2}\,, \qquad h_4 = \frac{1-m^2}{4}\,; \tag{30}$$

$$\phi = \mathrm{ns}\,(\xi) \pm \mathrm{ds}\,(\xi)\,, \qquad h_0 = \frac{m^2}{4}\,, \qquad h_2 = \frac{m^2-2}{2}\,, \qquad h_4 = \frac{1}{4}\,; \tag{31}$$

$$\phi = \mathrm{sn}\,(\xi) \pm \mathrm{i}\,\mathrm{cn}\,(\xi)\,, \qquad h_0 = \frac{m^2}{4}\,, \qquad h_2 = \frac{m^2-2}{2}\,, \qquad h_4 = \frac{m^2}{4}\,, \tag{32}$$

where m is a modulus. The Jacobi elliptic functions are doubly periodic and possess properties of triangular functions:

$$\mathrm{sn}^2(\xi) + \mathrm{cn}^2(\xi) = 1\,, \qquad \mathrm{dn}^2(\xi) = 1 - m^2\mathrm{sn}^2\xi\,,$$

$$\mathrm{sn}'(\xi) = \mathrm{cn}\,(\xi)\,\mathrm{dn}\,(\xi)\,, \qquad \mathrm{cn}'(\xi) = -\,\mathrm{sn}\,(\xi)\,\mathrm{dn}\,(\xi)\,,$$

$$\mathrm{dn}'(\xi) = -m^2\,\mathrm{sn}\,(\xi)\,\mathrm{cn}\,(\xi)\,.$$

When $m \to 1$, the Jacobi functions degenerate to the hyperbolic functions, i.e.

$$\mathrm{sn}\,(\xi) \to \tanh(\xi)\,, \qquad \mathrm{cn}\,(\xi) \to \mathrm{sech}\,(\xi)\,.$$

When $m \to 0$, the Jacobi functions degenerate to the triangular functions, i.e.

$$\mathrm{sn}\,(\xi) \to \sin(\xi)\,, \qquad \mathrm{cn}\,(\xi) \to \cos(\xi)\,.$$

The more detailed notations for the Weierstrass and Jacobi elliptic functions can be found in Refs. [11] and [12].

Case C If $h_4 = 0$, equation (5) possesses the following solutions:

$$\phi = -\frac{h_2}{h_3}\mathrm{sech}^2\Big(\frac{\sqrt{h_2}}{2}\,\xi\Big)\,, \quad h_0 = h_1 = 0\,, \quad h_2 > 0\,; \tag{33}$$

$$\phi = -\frac{h_2}{h_3}\sec^2\Big(\frac{\sqrt{-h_2}}{2}\,\xi\Big)\,, \quad h_0 = h_1 = 0\,, \quad h_2 < 0\,; \tag{34}$$

$$\phi = \frac{4}{h_3\xi^2}\,, \qquad h_0 = h_1 = h_2 = 0\,; \tag{35}$$

$$\phi = \wp\Big(\frac{\sqrt{h_3}}{2}\,\xi, g_2\,, g_3\Big)\,, \qquad h_2 = 0\,, \quad h_3 > 0\,, \tag{36}$$

where $g_2 = -4h_1/h_3$ and $g_3 = -4h_0/h_3$ are called invariants of Weierstrass elliptic function.

Case D If $h_0 = h_1 = 0$, equation (5) possesses the following solutions:

$$\phi = -\frac{h_2\sec^2(\sqrt{-h_2}\,\xi/2)}{2\sqrt{-h_2h_4}\tan(\sqrt{-h_2}\,\xi/2) + h_3}\,, \qquad h_2 < 0\,; \tag{37}$$

$$\phi = \frac{h_2\mathrm{sech}^2(\sqrt{h_2}\,\xi/2)}{2\sqrt{h_2h_4}\tanh(\sqrt{h_2}\,\xi/2) - h_3}\,, \qquad h_2 > 0\,, \tag{38}$$

Thus according to Eqs. (2), (4), (7) ∼ (38) and the conclusions in ***Step 5***, we can obtain some rational formal travelling-wave solutions of Eq. (1).

Remark If we set the parameters in Eq. (4) to different values, the Fan's method can be recovered by the EERE method. The concrete case is as follows: Setting $\mu = 0$, we just recover the solutions obtained by Fan's method.[1]

3 Exact Solutions of the (1+1)-Dimensional Dispersive Long Wave Equation

Let us consider the (1+1)-dimensional dispersive long wave equation (DLWE), i.e.,

$$v_t + vv_x + w_x = 0\,, \qquad w_t + (wv)_x + \frac{1}{3}v_{xxx} = 0\,, \tag{39}$$

where $w - 1$ is the elevation of the water wave, v is the surface velocity of water along x-direction. The equation system (39) can be traced back to the works of Broer,[13] Kaup,[14] Jaulent-Miodek,[15] Martinez,[16] Kupershmidt, etc.[17] A good understanding of all solutions of Eq. (39) is very helpful for coastal and civil engineers to apply the nonlinear water wave model in a harbor and coastal design. Therefore, finding more types of exact solutions of Eq. (39) is of fundamental interest in fluid dynamics. There are an amount of papers devoted to this equations.[18−21]

By considering the wave transformations $v(x,t) = V(\xi)$, $w(x,t) = W(\xi)$, and $\xi = k(x + \lambda t)$, we change Eq. (39) to the form

$$\lambda V' + VV' + W' = 0\,, \quad \lambda W' + (WV)' + \frac{1}{3}k^2V''' = 0\,. \tag{40}$$

According to the proposed method, we expand the solution of Eq. (40) in the form,

$$V(\xi) = a_0 + \sum_{j=1}^{m_v}\frac{a_j\phi^j(\xi)}{(\mu\phi(\xi)+1)^j}\,,$$

$$W(\xi) = A_0 + \sum_{j=1}^{m_w}\frac{A_j\phi^j(\xi)}{(\mu\phi(\xi)+1)^j}\,,$$

where $\phi(\xi)$ satisfies Eq. (5). Balancing the term V''' with term $(WV)'$ and the term W' with term VV' in Eq. (40)

gives $m_v = 1$ and $m_w = 2$. So we have

$$V(\xi) = a_0 + \frac{a_1\phi(\xi)}{\mu\phi(\xi)+1}\,,$$
$$W(\xi) = A_0 + \frac{A_1\phi(\xi)}{\mu\phi(\xi)+1} + \frac{A_2\phi^2(\xi)}{(\mu\phi(\xi)+1)^2}\,, \tag{41}$$

where $\phi(\xi)$ satisfies Eq. (5).

With the aid of *Maple*, substituting Eq. (4) along with Eq. (3) into Eq. (1), yields a set of algebraic equations for $\phi^p(\xi)(\sqrt{\sum_{\rho=0}^4 h_\rho\phi^\rho}\ )^q$, $(p = 0,1,\ldots; q = 0,1)$. Setting the coefficients of these terms $\phi^p(\xi)(\sqrt{\sum_{\rho=0}^4 h_\rho\phi^\rho}\ )^q$ to zero yields a set of over-determined algebraic equations with respect to a_0, a_1, A_0, A_1, A_2, μ, and k.

By use of the *Maple* software package "Charsets" by Dongming Wang, which is based on the Wu-elimination method,[22] solving the over-determined algebraic equations, we get the following results:

$$k = k\,, \qquad a_1 = \pm\frac{2}{3}\sqrt{3\,h_4 - 3\,h_3\mu + 3h_2\mu^2 - 3\mu^3h_1 + 3\,\mu^4h_0}k\,,$$
$$a_0 = \frac{\pm(-kh_3 + 2kh_2\mu - 3k\mu^2h_1 + 4\,kh_0\mu^3) - 2\sqrt{3\,h_4 - 3\,h_3\mu + 3_2\mu^2 - 3\,\mu^3h_1 + 3\mu^4h_0}\ \lambda}{2\sqrt{3h_4 - 3h_3\mu + 3h_2\mu^2 - 3\mu^3h_1 + 3\,\mu^4h_0}}\,,$$
$$A_0 = -\frac{k^2(-12\,h_1\mu h_4 + 24\mu^2h_0h_4 + 4h_2h_4 + 3\mu^4{h_1}^2 + 8{h_0}^2\mu^6 + 6h_3\mu^2h_1)}{12(h_4 - h_3\mu + h_2\mu^2 - \mu^3h_1 + \mu^4h_0)}$$
$$\frac{k^2(-16\,h_3h_0\mu^3 - 4\,h_2\mu^3h_1 + 12\,h_2\mu^4h_0 - 12\,\mu^5h_1h_0 - {h_3}^2)}{12(h_4 - h_3\mu + h_2\mu^2 - \mu^3h_1 + \mu^4h_0)}\,,$$
$$A_1 = \frac{1}{3}\,k^2(-h_3 + 2h_2\mu - 3\,\mu^2h_1 + 4h_0\mu^3)\,, \qquad A_2 = -\frac{2}{3}k^2(h_4 - h_3\mu + h_2\mu^2 - \mu^3h_1 + \mu^4h_0)\,. \tag{42}$$

Then according to Eqs. (42), we obtain the following solutions of DLWE (***Note*** Since tan- and cot-type solutions appear in pairs with tanh- and coth-type solutions, respectively, we omit them in this paper. In addition, some rational solutions are also omitted).

Family 1 When $h_3 = h_4 = 0$ and $h_0 = -{h_1}^2/4h_2$, we obtain the following solutions for the DLWE:

$$v_1 = a_0 \pm \frac{2\sqrt{3\,h_2\mu^2 - 3\mu^3h_1 + 3\mu^4{h_1}^2/4h_2}\ k\phi}{3(\mu\,\phi + 1)}\,, \tag{43a}$$
$$w_1 = A_0 + \frac{k^2(2\,h_2\mu - 3\mu^2h_1 + {h_1}^2\mu^3/h_2)\phi}{3(\mu\,\phi+1)} - \frac{2k^2(h_2\mu^2 - \mu^3h_1 + \mu^4{h_1}^2/4h_2)\phi^2}{3(\mu\,\phi+1)^2}\,, \tag{43b}$$

where $\phi(\xi) = -h_1/2h_2 + \exp(\sqrt{h_2}\ \xi)$, $\xi = k(x+\lambda t)$, k, a_0, and A_0 are determined by Eq. (42), μ, $h_2 > 0$, h_1, a_1, and λ are arbitrary constants.

Family 2 When $h_0 = h_3 = h_4 = 0$, we obtain the following solutions of DLWE:

$$v_2 = \frac{\pm(2kh_2\mu - 3k\mu^2h_1) - 2\sqrt{3h_2\mu^2 - 3\mu^3h_1}\ \lambda}{2\sqrt{3h_2\mu^2 - 3\mu^3h_1}} \pm \frac{2\sqrt{3h_2\mu^2 - 3\mu^3h_1}\ k\phi}{3(\mu\phi+1)}\,, \tag{44a}$$
$$w_2 = A_0 + \frac{k^2(2h_2\mu - 3\mu^2h_1)\phi}{3(\mu\phi+1)} - \frac{2k^2(h_2\mu^2 - \mu^3h_1)\phi^2}{3(\mu\phi+1)^2}\,, \tag{44b}$$

where

$$\phi(\xi) = -\frac{h_1}{2h_2} + \frac{h_1\sinh(\sqrt{h_2}\ \xi)}{2h_2}\,, \qquad \xi = k(x+\lambda t)\,,$$

k and A_0 are determined by Eq. (42), $h_2 > 0$, h_1, μ, a_1, and λ are arbitrary constants.

Family 3 When $h_0 = h_1 = h_3 = 0$, we obtain the following solutions of DLWE:

$$v_3 = a_0 \pm \frac{2\sqrt{3\,h_4 + 3\,h_2\mu^2}k\sqrt{-h_2/h_4}\ \mathrm{sech}\,(\sqrt{h_2}\ \xi)}{3(\mu\sqrt{-h_2/h_4}\ \mathrm{sech}\,(\sqrt{h_2}\ \xi) + 1)}\,, \tag{45a}$$
$$w_3 = A_0 + \frac{2k^2h_2\mu\sqrt{-h_2/h_4}\ \mathrm{sech}\,(\sqrt{h_2}\ \xi)}{3(\mu\sqrt{-h_2/h_4}\ \mathrm{sech}\,(\sqrt{h_2}\ \xi)+1)} + \frac{2\,k^2(h_4 + h_2\mu^2)h_2\,\mathrm{sech}^2(\sqrt{h_2}\ \xi)}{3h_4(\mu\sqrt{-h_2/h_4}\ \mathrm{sech}\,(\sqrt{h_2}\ \xi)+1)^2}\,, \tag{45b}$$

where $\xi = k(x+\lambda t)$, k is determined by Eq. (42), $h_2 > 0$, $h_4 < 0$, μ, a_1, and λ are arbitrary constants.

Family 4 When $h_1 = h_3 = 0$ and $h_0 = -{h_2}^2/4h_4$, we obtain the following solutions of DLWE,

$$v_4 = a_0 \pm \frac{\sqrt{12\,h_4 + 12\,h_2\mu^2 + 3\mu^4{h_2}^2/h_4}\;k\sqrt{-2\,h_2/2h_4}\;\tanh(\sqrt{-2\,h_2}\;\xi/2)}{\mu\sqrt{-2h_2/h_4}\;\tanh(\sqrt{-2\,h_2}\;\xi/2)+2}\,, \tag{46a}$$

$$w_4 = A_0 - \frac{k^2(-2\,h_2\mu - {h_2}^2\mu^3/h_4)\sqrt{-2h_2/h_4}\;\tanh(\sqrt{-2\,h_2}\;\xi/2)}{3(\mu\sqrt{-2h_2/h_4}\;\tanh(\sqrt{-2\,h_2}\;\xi/2)+2)} - \frac{k^2(-4h_4 - 4h_2\mu^2 - \mu^4{h_2}^2/h_4)h_2\tanh^2(\sqrt{-2\,h_2}\;\xi/2)}{3h_4(\,\mu\,\sqrt{-2h_2/h_4}\;\tanh(\sqrt{-2h_2}\;\xi/2)+2)^2}\,, \tag{46b}$$

where $\xi = k(x + \lambda t)$, k, a_0, A_0 are determined by Eq. (42), $h_2 < 0$, $h_4 > 0$, μ, a_1, and λ are arbitrary constants.

Family 5 When $h_0 = h_1 = 0$, we obtain the following solutions of DLWE:

$$v_5 = a_0 \pm \frac{2\sqrt{3\,h_4 - 3h_3\mu + 3h_2\mu^2}\;kh_2\,\mathrm{sech}^2(\sqrt{h_2}\;\xi/2)}{3(h_2\mu\,\mathrm{sech}^2(\sqrt{h_2}\;\xi/2) + 2\sqrt{h_2h_4}\;\tanh(\sqrt{h_2}\;\xi/2) - h_3)}\,, \tag{47a}$$

$$w_5 = A_0 + \frac{k^2(-h_3 + 2\,h_2\mu)h_2\,\mathrm{sech}^2(\sqrt{h_2}\;\xi/2)}{3(h_2\mu\,\mathrm{sech}^2(\sqrt{h_2}\;\xi/2) + 2\sqrt{h_2h_4}\;\tanh(\sqrt{h_2}\;\xi/2) - h_3)} - \frac{2k^2(h_4 - h_3\mu + h_2\mu^2){h_2}^2\,\mathrm{sech}^4(\sqrt{h_2}\xi/2)}{3(h_2\mu\,\mathrm{sech}^2(\sqrt{h_2}\;\xi/2) + 2\sqrt{h_2h_4}\;\tanh(\sqrt{h_2}\;\xi/2) - h_3)^2}\,, \tag{47b}$$

where $\xi = k(x + \lambda t)$, k, a_0, and A_0 are determined by Eq. (42), $h_2 > 0$, h_3, h_4, μ, a_1, and λ are arbitrary constants.

Family 6 When $h_2 = h_4 = 0$, we obtain the following solutions for the DLWE:

$$v_6 = a_0 \pm \frac{2\sqrt{-3h_3\mu - 3\mu^3h_1 + 3\mu^4h_0}\;k\phi}{3(\mu\,\phi+1)}\,, \tag{48a}$$

$$w_6 = A_0 + \frac{k^2(-h_3 - 3\mu^2h_1 + 4h_0\mu^3)\phi}{3(\mu\phi+1)} - \frac{2k^2(-h_3\mu - \mu^3h_1 + \mu^4h_0)\phi^2}{3(\mu\phi+1)^2}\,, \tag{48b}$$

where

$$\phi(\xi) = \wp\Big(\frac{\sqrt{h_3}\,\xi}{2}, g_2, g_3\Big)\,, \quad \xi = k(x+\lambda t)\,, \quad g_2 = -4\frac{h_1}{h_3}\,, \quad g_3 = -4\frac{h_0}{h_3}\,,$$

and k, A_0 are determined by Eq. (42), $h_3 > 0$, h_0, h_1, μ, a_1, and λ are arbitrary constants.

Family 7 When $h_1 = h_3 = 0$, $h_0 = 1$, $h_2 = -(m^2+1)$, and $h_4 = m^2$, we obtain the following solutions for the DLWE:

$$v_7 = a_0 \pm \frac{2\sqrt{3m^2 + 3(-m^2-1)\mu^2 + 3\mu^4}\;k\,\mathrm{sn}\,(\xi)}{3(\mu\,\mathrm{sn}\,(\xi)+1)}\,, \tag{49a}$$

$$w_7 = A_0 + \frac{k^2(2(-m^2-1)\mu + 4\mu^3)\,\mathrm{sn}\,(\xi)}{3(\mu\,\mathrm{sn}\,(\xi)+1)} - \frac{2k^2(m^2 + (-m^2-1)\mu^2 + \mu^4)\,\mathrm{sn}^2(\xi)}{3(\mu\;\mathrm{sn}\,(\xi)+1)^2}\,, \tag{49b}$$

where $\xi = k(x + \lambda t)$, k, and A_0 are determined by Eq. (42), μ, a_1, and λ are arbitrary constants.

Family 8 When $h_1 = h_3 = 0$, $h_0 = 1$, $h_2 = -(m^2+1)$, and $h_4 = m^2$, we obtain the following solutions for the DLWE,

$$v_8 = a_0 \pm \frac{2\sqrt{3m^2 + 3(-m^2-1)\mu^2 + 3\mu^4}\;k\,\mathrm{cd}\,(\xi)}{3(\mu\,\mathrm{cd}\,(\xi)+1)}\,, \tag{50a}$$

$$w_8 = A_0 - \frac{k^2(-2\mu(-m^2-1) - 4\,\mu^3)\,\mathrm{cd}\,(\xi)}{3(\mu\,\mathrm{cd}\,(\xi)+1)} + \frac{2k^2(-m^2 - (-m^2-1)\mu^2 - \mu^4)\,\mathrm{cd}\,(\xi)}{3(\mu\,\mathrm{cd}\,(\xi)+1)^2}\,, \tag{50b}$$

where $\xi = k(x + \lambda t)$, k, and A_0 are determined by Eq. (42), μ, a_1, and λ are arbitrary constants.

Family 9 When $h_1 = h_3 = 0$, $h_0 = 1 - m^2$, $h_2 = 2m^2 - 1$, and $h_4 = -m^2$, we obtain the following solutions for the DLWE:

$$v_9 = a_0 \pm \frac{2\sqrt{-3m^2 + 3(2m^2-1)\mu^2 + 3\mu^4(1-m^2)}\;k\,\mathrm{cn}\,(\xi)}{3(\mu\;\mathrm{cn}\,(\xi)+1)}\,, \tag{51a}$$

$$w_9 = A_0 - \frac{k^2(-2(2m^2-1)\mu - 4(1-m^2)\mu^3)\,\mathrm{cn}\,(\xi)}{3(\mu\,\mathrm{cn}\,(\xi)+1)} + \frac{2k^2(m^2 - (2m^2-1)\mu^2 - \mu^4(1-m^2))\,\mathrm{cn}\,(\xi)}{3(\mu\,\mathrm{cn}\,(\xi)+1)^2}\,, \tag{51b}$$

where $\xi = k(x + \lambda t)$, k, a_0, and A_0 are determined by Eq. (42), μ, a_1, and λ are arbitrary constants.

Family 10 When $h_1 = h_3 = 0$, $h_0 = m^2 - 1$, $h_2 = 2 - m^2$, and $h_4 = -1$, we obtain the following solutions for the DLWE:

$$v_{10} = a_0 \pm \frac{2\sqrt{-3 + 3(2 - m^2)\mu^2 + 3\mu^4(m^2 - 1)}\, k\,\mathrm{dn}\,(\xi)}{3(\mu\,\mathrm{dn}\,(\xi) + 1)}\,, \tag{52a}$$

$$w_{10} = A_0 - \frac{k^2(-2(2 - m^2)\mu - 4\,(m^2 - 1)\mu^3)\,\mathrm{dn}\,(\xi)}{3(\mu\,\mathrm{dn}\,(\xi) + 1)} + \frac{2k^2(1 - (2 - m^2)\mu^2 - \mu^4(m^2 - 1))\,\mathrm{dn}\,(\xi)}{3(\mu\;\mathrm{dn}\,(\xi) + 1)^2}\,, \tag{52b}$$

where $\xi = k(x + \lambda t)$, k, a_0, and A_0 are determined by Eq. (42), μ, a_1, and λ are arbitrary constants.

Family 11 When $h_1 = h_3 = 0$, $h_0 = m^2$, $h_2 = -(1 + m^2)$, and $h_4 = 1$, we obtain the following solutions for the DLWE:

$$v_{11} = a_0 \pm \frac{2\sqrt{3 + 3(-m^2 - 1)\mu^2 + 3\mu^4 m^2}\, k\,\mathrm{ns}\,(\xi)}{3(\mu\,\mathrm{ns}\,(\xi) + 1)}\,, \tag{53a}$$

$$w_{11} = A_0 - \frac{k^2(-2\,\mu\,(-m^2 - 1) - 4m^2\mu^3)\,\mathrm{ns}\,(\xi)}{3(\mu\,\mathrm{ns}\,(\xi) + 1)} + \frac{2k^2(-1 - (-m^2 - 1)\mu^2 - \mu^4 m^2)\,\mathrm{ns}^2(\xi)}{3(\mu\,\mathrm{ns}\,(\xi) + 1)^2}\,, \tag{53b}$$

where $\xi = k(x + \lambda t)$, k, and A_0 are determined by Eq. (42), μ, a_1, and λ are arbitrary constants.

Family 12 When $h_1 = h_3 = 0$, $h_0 = m^2$, $h_2 = -(1 + m^2)$, and $h_4 = 1$, we obtain the following solutions for the DLWE:

$$v_{12} = a_0 \pm \frac{2\sqrt{3 + 3(-m^2 - 1)\mu^2 + 3\mu^4 m^2}\, k\,\mathrm{dc}\,(\xi)}{3(\mu\,\mathrm{dc}\,(\xi) + 1)}\,, \tag{54a}$$

$$w_{12} = A_0 - \frac{k^2(-2(-m^2 - 1)\mu - 4m^2\mu^3)\,\mathrm{dc}\,(\xi)}{3(\mu\,\mathrm{dc}\,(\xi) + 1)} + \frac{2k^2(-1 - (-m^2 - 1)\mu^2 - \mu^4 m^2)\,\mathrm{dc}^2(\xi)}{3(\mu\,\mathrm{dc}\,(\xi) + 1)^2}\,, \tag{54b}$$

where $\xi = k(x + \lambda t)$, k, a_0, and A_0 are determined by Eq. (42), μ, a_1, and λ are arbitrary constants.

Family 13 When $h_1 = h_3 = 0$, $h_0 = -m^2$, $h_2 = 2m^2 - 1$, and $h_4 = 1 - m^2$, we obtain the following solutions for the DLWE:

$$v_{13} = a_0 \pm \frac{2\sqrt{3 - 3m^2 + 3(2m^2 - 1)\mu^2 - 3\mu^4 m^2}\, k\,\mathrm{nc}\,(\xi)}{3(\mu\,\mathrm{nc}\,(\xi) + 1)}\,, \tag{55a}$$

$$w_{13} = A_0 + \frac{k^2(2(2m^2 - 1)\mu - 4m^2\mu^3)\,\mathrm{nc}\,(\xi)}{3(\mu\,\mathrm{nc}\,(\xi) + 1)} - \frac{2k^2(1 - m^2 + (2m^2 - 1)\mu^2 - \mu^4 m^2)\,\mathrm{nc}^2(\xi)}{3(\mu\,\mathrm{nc}\,(\xi) + 1)^2}\,, \tag{55b}$$

where $\xi = k(x + \lambda t)$, k, a_0, and A_0 are determined by Eq. (42), μ, a_1, and λ are arbitrary constants.

Family 14 When $h_1 = h_3 = 0$, $h_0 = -1$, $h_2 = 2 - m^2$, and $h_4 = m^2 - 1$, we obtain the following solutions for the DLWE:

$$v_{14} = a_0 \pm \frac{2\sqrt{3m^2 - 3 + 3(2 - m^2)\mu^2 - 3\mu^4}\, k\,\mathrm{nd}\,(\xi)}{3(\mu\,\mathrm{nd}\,(\xi) + 1)}\,, \tag{56a}$$

$$w_{14} = A_0 - \frac{k^2(-2(2 - m^2)\mu + 4\mu^3)\,\mathrm{nd}\,(\xi)}{3(\mu\,\mathrm{nd}\,(\xi) + 1)} + \frac{2k^2(-m^2 + 1 - (2 - m^2)\mu^2 + \mu^4)\,\mathrm{nd}^2(\xi)}{3(\mu\,\mathrm{nd}\,(\xi, m) + 1)^2}\,, \tag{56b}$$

where $\xi = k(x + \lambda t)$, k, and A_0 are determined by Eq. (42), μ, a_1, and λ are arbitrary constants.

Family 15 When $h_1 = h_3 = 0$, $h_0 = 1 - m^2$, $h_2 = 2 - m^2$, and $h_4 = 1$, we obtain the following solutions for the DLWE:

$$v_{15} = a_0 \pm \frac{2\sqrt{3 + 3(2 - m^2)\mu^2 + 3\mu^4(1 - m^2)}\, k\mathrm{cs}\,(\xi)}{3(\mu\mathrm{cs}\,(\xi) + 1)}\,, \tag{57a}$$

$$w_{15} = A_0 + \frac{k^2(2(2 - m^2)\mu + 4(1 - m^2)\mu^3)\mathrm{cs}\,(\xi)}{3(\mu\mathrm{cs}\,(\xi) + 1)} - \frac{2k^2(1 + (2 - m^2)\mu^2 + \mu^4(1 - m^2))\mathrm{cs}^2(\xi)}{3(\mu\mathrm{cs}\,(\xi) + 1)^2}\,, \tag{57b}$$

where $\xi = k(x + \lambda t)$, k, a_0, and A_0 are determined by Eq. (42), μ, a_1, and λ are arbitrary constants.

Family 16 When $h_1 = h_3 = 0$, $h_0 = 1$, $h_2 = 2 - m^2$, and $h_4 = 1 - m^2$, we obtain the following solutions for the DLWE:

$$v_{16} = a_0 \pm \frac{2\sqrt{3 - 3m^2 + 3(2 - m^2)\mu^2 + 3\mu^4}\, k\,\mathrm{sc}\,(\xi)}{3(\mu\,\mathrm{sc}\,(\xi) + 1)}\,, \tag{58a}$$

$$w_{16} = A_0 + \frac{k^2(2(2-m^2)\mu + 4\mu^3)\,\mathrm{sc}\,(\xi)}{3(\mu\,\mathrm{sc}\,(\xi)+1)} - \frac{2k^2(1-m^2+(2-m^2)\mu^2+\mu^4)\,\mathrm{sc}^2(\xi)}{3(\mu\,\mathrm{sc}\,(\xi)+1)^2}, \tag{58b}$$

where $\xi = k(x+\lambda t)$, k, and A_0 are determined by Eq. (42), μ, a_1, and λ are arbitrary constants.

Family 17 When $h_1 = h_3 = 0$, $h_0 = 1$, $h_2 = 2m^2 - 1$, and $h_4 = m^2(m^2-1)$, we obtain the following solutions for the DLWE:

$$v_{17} = a_0 \pm \frac{2\sqrt{3m^2(m^2-1)+3(2m^2-1)\mu^2+3\mu^4}\,k\,\mathrm{sd}\,(\xi)}{3(\mu\,\mathrm{sd}\,(\xi)+1)}, \tag{59a}$$

$$w_{17} = A_0 + \frac{k^2(2(2m^2-1)\mu+4\mu^3)\,\mathrm{sd}\,(\xi)}{3(\mu\,\mathrm{sd}\,(\xi)+1)} - \frac{2k^2(m^2(m^2-1)+(2m^2-1)\mu^2+\mu^4)\,\mathrm{sd}^2(\xi)}{3(\mu\,\mathrm{sd}\,(\xi)+1)^2}, \tag{59b}$$

where $\xi = k(x+\lambda t)$, k, a_0, and A_0 are determined by Eq. (42), μ, a_1, and λ are arbitrary constants.

Family 18 When $h_1 = h_3 = 0$, $h_0 = m^2(m^2-1)$, $h_2 = 2m^2-1$, and $h_4 = 1$, we obtain the following solutions for the DLWE:

$$v_{18} = a_0 \pm \frac{2\sqrt{3+3(2m^2-1)\mu^2+3\mu^4 m^2(m^2-1)}\,k\,\mathrm{ds}\,(\xi)}{3(\mu\,\mathrm{ds}\,(\xi)+1)}, \tag{60a}$$

$$w_{18} = A_0 - \frac{k^2(-2(2m^2-1)\mu - 4m^2(m^2-1)\mu^3)\,\mathrm{ds}\,(\xi)}{3(\mu\,\mathrm{ds}\,(\xi)+1)} + \frac{2k^2(-1-(2m^2-1)\mu^2-\mu^4 m^2(m^2-1))\,\mathrm{ds}^2(\xi)}{3(\mu\,\mathrm{ds}\,(\xi)+1)^2}, \tag{60b}$$

where $\xi = k(x+\lambda t)$, k, a_0, and A_0 are determined by Eq. (42), μ, a_1, and λ are arbitrary constants.

Family 19 When $h_1 = h_3 = 0$, $h_0 = 1/4$, $h_2 = (1-2m^2)/2$, and $h_4 = 1/4$, we obtain the following solutions for the DLWE:

$$v_{19} = a_0 \pm \frac{\sqrt{3+6(1-m^2)\mu^2+3\mu^4}\,k(\,\mathrm{ns}\,(\xi)\pm\mathrm{cs}\,(\xi))}{3(\mu(\,\mathrm{ns}\,(\xi)\pm\mathrm{cs}\,(\xi))+1)}, \tag{61a}$$

$$w_{19} = A_0 + \frac{k^2((1-m^2)\mu+\mu^3)(\,\mathrm{ns}\,(\xi)\pm\mathrm{cs}\,(\xi))}{3(\mu(\,\mathrm{ns}\,(\xi)\pm\mathrm{cs}\,(\xi))+1)} - \frac{k^2(1+2(1-m^2)\mu^2+\mu^4)(\,\mathrm{ns}\,(\xi)\pm\mathrm{cs}\,(\xi))^2}{6(\mu(\,\mathrm{ns}\,(\xi)\pm\mathrm{cs}\,(\xi))+1)^2}, \tag{61b}$$

where $\xi = k(x+\lambda t)$, k, a_0, and A_0 are determined by Eq. (42), μ, a_1, and λ are arbitrary constants.

Family 20 When $h_1 = h_3 = 0$, $h_0 = (1-m^2)/4$, $h_2 = (1+m^2)/2$, and $h_4 = (1-m^2)/4$, we obtain the following solutions for the DLWE:

$$v_{20} = a_0 \pm \frac{\sqrt{3-3m^2+6(1+m^2)\mu^2+3\mu^4(1-m^2)}k(\,\mathrm{nc}\,(\xi)\pm\,\mathrm{sc}\,(\xi))}{3(\mu(\,\mathrm{nc}\,(\xi)\pm\,\mathrm{sc}\,(\xi))+1)}, \tag{62a}$$

$$w_{20} = A_0 - \frac{k^2(-(1+m^2)\mu-(1-m^2)\mu^3)(\,\mathrm{nc}\,(\xi)\pm\,\mathrm{sc}\,(\xi))}{3(\mu\,(\,\mathrm{nc}\,(\xi)\pm\,\mathrm{sc}\,(\xi))+1)} + \frac{k^2(-1+m^2-2(1+m^2)\mu^2-\mu^4(1-m^2))(\,\mathrm{nc}\,(\xi)\pm\,\mathrm{sc}\,(\xi))^2}{6(\mu(\,\mathrm{nc}\,(\xi)\pm\,\mathrm{sc}\,(\xi))+1)^2}, \tag{62b}$$

where $\xi = k(x+\lambda t)$, k, a_0, and A_0 are determined by Eq. (42), μ, a_1, and λ are arbitrary constants.

Family 21 When $h_1 = h_3 = 0$, $h_0 = m^4/4$, $h_2 = (m^2-2)/2$, and $h_4 = 1/4$, we obtain the following solutions for the DLWE:

$$v_{21} = a_0 \pm \frac{\sqrt{3+6(m^2-2)\mu^2+3\mu^4 m^4}\,k(\,\mathrm{ns}\,(\xi)\pm\,\mathrm{ds}\,(\xi))}{3(\mu(\,\mathrm{ns}\,(\xi)\pm\,\mathrm{ds}\,(\xi))+1)}, \tag{63a}$$

$$w_{21} = A_0 - \frac{k^2(-(m^2-2)\mu-m^4\mu^3)(\,\mathrm{ns}\,(\xi)\pm\,\mathrm{ds}\,(\xi))}{3(\mu(\,\mathrm{ns}\,(\xi)\pm\,\mathrm{ds}\,(\xi))+1)} + \frac{k^2(-1-2(m^2-2)\mu^2-\mu^4 m^4)(\,\mathrm{ns}\,(\xi)\pm\,\mathrm{ds}\,(\xi))^2}{6(\mu(\,\mathrm{ns}\,(\xi)\pm\,\mathrm{ds}\,(\xi))+1)^2}, \tag{63b}$$

where $\xi = k(x+\lambda t)$, k, a_0, and A_0 are determined by Eq. (42), μ, a_1, and λ are arbitrary constants.

Family 22 When $h_1 = h_3 = 0$, $h_0 = m^2/4$, $h_2 = m^2-2/2$, and $h_4 = m^2/4$, we obtain the following solutions for the DLWE:

$$v_{22} = a_0 \pm \frac{\sqrt{3m^2+6(m^2-2)\mu^2+3\mu^4 m^2}\,k(\,\mathrm{sn}\,(\xi)\pm\mathrm{i}\,\mathrm{cn}\,(\xi))}{3(\mu(\,\mathrm{sn}\,(\xi)\pm\mathrm{i}\,\mathrm{cn}\,(\xi))+1)}, \tag{64a}$$

$$w_{22} = A_0 - \frac{k^2(-(m^2-2)\mu - \mu^3 m^2)(\operatorname{sn}(\xi) \pm \mathrm{i}\operatorname{cn}(\xi))}{3(\mu(\operatorname{sn}(\xi) \pm \mathrm{i}\operatorname{cn}(\xi)) + 1)} - \frac{k^2(m^2 + 2(m^2-2)\mu^2 + \mu^4 m^2)(\operatorname{sn}(\xi) \pm \mathrm{i}\operatorname{cn}(\xi))^2}{6(\mu(\operatorname{sn}(\xi) \pm \mathrm{i}\operatorname{cn}(\xi)) + 1)^2}, \quad \text{(64b)}$$

where $\xi = k(x + \lambda t)$, k, a_0, and A_0 are determined by Eq. (42), μ, a_1, and λ are arbitrary constants.

4 Summary and Conclusion

Based on the EERE method and symbolic computation, we obtain many types of solutions including rational form solitary wave solutions, rational form Jacobi, and Weierstrass doubly periodic solutions for the (1+1)-DLWE. The success of the EERE method lies in the following fact: We introduce a new ansätz in terms of finite rational formal expansion (4), which is more general than known ansätz in tanh method[1−3,5−9] for seeking travelling wave solutions of NLEEs. Writing the soliton solutions of a nonlinear equation as the polynomials of auxiliary variables of the elliptic equation, the equation can be changed into a nonlinear system of algebraic equations. The system can be solved with the help of symbolic computation system such as *Maple*. The algorithm can be also applied to a wide class of NLEEs in mathematical physics. The method can be extended to find rational form soliton-like solutions and more types of rational form double periodic solutions of Eq. (1). Only will the restriction on $\xi(x, y, t)$ in Eq. (4) as merely a linear function x, y, t, and the restrictions on the coefficients h_ρ, a_{i0} and a_{ij} ($\rho = 0, 1, \ldots, 4$; $i = 1, 2, \ldots$; $j = 1, 2, \ldots, m_i$) as constants be removed.

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